THE MATHEMATICAL THEORY OF

PERFECTLY ELASTIC SOLIDS

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AN ELEMENTARY TREATISE

ON THE

MATHEMATICAL THEORY OF

PERFECTLY ELASTIC SOLIDS

WITH A SHORT ACCOUNT OF

VISCOUS FLUIDS

BY

WILLIAM JOHN IBBETSON, M.A.,

FELLOW OF THE ROYAL ASTRONOMICAL SOCIETY AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY, MEMBER OF THE LONDON MATHEMATICAL SOCIETY; LATE SENIOR SCHOLAR OF CLARE COLLEGE, CAMBRIDGE.

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PREFACE.

In the present Volume I have attempted to present to the English student a continuous and fairly complete analysis of the Mathematical Theory of Elasticity, as it stands at present, together with a brief account of the physical basis on which the theory rests, and of the considerations which limit its practical application to natural materials.

It would, of course, have been impossible to exhaust so wide a subject within the limits of an elementary text-book, and my endeavour has rather been, after giving a very full and clear account of the properties of Strain and Stress, considered separately and in their relations to one another, to indicate to the student as many as possible of the various modes of further advance, in order that he may be able to read without difficulty any of the more specialised memoirs, both theoretical and practical, that constitute the already enormous literature of Elasticity.

The labour involved in the collection and arrangement of the materials for such a work can only be appreciated by those who have fully studied the subject for themselves, and it would have been largely increased had I undertaken to acknowledge in foot-notes the sources from which each theorem or formula was derived. My original intention was to complete the Volume by a Bibliographical and Historical Chapter, but during the twenty-one months that this book has been in the press the announcement of the late Dr. Todhunter's great work on the history of the

subject, and ultimately the appearance of its first volume under the editorship of Prof. K. Pearson, have led me to abandon that design, though very unwillingly. Such references as have been inserted are intended chiefly as guides to further reading.

A portion of the projected scheme has however been retained as Appendix III. (pages 162-168), on the history of Hooke's Law, and this perhaps suffers from its isolation. It must be understood that all the statements and remarks contained in it refer exclusively to its subject, and not at all to the general question of Green's Theory and the minimum number of Elastic Coefficients, on which I hold the orthodox opinion, though I cannot regard the matter as finally closed to discussion.

I have adopted the notation of Thomson & Tait's "Natural Philosophy" for Strain and Stress, in spite of its obvious theoretical deficiencies, partly because it is the one most familiar to English readers, and partly because it is so eminently readable and speakable. I am inclined personally to prefer the double-suffix notation on all other accounts, and I would suggest the following system as the most generally useful (the symbols in parentheses being those employed in the present work, and the suffixes referring to the generalised coördinate notation of Chapter V.):—Strains, $e_{\xi\xi}(e)$, $e_{\eta\eta}(f)$, $e_{\zeta\zeta}(g)$, $s_{\eta\zeta}(a)$, $s_{\zeta\xi}(b)$, $s_{\xi\eta}(c)$, $e_1(\epsilon_1)$, $e_2(\epsilon_2)$, $e_3(\epsilon_3)$; Rotations, $\theta_\xi(\Theta_1)$, $\theta_\eta(\Theta_2)$, $\theta_\zeta(\Theta_3)$; Stresses $N_{\xi\xi}(P)$, $N_{\eta\eta}(Q)$, $N_{\zeta\zeta}(R)$, $T_{\eta\zeta}(S)$, $T_{\zeta\xi}(T)$,$T_{\xi\eta}(U)$, $N_1(N_1)$, $N_2(N_2)$, $N_3(N_3)$.

I fail to see any adequate reason for modifying the established nomenclature of the subject, except it be to amplify it. It must be owned that it is largely Latin in origin, but that very fact has its historical interest, recalling as it does the magnificent series of memoirs produced in succession by the great French mathematicians who were practically the creators of the theory.

It is with great pleasure that I record my obligations to Professors Sir W. Thomson, P. G. Tait, and J. J. Thomson for the

ready kindness with which they assisted me in the most difficult portion of my task—the revision of Chapter I.—as well as for their expressions of sympathy and encouragement for my undertaking as a whole. I am also much indebted to Professors Alex. B. W. Kennedy and A. G. Greenhill for permission to make free use of their original papers; to my friend, Mr. H. M. Elder, B.A., late Assistant Master at Wellington College, and formerly Scholar of Trinity, for his skill and care in photographing Figures 37, 39, 40, 41, and for assistance in revising some of the proofs; and to my Publishers, for kindly lending me the blocks of Figures 63, 64, 65.

It is almost inevitable that, in a work of this kind, many errors must remain undetected, in spite of every care. I shall be grateful for notice of any such that my readers may discover, as well as for suggestions as to notation, arrangement, and other matters of opinion.

W. J. IBBETSON.

EASTERN HOUSE,
CAMBRIDGE, *March 9th*, 1887.

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ERRATA.

Page 22,	line 7,	*for* $\lambda\lambda(\ldots$	*read*	$\lambda(\lambda\ldots$
,, 24,	,, 35,	,, γ_3	,,	g
,, 29,	,, 32,	,, $\lambda=$	,,	$\lambda'=$
,, 29,	,, 33,	,, $\mu=$	,,	$\mu'=$
,, 51,	,, 26,	,, (51)	,,	(52)
,, 51,	,, 34,	,, axis	,,	axes
,, 134,	,, 9,	*dele* (See Chapter XII.)		
,, 154,	,, 3,	*for* sum	*read*	mean

,, 186, Figure 24. The letters a, b, c, d, used on pages 185, 187 to denote the stages AB, BC, C_1D, DE are omitted from the Figure.

Page 211,	lines 1 and 5,	*for* $\xi\eta$, $\xi\zeta$	*read*	s_2, s_3.
,, 235,	line 16,	for $\dfrac{\partial\Theta}{\partial\omega}$	,,	$\dfrac{\partial\Theta_2}{\partial\omega}$
,, 239,	,, 13,	,, $\mathfrak{h}'\mathrm{H}$	,,	$\mathfrak{h}\mathrm{H}'$
,, 268,	,, 6,	,, $\dfrac{\partial'}{\partial z}$	,,	$\dfrac{\partial f'}{\partial z}$
,, 270,	,, 24,	,, $\mathfrak{i}$	,,	$-\rho\mathfrak{i}$
,, 273,	,, 20,	,, $\mathbf{I}$	,,	$\mathbf{I}_{ij}$
,, 283,	,, 34,	,, $\Sigma\iiint V dxdydz$	,,	$\Sigma\iiint V_i dxdydz$.
,, 283,	,, 36,	,, $\Sigma(V)$	,,	$\Sigma(V_i)$
,, 302,	,, 24,	,, $\phi=\dot{\Phi}$	,,	$\dot{\phi}=\Phi$
,, 310,	,, 21,	,, $-\phi$	,,	$-\ddot{\phi}$
,, 331,	end of last line,	*for* r^{-i-}	,,	r^{-i-1}
,, 350,	line 27,	*for* 20	,,	21

,, 372, Figure 43. The upper dotted curve was inserted by a mistake, and has no connection with the problem of § 320.

CHAPTER I.

PROPERTIES OF ELASTIC SOLIDS.

Solid Matter as it really is.

1.] **Molecular Structure of Matter.** To superficial observation matter presents innumerable gradations of "coarseness" or "fineness" of structure, from the obvious "granularity" of sandstone and other rocks to the apparently perfect "continuity" of crystals, jellies and liquids.

A little reflection, however, will show that these terms have a purely relative application, depending upon the magnitude of the smallest constituent portions of matter which are perceptible to our senses as individually distinct. The paper upon which these words are printed appears to the eye perfectly smooth and uniform; but under a microscope of even moderate power it is seen to be really a more or less compact mass of tangled linen threads. If a mass of sandstone as large as the earth could be reduced to the size of a cricket-ball, each of its parts shrinking in the same proportion, it would possess no more real uniformity than before, yet none of the optical means at our disposal would enable us to detect any granularity of structure. Similarly, if a pint of water, or a globe of glass of equal volume, could be magnified in the same way to the size of the earth, it is quite possible that it might appear to us vastly more coarse-grained in structure than a mass of sandstone or gravel as we see it in Nature. That is to say, the smallest constituents which we could then distinguish from one another might be larger than the crystals in the sandstone, or even than the pebbles in the gravel (see § 36, below).

2.] We have, in fact, strong reason to believe that all kinds of matter, however apparently continuous, are ultimately granular in structure, being composed of very minute (but not infinitely small) material particles or **molecules**, which perform incessant motions so long as the matter contains any heat; their capacity for relative motion, as well as their size, mass and closeness of aggregation, varying considerably in different kinds of matter.

3.] **Intermolecular Forces.** These molecules exert upon one another mutual forces, to which the cohesiveness of matter is due. Of their nature little or nothing is known with certainty, except that their intensity in the natural arrangement of the molecules varies within very wide limits for different kinds of matter, while, if the molecules be artificially separated by appreciable distances, it is impossible to detect their existence by the most delicate instruments.* It appears, therefore, that we are justified in assuming their sphere of action to be exceedingly limited.

4.] **Impressed Forces.** The molecules are also liable to be influenced by external "impressed" or "applied" forces, such as Gravitation and other natural forces of attraction and repulsion.

5.] **Natural State.** When matter is entirely free from the action of such external forces, it is said to be in its "natural state." This term does not imply that matter is ever found, or can even be conceived to be in this state under natural conditions; but that in this state, and in this only, it may be supposed isolated from all co-existing matter, so that all the phenomena it presents depend only on its individual nature.

6.] **Solid Matter.** In the kind of matter called solid each molecule performs small vibrations about a *mean position*, which, so long as the body is in its natural state and maintained at constant temperature, may be regarded as fixed. Under the same conditions the vibrations of each molecule may be assumed strictly periodic, and the mean value of the amplitudes of the vibrations of any considerable number of molecules may be supposed constant.

7.] **Homogeneity and uniform density.** If, when the matter is in its natural state, and at any uniform temperature, the mean positions of the molecules are uniformly distributed, and if their masses and the periods and mean amplitudes of their vibrations are the same throughout, the matter is said to be "naturally homogeneous."

It follows that a closed surface of given volume, but of any form, whose least dimension is very large in comparison with the greatest mean distance of two adjacent molecules, will, if drawn anywhere within the substance of homogeneous matter, always include the same number of molecules,—and therefore the same total mass. The mass thus enclosed by a surface of unit volume is called the *Density* of the matter, in any given system of units.

* See, however, Note at end of volume.

8.] **A Homogeneous Solid Body** is a continuous portion of homogeneous solid matter, bounded by a surface consisting of one or more completely closed sheets, each of which has at every instant a definite form and volume. The form of the bounding surface, and the volume enclosed by it (or between its sheets) in the natural state of the body, at any given uniform temperature, are called the *natural form and volume of the body at the given temperature.* The relative arrangement of the mean positions of the molecules under the same conditions will be called the *natural configuration of the molecules* at the given temperature.

9.] We have seen in § 3 that the only property which we can ascribe with certainty to the intermolecular forces is that they depend in some way on the molecular configuration; the law of dependence varying for different kinds of matter. Since, however, when the body is free from the action of external forces, we can hardly conceive of their being affected by any other consideration, we shall *assume* that, in the natural state, they depend solely on the configuration of the molecules, and on the temperature.

It is obvious that the form and volume of the bounding surface, which is merely the envelope of the external layer of molecules, must in all states of the body depend solely and entirely on the molecular configuration.

10.] **Definition of Strain.** Any departure from the natural configuration is called a Strain. Thus a Strain may be defined as any **change in the relative arrangement** of the mean positions of the molecules from that which is natural to the body at the given temperature.

11.] **Elastic Properties.** It is found by experiment that all solid bodies possess, in a greater or less degree, the properties included under the title of Elasticity. These may be summed up, in general terms, as follows:—

(*i.*) The natural form and volume of the body (and therefore also the natural configuration of the molecules) are always the same when the body is at the same uniform temperature, through whatever cycles of *gradual* changes of temperature (*within certain limits*) the body may be brought, *so long as it is not subjected to external force.*

(*ii.*) Hence it has a perfectly definite and characteristic natural or "unstrained" configuration at each temperature *within these limits,* which cannot be altered, while the temperature remains the same, except by the application of external force.

Or, in other words, *it always requires the application of external force to produce strain.*

(*iii.*) Given the type of the external forces applied, the greater they are the greater will be the strain produced; and, conversely, the greater the strain to be produced, the greater the external forces which must be applied.

(*iv.*) If the applied forces and the consequent strain be confined *within certain limits*, the body offers continuous resistance to the strain, so that *it requires the continued exertion of external force* to maintain the body *in a given state of strain;* and when this force is removed the body *tends to return* to its natural state at its ultimate temperature.

12.] **Limits of Elasticity.** All these elastic properties are exhibited in very different degrees, and subject to many limitations, by different classes of natural solids.

Short of the strain required to produce absolute rupture (called the *proof-strain* of the material) there is always a limit to the elasticity of every natural substance. So long as the applied forces are such as to produce a strain well within this limit the resistance increases steadily with the strain, while it always requires sensibly the same force to maintain the same strain at the same temperature; and on the removal of this force the body returns to a state sensibly identical with its natural state.

When, however, the strain exceeds the elastic limits of the material the properties of the body undergo a marked change, and it passes into what is known as the *ductile* state. In this condition the resistance still increases with the strain, but much less rapidly than before the limit was passed, and the tendency to return towards the natural state is much diminished, so that, when the external force is removed, the body is found to have acquired a "set" or *permanent strain.*

13.] **Ductility and Brittleness.** Those materials whose elastic limit is separated by a considerable interval from the point of rupture, and whose state of ductility therefore has a distinct range, are called *ductile*, malleable, or plastic. To this class belong most of the natural metals, as well as steel gradually cooled.

Thus under the enormous pressures applied in the Mint, the density of gold is permanently altered from 19·258 to 19·367, and that of copper from 8·535 to 8·916.

At the bottom of this class are various soft solids (of which putty or tallow may be taken as a familiar example) whose elasticity is almost imperceptible, and which are for all practical purposes wholly ductile.

On the other hand, crystalline bodies, glass (when cold), jellies, and steel suddenly chilled from a red heat have extremely little ductility, so that, practically, breakage is the first intimation we receive of having reached the elastic limits. Such materials are called *brittle*.

The two classes, however, are not separated by any hard and fast line, the various gradations of tempered steel, for example, forming a series of connecting links.

14.] **Elasticities of Shape and Bulk.** The elastic resistances of a solid may be roughly divided into resistances to Distortion, Expansion, and Compression respectively.

The limits of these are often very different in the same solid, the first having generally a very small range.

15.] **Tempering.** The reason for the limitation imposed in § 11 (*i.*) on the changes of temperature to which the body may be supposed subjected, is that, by sudden and violent changes of temperature, many substances, and notably metals and glass, may be entirely altered in all their elastic properties.

The brittleness of glass (Prince Rupert's drops) and of steel (glass-hard steel) when heated to redness and suddenly chilled in water is proverbial. But glass may be "toughened" by gradual cooling in hot oil, and steel by gradual and cautious reheating may acquire a vast number of degrees of "temper" intermediate between brittleness and ductility. All these different qualities of steel must be regarded as distinct materials, none of whose elastic properties are absolutely identical.

The change produced in a metal by tempering is obviously analogous to that produced by straining it beyond its elastic limits; and some very striking results have been obtained in the way of tempering wires by giving them a permanent strain. Mr. J. T. Bottomley has shown that the tensile strength of soft-iron wire may be increased more than 25 per cent. by prolonged tension: while Messrs. A. & T. Gray find that the power of copper wire to resist twisting about its axis may be reduced to $\frac{1}{6}$ of its natural value by giving it a permanent twist.

16.] **Viscosity and Fatigue.** Besides the above well-known restrictions, two remarkable irregularities have been discovered by Sir William Thomson in the elasticity of metals, trained within their elastic limits, which are probably common to all natural solids.

In the first place the resistance to strain is found to vary with the rate at which the strain is imposed.

This proves the existence of a property of solid matter analogous to the "viscosity" of fluids, in virtue of which the latter oppose to change of shape a resistance proportional to the

rapidity of the change. The law by which the increase of resistance in the case of solids depends on the increase of the rate of straining is certainly not so simple, but the analogy justifies the application of the term *solid viscosity* to this property.

Secondly, it was found that wires which had been frequently and *recently* strained, well within their elastic limits, exhibited less marked tendency to elastic recovery, and much greater viscosity than when they had been left at rest in the natural state for some days before the experiment.

This result shows that the elastic properties of a natural solid may suffer diminution or *Fatigue* by frequent exercise, and that these properties may be more or less fully restored by repose.

17.] All these limitations and imperfections in the Elasticity of natural solids present insurmountable difficulties in the way of an analytical theory; and for the purposes of a first approximation they must be eliminated.

If we class the more or less "imperfectly elastic" substances, which we find in nature, according to the range of their elasticity and the degree of perfection in which they exhibit its characteristic properties within these limits, they are seen to form an ascending scale suggesting an ideal summit which is never actually reached in nature, but only more or less closely approximated to under favourable circumstances.

This ideal, which we shall adopt as the subject of our investigations, we define as a Perfectly Elastic Solid.

Real Matter with ideally perfect Elasticity.

18.] **A Perfectly Elastic Solid** is characterized by the following properties up to the point of breakage:—

(*i.*) In its natural state at any temperature the molecular configuration, together with the form and volume of the bounding surface, are perfectly definite, and characteristic of that temperature.

(*ii.*) If the temperature (supposed always uniform throughout the body) be changed, the solid passes continuously to the natural state for the new temperature, through all the intermediate states natural to the intermediate temperatures.

(*iii.*) It requires the application of external force to produce a strain at any temperature; and it requires the continued application of the same force (or system of forces) to maintain the strain.

(*iv.*) It always requires the same force (or system of forces) to maintain the same strain at the same temperature, through whatever intermediate states of temperature and strain it may

have been brought to the given state, and at whatever rate these intermediate changes may have been passed through.

(*v.*) When all external forces are removed it returns to its natural configuration for the temperature at which it is left.

19.] **Approximation of natural solids to Perfect Elasticity.** Under very small strains which do not approach the *Elastic Limits* of the material; produced so gradually and maintained for so short a time as never to call *Viscosity* into play, or to produce *Elastic Fatigue;* and subject to changes of temperature too limited and gradual to impart a *Temper:* the metals, crystals, glass, jellies, indiarubber, etc., may all be said to have approximately perfect elasticity, as defined in the last article.

20.] **Intrinsic Energy in the Natural State.** Assuming, as we shall do throughout, that the body is free from all influences due to electrification, magnetisation, etc., it is obvious that the energy possessed by it in its natural state, at any given temperature, must consist of two parts:—

(*i.*) The intrinsic potential energy, due to the configuration of the mean positions of the molecules under the intermolecular forces: and

(*ii.*) The kinetic energy due to the vibrations of the molecules about these mean positions.

The intermolecular forces being supposed governed by fixed laws which, in a given body in its natural state, depend$ solely (§ 9) on the configuration and temperature, it follows that the potential energy in the natural state also depends only on the configuration and temperature. But by § 18 (*i.*), the natural configuration depends only on the temperature. Hence we may state, in mathematical language, that, *in the natural state, the potential energy is a function of the temperature only.* Again, according to the most modern theory of gases, the kinetic energy due to the motion of their molecules is simply proportional to the absolute temperature. The relation is probably not so simple in the case of elastic solids, but we are justified in assuming, as in the theory of Thermodynamics, that *the kinetic energy of the molecules is some function of the absolute temperature,* so that neither can be altered without altering the other; the form of the relation being such that both increase r diminish together.

21.] **Stability of the Natural State.** According to § 18 (*iii., iv.*) it requires the application of external force to disturb the body from a given natural state, and to hold it in any given state of strain, *the temperature remaining unchanged:* and when the force is removed, the body returns to the state from which it was disturbed. Hence the natural configuration

at each temperature is one of *stable equilibrium* for straining disturbances without change of temperature. And since, by the last Article, the kinetic energy of the molecules is the same in every state at the same temperature, it follows by a well-known theorem in Statics, that the natural configuration at each temperature is such that the potential energy has its least possible value for that temperature under the given law of intermolecular force.

Hence it follows that if the body be strained in any manner, while the temperature is kept constant, the potential energy will be increased. And since in this case the kinetic energy remains constant, *the increase of potential energy is necessarily equal to the work done on the body by the external forces in producing the strain.*

If now, the temperature still being maintained constant, the body be allowed to work against the external forces, it will, in returning to its natural state, lose all the additional potential energy which it acquired by the strain. This then must be the exact measure of the work done by it against the external forces, which is thus equal and opposite to the work done upon it by them in producing the strain.

This result may obviously be extended to a body starting from equilibrium in any given state of strain, and passing, *at constant temperature*, through any cycles of strain back again to its initial state, the total sum of work done on or by the body being identically zero.

Thus a perfectly elastic body maintained at constant temperature forms with any system of external straining forces a perfectly conservative system, the excess of the body's potential energy over that natural to the temperature being a function only of the strain and of the temperature, and *vanishing with the strain.*

22.] **Temperature free to vary.** In general, when the temperature of the body is left free to vary, energy communicated to the body, either in the form of heat or of mechanical work done by external forces, will be distributed in both forms.

Thus, the primary effect of the addition of heat is to raise the temperature of the body, and thus to increase the molecular kinetic energy. But, since no external forces are applied, we know by § 18 (*ii.*) that the configuration of the molecules must change to that natural to the new temperature.

Hence, if the law of intermolecular force be such that the potential energy of the natural configuration increases with the rise of temperature, some of the heat will be expended in producing this increase, so that the resultant rise of temperature will be that due to an increase of kinetic energy less than the full

energy-equivalent of the added heat. On the other hand, if the natural potential energy diminishes with the rise of temperature, the effect of adding heat will be to convert some of it into kinetic, and the resultant rise of temperature will be in excess.

Again, if mechanical work be done on the body by external forces, so as to produce a strain, the mode of the molecular vibrations (which must depend upon the configuration) may be altered, and consequently the kinetic energy and the temperature may suffer change. Here again the increase of potential energy due to the strain will not be the exact equivalent of the work done in straining the body.

23.] **Dissipation of Energy**. The availability of heat for conversion into mechanical work, or potential energy, depends entirely on its distribution as to temperature. Every *reversible* conversion of mechanical work into heat is accompanied by the removal of a proportional quantity of heat from a body or portions of a body at a lower temperature to a body or portions of the same body at a higher temperature; and if the distribution so produced could be maintained indefinitely, the process could at any time be reversed without the addition or removal of heat from the body as a whole, the work done by the body in recovering its original thermal and mechanical state being precisely equal to that done on it by external forces in producing the first change of state.

But, under natural conditions, it is impossible to maintain, for any length of time, a non-uniform distribution of temperature without constantly supplying heat to some portions of the surface, and constantly removing it from others. If the body be supposed guarded from loss or gain of heat by radiation, the process of gradual conduction is constantly tending to equalize the temperature throughout its mass, and thus to *dissipate* its intrinsic energy for mechanical purposes, by rendering its heat unavailable for reconversion into potential energy or mechanical work.

Now, by § 22, Strain produces a change of temperature which varies with the strain; hence a non-uniform straining of the body will produce a non-uniform distribution of temperature, and consequently the energy of a body so strained will be liable to dissipation by means of conduction.

24.] **Conditions for a Conservative System**. In order, therefore, that a perfectly elastic solid may form with external mechanical forces a perfectly conservative system, we must assume one of two conditions: either

(*i.*) That the body is perfectly guarded from loss or gain of heat by radiation or surface-conduction, and that all the stages of strain and recovery are passed through so rapidly as to prevent all possibility of dissipation by conduction in its interior; or

(*ii.*) That the straining is so gradually performed, that heat may be constantly communicated to or taken from the different parts of the body, by suitable means, in such a manner as to maintain every portion uniformly at the initial temperature.

25.] The two cases are perhaps of equal practical importance, and the former is certainly the more interesting theoretically, but the relations between temperature, kinetic energy, and intermolecular force are at present so hopelessly obscure that but little can be done towards its development.*

It may be observed that, even if conditions (*i.*) were exactly fulfilled, a *natural* solid would still be found to dissipate energy irrecoverably by reason of its viscosity; (see the second condition of § 19).

26.] **Theory Adopted.** To simplify our theory, and eliminate as many unknown physical relations as possible, we shall assume that the conditions of § 24 (*ii.*) are always satisfied. We may observe that all the conditions of § 19 will be satisfied at the same time, if the strain be small; so that results obtained for small strains on this assumption will be very approximately true for many natural solids.

The body is then to be supposed always maintained at one constant temperature, uniform throughout, and thus the results of § 21 may be accepted as rigorously true.

The kinetic energy of the molecules will be constant, and so also will the natural potential energy, or that possessed by the body when free from strain.

27.] **Energy of the Strain.** Since we are only concerned with the Strain and its effects, we leave these constant terms in the energy of the strained body altogether out of account; and it is the *excess* of the potential energy of the strained body over its potential energy in the natural state which we shall in future refer to indifferently as the Potential Energy of *or* due to the Strain *or* of the strained body, or, more briefly, as the Energy of the Strain.

By § 21, the Energy of the Strain is in all cases equal to the mechanical work done on the body by the external forces in producing the strain.

Now, by § 18 (*iv.*), the same system of external forces, applied to the body in its natural state, invariably produces the same strain. Hence, if the strain be given, the forces to be applied, and also the displacements of their points of application are fully specified.

Thus the Energy of a given Strain, being equal to the work done in producing it, is completely determined when the strain

* See Sir W. Thomson's *Reprinted Papers*, Volume I., pages 293-313.

is specified; or, in mathematical language, *is a function solely of the given strain, and absolutely independent of any intermediate states of strain through which the body may have been brought.*

28.] **Stress defined.** The effect of Strain, or *change* in the *relative* positions of the molecules, is to call into play Stress, or **change in the mutual forces** between the molecules.

In the natural state, the molecules form a system of bodies performing small oscillations about mean positions under purely mutual forces. It follows, not only that the resultant of all these mutual forces acting within the body is identically zero, but also that if all the molecules were placed at rest in their mean positions, the resultant of the intermolecular forces acting on any one molecule would be identically zero.

The system of intermolecular forces in the natural state must therefore be regarded, whether with reference to individual molecules or to the body as a whole, as an equilibrating system.

29.] Similarly, when the body is held in equilibrium in a given state of strain by suitable applied forces, the intermolecular forces—altered from their natural values by the change of configuration, but still purely mutual—together with all the applied forces on the several molecules must form an equilibrating system on the body as a whole. And the altered intermolecular forces on any individual molecule, together with the applied force on that molecule, likewise form an equilibrating system.

Now the altered intermolecular forces, being still purely mutual, must, as before, have by themselves a null or zero effect on the body as a whole. Hence it follows that any system of applied forces capable of holding an elastic body in equilibrium in a given state of strain must be such that its component forces, acting at the points to which the strain has displaced their original points of application, form by themselves an equilibrating system.

30.] Defining then, in accordance with § 28, the *stress* between two molecules as the *change* in their *mutual* action due to the change in their relative position, we see that the effects of applying any system of forces to an elastic solid are:—

(*i.*) To produce such a strain that the external forces acting on the molecules in their new positions shall satisfy the ordinary conditions of an equilibrating system, such as would hold the body in equilibrium if the molecules were to become rigidly connected in their new positions; and

(*ii.*) In so doing, to call into play stresses between the molecules, such that the resultant force on any one molecule due to stress is equal and opposite to the applied force. The stresses, therefore, always resist further strain, and on any

relaxation of the applied forces tend to restore the body to its natural state, diminishing continuously as the potential energy of the strain is expended in the process, and finally vanishing together with the strain.

31.] **Work done by Stress.** Since the stress on each molecule is always equal and opposite to the applied force, while the displacement of their common point of application is necessarily the same, it follows that all work done *by* the applied forces may be reckoned as work done *against* the stresses, and *vice versâ.*

Thus, in passing from a state of strain in which the potential energy (§ 27) is W, to a second state in which it is increased to $W+\delta W$, the work done on the body by the applied forces in opposition to the stresses is δW; while, if the stresses be allowed to restore the body to its original state, they will do work δW against the applied forces.

32.] **Strain-Coördinates.** Let us suppose that any changes in the relative configuration of the molecules may be represented by variations of a certain number of independent coördinates θ, $\phi, \chi, \psi,\ldots$, the word being used in its generalised Lagrangian sense.

Then, since the Potential Energy of the strain depends only on these changes, it must be capable of being expressed as a function of the strain-coördinates.

Similarly, if V be the mutual potential energy of any two molecules, due to the stresses they exert upon one another, V must be a function of the differences between the actual values of θ, $\phi,\ldots$, defining their relative positions, and their initial values in the natural state.

If then δV be the small increase of V due to a small increase of strain, which changes θ, $\phi,\ldots$, to $\theta+\delta\theta$, $\phi+\delta\phi,\ldots\ldots$, we must have *

$$\delta V = \frac{\partial V}{\partial \theta}\,.\,\delta\theta + \frac{\partial V}{\partial \phi}\,.\,\delta\phi + \frac{\partial V}{\partial \chi}\,.\,\delta\chi + \ldots\ldots$$

Now, if W be the total potential energy of the whole body, it is obvious that $W=\frac{1}{2}\Sigma\Sigma(V)$, the summation being taken twice through all the molecules.

Hence

$$\begin{aligned}\delta W &= \tfrac{1}{2}\Sigma\Sigma(\delta V)\\ &= \tfrac{1}{2}\Sigma\Sigma\left\{\frac{\partial V}{\partial \theta}\,.\,\delta\theta + \frac{\partial V}{\partial \phi}\,.\,\delta\phi + \ldots\ldots\right\}.\end{aligned}$$

But if Θ, Φ, X, $\Psi,\ldots$ be the stresses respectively resisting increase of the relative coördinates θ, ϕ, χ, $\psi,\ldots$ of any one pair

* The symbol ∂ is used throughout this work to denote *partial* differentiation; *d* being reserved exclusively for *total* differentiation. The usual flux-notation is also frequently employed for partial differentiation *as to time.*

of molecules, the work done against stress in producing a small increase of strain in the relative positions of that pair is

$$\{\Theta\,.\,\delta\theta + \Phi\,.\,\delta\phi + X\,.\,\delta\chi + \ldots\ldots\}.$$

Hence the whole work done on the body against stress is

$$\tfrac{1}{2}\Sigma\Sigma\{\Theta\,.\,\delta\theta + \Phi\,.\,\delta\phi + X\,.\,\delta\chi + \ldots\ldots\} = \delta W, \text{ by § 31.}$$

Thus it follows that

$$\Theta = \partial V/\partial\theta;\ \Phi = \partial V/\partial\phi;\ \ldots\ldots$$

33.] **Simple Strains.** A strain which consists in the variation of only one of the coördinates, such as θ, is called a Simple Strain of the type defined by θ. Similarly, Θ is called the simple stress of the same type. In the case of a simple strain we have evidently

$$V = \int_{\theta_0}^{\theta} \Theta d\theta,$$

$$W = \tfrac{1}{2}\Sigma\Sigma\int_{\theta_0}^{\theta} \Theta d\theta,$$

θ_0 being the value of θ in the natural state for the pair of molecules to which V belongs.

A complex strain, in which more than one of the coördinates suffer change, is said to be compounded of, or to have for its components, the simple strains

$$\theta - \theta_0,\ \phi - \phi_0, \ldots\ldots$$

Two complex strains are said to be of the same type when their simple components differ only by a constant factor. Thus, if the first strain changes the coördinates to $\theta_1, \phi_1, \chi_1, \ldots$, and the second to $\theta_2, \phi_2, \chi_2, \ldots$, the conditions that they may be of the same type are

$$\frac{\theta_1 - \theta_0}{\theta_2 - \theta_0} = \frac{\phi_1 - \phi_0}{\phi_2 - \phi_0} = \frac{\chi_1 - \chi_0}{\chi_2 - \chi_0} = \ldots\ldots$$

34.] **Summary.** We have now shown that if a perfectly elastic homogeneous solid body be strained by external forces, while always maintained at the same temperature—

(*i.*) The potential energy of the strain will always be equal to V, the work done by the external forces in producing the strain.

(*ii.*) The strain calls into play internal elastic forces or stresses, which are of the nature of purely mutual reactions between the molecules; the stresses between any pair of molecules having for their potential the mutual potential energy of the pair, and consequently tending to resist further strain and to restore the body to its natural state; while the resultant of all the forces on any one molecule due to stress is always equal and opposite to the applied force.

(*iii.*) The potential energy and the stresses are functions solely of the actually existing state of strain, and absolutely independent of all intermediate states through which the body may have been brought.

(*iv.*) As the external forces are relaxed, the stresses experience less and less opposition, so that they diminish continually as they restore the body to its natural state, expending on that process precisely the amount W of work which was done against them in straining the body, and finally vanishing with the strain.

(*v.*) It is obvious that when the molecules are in motion under external forces the effective force to which the motion of the *mean position* of any molecule in the direction opposed to the stress is due, together with the resultant stress on that molecule, is equal to the applied force.

Ideal Continuous Matter with Perfect Elasticity.

35.] **Difficulty of further developing the Theory.** We have thus deduced, from what we believe to be the true properties of matter, the laws of equilibrium and motion of the molecules of a perfectly elastic solid. In order to develop our Theory analytically, we must be able to follow the movements of each molecule throughout the strain, and to discover all the mechanical conditions to which it individually is subjected.

For this purpose we require to know the absolute mass and dimensions of the molecules of the body under consideration; the law of distribution of their mean positions in the natural state; the law of intermolecular force—the manner in which it depends upon, and varies with, both the configuration and the temperature; the limits of its sphere of action; and, lastly, the connection between mean configuration, period and amplitude of vibration and the temperature.

36.] Unfortunately, on almost all these points, our ignorance is at present absolute; and where we have any means of forming an opinion, the conclusions arrived at are so vague as to be valueless for our purpose.

For instance, as to the dimensions of the molecules the latest conclusions of science are summarised as follows by Sir William Thomson * :—

"The four lines of argument which I have now indicated lead all to substantially the same estimate of the dimensions of molecular structure. Jointly they establish, with what we cannot but regard as a very high degree of probability, the conclusion that, in any ordinary liquid, transparent solid, or seemingly opaque

* Lecture on the Size of Atoms, Royal Institution, February 3, 1883.

solid, the mean distance between the centres of contiguous molecules is less than the five-millionth and greater than the thousand-millionth of a centimetre.

"To form some conception of the degree of coarse-grainedness indicated by this conclusion, imagine a globe of water or glass, as large as a football (or say a globe of sixteen centimetres diameter), to be magnified up to the size of the earth, each constituent molecule being magnified in the same proportion. The magnified structure would be more coarse-grained than a heap of small shot, but probably less coarse-grained than a heap of foot-balls."

37.] **Boscovitch's Theory.** As to the law of intermolecular force, we are in still more complete ignorance.

The most natural assumption is that the action between any two molecules is reducible to a single force acting between their centres of mass, and varying only with the distance between these points. This theory is always referred to as Boscovitch's, after the Jesuit Father who first formally stated it. It was adopted by all the earlier workers in Elasticity, who drew from it deductions leading to a simple and consistent theory, which however, was unable to bear the light of experiment. It was first disproved by Stokes, and has come to be regarded as an absurdity by all living physicists of any eminence (see § 208, below).

Sir William Thomson has however quite recently shown * that the points in this theory which have had to be rejected are not legitimate deductions from Boscovitch's principle.

38.] As to the greatest distance at which the intermolecular forces are appreciable, Cauchy deduced from his Theory of the Dispersion of Light that it must be comparable with the length of a luminous wave—the mean value of which may be taken as about one fifty-thousandth of a centimetre; and, although his theory was based upon Boscovitch's hypothesis, yet this result seems to hold good.

Here we practically reach the limits of our knowledge of solid matter.†

39.] **Conventional Theory substituted.** Our ignorance of its intimate dynamical properties placing it out of our power to deal analytically with matter as it really is, it becomes necessary to substitute a hypothetical substance which will lend itself to mathematical treatment: attributing to it such arbitrary properties as will approximate the results of our analytical theory

* Lectures on Molecular Dynamics, John Hopkins University, Baltimore, U.S.A., pages 124-132 of the papyrograph reprint.

† See however Quincke's and Plateau's results, quoted in Tait's "Properties of Matter," Article 293.

to the deductions we have drawn from experiments on real matter.

Our theory will then take its place as the last in the series formed by the various branches of Dynamics, which must be regarded as successive steps, each approaching nearer than the preceding to the true state of things, but none of them actually realised in nature.

40.] **Dynamics of a Particle.** The smallest "element of volume" which the refinement of analysis can reach must still, for the purposes of that very analysis, be held to have three linear dimensions, so that if it be occupied by an "element of mass" subject to forces which vary from point to point throughout space, this mass must in general be acted upon both by a force and by a couple; both of them elementary, of course, but yet measurable by analysis.

Hence we have recourse, for our first and simplest conception of dynamics, to the purely abstract idea of a "**Material Particle**," which we define as a very minute but still *finite mass*, so condensed that its linear dimensions are inappreciable to our analysis, and therefore *infinitely small, even when compared with our smallest "element of volume."*

Such a particle cannot, of course, be subjected to couples, and therefore Dynamics is reduced to its simplest form.

41.] **Dynamics of a Rigid Body.** We next advance to the conception of a "Rigid Body," which we regard as an aggregation of such particles, so connected as to be entirely incapable of relative motion.

The particles are supposed to be uniformly distributed, and, in the case of a homogeneous body, to be all of equal mass.

Since the particles of the body remain in an invariable state of relative equilibrium, the mutual forces exerted by them upon one another, must under all circumstances of equilibrium or motion of the body as a whole, form by themselves an equilibrating system (**D'Alembert's Principle**).

They consequently cannot possibly do any work, and therefore do not enter into the equations of energy. In fact, we only owe to them the Kinematical or Geometrical equations which express in various analytical forms the fundamental fact that the body always moves as a whole, without relative motion of its particles.

Moreover the external action on each particle takes the form of a single force, and these various forces can always be compounded into a single Resultant Force and a single Resultant Couple, which may be regarded as acting upon the body as a whole.

Thus for all mechanical purposes the supposed structure of

the body may be, and is, altogether left out of consideration, and it may indifferently be regarded as a portion of perfectly *continuous or structureless matter.*

42.] **Continuous Elastic Matter.** We now take our last step in advance, and recognise the possibility of relative motion between the constituent parts of a body.

Replacing the molecules of our perfectly elastic solid by the abstract particles, of which an infinite number are contained in the smallest element, we transform it into a portion of *continuous elastic matter*, capable of experiencing, and of offering a certain resistance to alterations of its form and volume.

43.] **Homogeneity of Continuous Matter.** In accordance with this view we now define a Homogeneous Body as such that any two *equal and similar* portions, *similarly situated* in the body, are precisely identical in all physical and mechanical properties, *however small they may be taken* within the limits of analytical refinement.

Thus we quite abandon the idea of granular or molecular structure, and, by diminishing the size of our particles indefinitely, extend that perfect degree of homogeneity which in nature is common to many substances, *taken in bulk* (*see* § 1), to the smallest volumes which we can conceive.

44.] **Points, Lines, and Surfaces in the Body.** Our body being now composed of perfectly continuous and naturally homogeneous matter, we must, for Geometrical purposes, replace the molecules by recognisable "points in the body" which are to be taken as necessarily coinciding with material particles, or infinitely small portions of the continuous matter.

Similarly, we define a "line in the body" as a cord of continuous matter, of any form and any length, and of infinitely small transverse dimensions; while a "surface in the body" is to be regarded as a sheet of continuous matter of any form, and of infinitely small thickness.

Points, lines, and surfaces in the body must, of course, when once chosen, be supposed to maintain their identity throughout all changes of form and position.

45.] **Heat-vibrations neglected.** In this transformation we entirely ignore the heat-vibrations of the molecules, because

(*i.*) The molecules being replaced by particles infinitely close together, either their amplitudes will be reduced to the vanishing point, or they will have the same phase, period, and amplitude for all points of the body, in which case they will not enter into the strain (*see* § 48):

(*ii.*) The temperature being constant, the kinetic energy is

also constant, and may be left out of consideration together with the constant part of the potential energy proper to the natural state (*see* § 27).

Thus every point in the body is to be supposed at rest, except in so far as its motion is due to change of strain.

46.] **Course of our Analysis.** Strain will now consist in relative displacements of points in the body, and consequent distortions of lines and surfaces, and changes in the form and volume of portions of the body enclosed by the latter.

Our analysis of Strains will therefore have for its aim to discover a simple system of independent strain coördinates, the variation of any one of which will constitute a Simple Strain; and to learn how to express *any* change of form or volume in terms of these as standard types.

We shall next investigate the corresponding Simple Stresses, (which will be of the nature of resistances offered by the body to the respective Simple Strains), and the relations which must exist between them and the applied forces, in order that the body may be held in equilibrium in any given state of strain by these two opposed systems.

To complete our general theory we shall then only require to know how to express stress in terms of the strain to which it is due. We shall then be able to calculate the potential energy due to any given strain, and the external forces required to produce it; or, conversely, the strain produced by any given system of applied forces; so that the solution of any problem will be reduced to a mere matter of analysis.

We shall, for the next five chapters, confine ourselves to the consideration of bodies whose dimensions are at least finite in all directions.

CHAPTER II.

ANALYSIS OF STRAINS.

47.] We have defined Strain as any *change in the relative positions* of points in the body, produced by external forces.

For purposes of analytical treatment it is of course necessary to assume that the relative displacements of points in a body undergoing strain follow some definite law depending on their relative positions in the natural or unstrained state. In other words, the displacement of any point Q in the body, relative to a given point P, must be some function of the initial position of Q relative to P; and it is further necessary to suppose, in order that the strain may produce no breach of continuity in the substance of the body, that the relative displacement is a *continuous* function of relative position, for by this limitation we secure that the increase of the distance between any two points is, at the most, of the same order of magnitude as their initial distance.

48.] Secondly, it is to be observed that Strain, as defined, depends solely on such *relative* displacements. Any displacement —whether linear or angular—which is the same for all points, and which therefore produces no alteration in their relative arrangement in the body, amounts merely to a translation or rotation *of the body as a whole*, such as might be suffered by any perfectly rigid body; and since such motions do not call into play any elastic forces, they are not included under the head of Strains.

Although, however, a rotation of the body as a whole does not by itself constitute a strain, and can add nothing to the energy of any true strain that may accompany it: yet, since in discussing strains which vary from point to point we only consider a small portion of the body at a time, and since a rotation which varies from one portion of the body to another *does* constitute a strain, we make a point of recording rotations, and only ignore such displacements as are parallel, equal, and of like sign for all points in the body.

49.] Now, let us take an unstrained body, and refer the positions of all points in it to a system of rectangular axes, fixed in space, whose origin O coincides with any point M in the body.

If the body be now strained in any manner the point M will in general suffer a displacement from its initial position at O, the amount and direction of which will depend upon its situation in the body. But it follows from the last Article that, without modifying in any manner the effects *of the Strain*, we may impress upon all points of the body displacements equal, parallel, and opposite to that of M; the effect of which will of course be to move the body back, parallel to itself, until M once more coincides with O.

50.] Thus we may, whenever it will simplify our analysis,* suppose that point of the body which coincides with our arbitrarily chosen origin to be absolutely fixed in space, without in the slightest degree restricting the perfectly general character of the strain.

Although, however, the origin may be regarded as fixed both in space and in the body, the axes are only fixed in space. That is to say, the straight lines in the unstrained body which coincide with the axes will no longer do so after the strain; and, in fact, they will in general be no longer straight lines, but continuous curves intersecting more or less obliquely in the origin.

Assuming then that the point of the body chosen as origin is fixed, the *absolute* displacement of any point in the body (and therefore also its component displacements parallel to the fixed axes) must be a continuous function of the *absolute* coördinates of the point; and these absolute displacements now constitute the strain.

Theory of Small Strains.

51.] **Equations of Displacement.** Let P be any point in the unstrained body, whose coördinates referred to the fixed axes are (x, y, z). Let the body be subjected to a very small strain, and let P in consequence be displaced to P' $(x+u, y+v, z+w)$.

Then u, v, w are the component displacements of P, parallel to the fixed axes, and we must have

$$\left.\begin{aligned} u &= \phi\,(x, y, z) \\ v &= \chi\,(x, y, z) \\ w &= \psi\,(x, y, z) \end{aligned}\right\},$$

where u, v, w are supposed very small, and ϕ, χ, ψ are arbitrary

* See Appendix I., at the end of this Chapter, on the advantage of regarding a point in the body as fixed.

functions, continuous throughout the body. We shall assume that all their partial derivatives as to x, y, and z are also continuous, so that none of them can become infinite.

52.] Let another point of the body, initially at Q $(x+h, y+k, z+l)$; very close to P so that h, k, l are small quantities of the first order in comparison with x, y, z; be displaced by the same strain to Q' $(x+h+u', y+k+v', z+l+w')$.

Then, as before,

$$\left.\begin{aligned} u' &= \phi\,(x+h,\ y+k,\ z+l) \\ v' &= \chi\,(x+h,\ y+k,\ z+l) \\ w' &= \psi\,(x+h,\ y+k,\ z+l) \end{aligned}\right\}.$$

Since ϕ, χ, ψ and their derivatives are supposed continuous, we may expand by Taylor's Theorem, and neglect squares and higher powers of h, k, l.

Thus

$$\left.\begin{aligned} u' &= u + h\frac{\partial u}{\partial x} + k\frac{\partial u}{\partial y} + l\frac{\partial u}{\partial z} \\ v' &= v + h\frac{\partial v}{\partial x} + k\frac{\partial v}{\partial y} + l\frac{\partial v}{\partial z} \\ w' &= w + h\frac{\partial w}{\partial x} + k\frac{\partial w}{\partial y} + l\frac{\partial w}{\partial z} \end{aligned}\right\}.$$

The coördinates of Q, relative to P, have been changed from (h, k, l) to $(h+u'-u, k+v'-v, l+w'-w)$; so that if δh, δk, δl be the increments of these relative coördinates, due to the strain,

$$\left.\begin{aligned} \delta h &= h\frac{\partial u}{\partial x} + k\frac{\partial u}{\partial y} + l\frac{\partial u}{\partial z} \\ \delta k &= h\frac{\partial v}{\partial x} + k\frac{\partial v}{\partial y} + l\frac{\partial v}{\partial z} \\ \delta l &= h\frac{\partial w}{\partial x} + k\frac{\partial w}{\partial y} + l\frac{\partial w}{\partial z} \end{aligned}\right\} \qquad (1)$$

53.] **Elongation**. If L be the length of any line in the unstrained body, and if this length be altered to L' by the strain, the ratio $(L'-L)/L$, or the increase of length per unit of initial length, is called the *Elongation* of the line.

If the line is diminished in length by the strain, it is said to suffer *negative* elongation, and the positive ratio $(L-L')/L$ is otherwise called its *Contraction*.

Let $PQ=\rho$, $P'Q'=\rho+\delta\rho$; then ρ is of the same order of magnitude as h, k, l, and $\delta\rho$ is of the same order as $\delta h, \delta k, \delta l$. And since $\rho^2=h^2+k^2+l^2$, we have to the order of approximation adopted, $\rho\delta\rho=h\delta h+k\delta k+l\delta l$.

But, if ϵ be the elongation produced by the strain in the elementary straight line PQ, $\epsilon = \delta\rho/\rho$.

$$\therefore \qquad \epsilon = \frac{h}{\rho} \cdot \frac{\delta h}{\rho} + \frac{k}{\rho} \cdot \frac{\delta k}{\rho} + \frac{l}{\rho} \cdot \frac{\delta l}{\rho} \quad \text{.......................... (2)}$$

Now if λ, μ, ν be the initial direction-cosines of PQ,

$$h/\lambda = k/\mu = l/\nu = \rho \quad \text{.......................... (3)}$$

Hence, substituting from (1) and (3) in (2) we find

$$\begin{aligned} \epsilon = \lambda\left(\lambda\frac{\partial u}{\partial x} + \mu\frac{\partial u}{\partial y} + \nu\frac{\partial u}{\partial z}\right) \\ + \mu\left(\lambda\frac{\partial v}{\partial x} + \mu\frac{\partial v}{\partial y} + \nu\frac{\partial v}{\partial z}\right) \\ + \nu\left(\lambda\frac{\partial w}{\partial x} + \mu\frac{\partial w}{\partial y} + \nu\frac{\partial w}{\partial z}\right) \quad \text{.................... (4)} \end{aligned}$$

or, re-arranging terms,

$$\begin{aligned} \epsilon = \lambda^2\frac{\partial u}{\partial x} + \mu^2\frac{\partial v}{\partial y} + \nu^2\frac{\partial w}{\partial z} + \mu\nu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ + \nu\lambda\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \lambda\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \quad \text{.................. (5)} \end{aligned}$$

54.] From the form (5) we see, by writing successively ($\lambda=1$, $\mu=0$, $\nu=0$), ($\lambda=0$, $\mu=1$, $\nu=0$), ($\lambda=0$, $\mu=0$, $\nu=1$), that $\partial u/\partial x$, $\partial v/\partial y$, $\partial w/\partial z$ are the elongations of elementary straight lines drawn from (x, y, z) parallel to Ox, Oy, Oz, respectively.

Again from the form (4) it is easily seen that ϵ may be written in the form

$$\epsilon = \left(\lambda\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial y} + \nu\frac{\partial}{\partial z}\right)(\lambda u + \mu v + \nu w),$$

if we assume λ, μ, ν constant as to x, y, z; that is, if we suppose the elementary straight line to be drawn in the *given* direction (λ, μ, ν) from different points of the body.

Now if ρ be regarded as a current coördinate, giving the initial distances from (x, y, z) of points situated in the given direction (λ, μ, ν), and if U be the displacement of (x, y, z), resolved along this line in the positive direction of ρ, we have

$$\left.\begin{aligned} U &= \lambda u + \mu v + \nu w \\ \frac{\partial}{\partial \rho} &= \lambda\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial y} + \nu\frac{\partial}{\partial z} \end{aligned}\right\}.$$

Thus

$$\epsilon = \partial U/\partial \rho \quad \text{.................................. (6)}$$

which gives the elongation of an elementary straight line drawn in any direction from any given point of the body.

55.] **Change of Direction.** Again, if (λ', μ', ν') be the direction-cosines of $P'Q'$,

$$\begin{aligned}\lambda' &= (h+\delta h)/(\rho+\delta\rho)\\ &= \frac{h}{\rho}+\frac{\delta h}{\rho}-\frac{h}{\rho}\cdot\frac{\delta\rho}{\rho}, \text{ etc.,}\end{aligned}$$

therefore substituting from (1)

$$\left.\begin{aligned}\lambda' &= \lambda\left(1-\epsilon+\frac{\partial u}{\partial x}\right)+\mu\frac{\partial u}{\partial y}+\nu\frac{\partial u}{\partial z}\\ \mu' &= \lambda\frac{\partial v}{\partial x}+\mu\left(1-\epsilon+\frac{\partial v}{\partial y}\right)+\nu\frac{\partial v}{\partial z}\\ \nu' &= \lambda\frac{\partial w}{\partial x}+\mu\frac{\partial w}{\partial y}+\nu\left(1-\epsilon+\frac{\partial w}{\partial z}\right)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(7)$$

Since λ, μ, ν, as well as h, k, l and ρ, for any elementary straight line in the body, are of the same order of magnitude after as before the small strain, it follows that all lines and surfaces in the body preserve not only their continuity, but also *the continuity of their curvature*, throughout the strain.

56.] **Permanence of Intersections of Lines and Surfaces.** It is easy to show that the points of intersection of lines, and the curves of intersection of surfaces, in the unstrained body become the points or curves of intersection of the same lines or surfaces in their strained state.

That is to say, if two curves in the unstrained body intersect in P, and if P be removed to P' by the strain, the curves will be strained into curves intersecting in P'. And similarly for the curves of intersection of surfaces.

For let the coördinates of P be (x, y, z) and those of P' (x', y', z') so that $x' = x + u$, $y' = y + v$, $z' = z + w$. Then if u, v, w are given as functions of x, y, z, we can express x', y', z' explicitly as functions of x, y, z; and therefore (theoretically at least) u, v, w as functions of x', y', z'.

Now the two equations

$$\left.\begin{aligned}f_1(x, y, z) &= 0\\ f_2(x, y, z) &= 0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(A)$$

taken separately represent two surfaces in the unstrained body, and, if these surfaces intersect, the same equations, regarded as simultaneous, represent their curve of intersection.

The equations of the surfaces into which these are strained are

$$\left.\begin{aligned}f_1(x'-u, y'-v, z'-w) &= 0\\ f_2(x'-u, y'-v, z'-w) &= 0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(B)$$

where u, v, w are supposed to be expressed explicitly in terms of x', y', z'.

But equations (B), regarded as simultaneous, may be taken as representing *either* the curve of intersection of the surfaces which they separately represent, *or* the curve which before the strain was represented by the simultaneous equations (A).

Thus the curve of intersection of any two strained surfaces in the body is the strained form of the curve of intersection of the same surfaces before the strain; and by a precisely similar method we can show that the point of intersection of any two strained lines is the strained position of the point of intersection of the same lines before the strain.

57. **General Effect of Strain.** We see from equations (5) and (7) that the magnitude and direction of every elementary straight line in the body are *in general* altered by the strain, and that these changes are *in general* different for different elements. Hence the general effect of the strain is both to shift and to distort all lines and surfaces in the body. We shall reserve the exceptional cases for future discussion.

58.] **Limitations of Small Strain.** From equation (5) it appears that the elongation of an elementary straight line, drawn in any direction from a given point, is of the same order of magnitude as the first derivatives of the component displacements of that point with regard to its initial coördinates. In future, unless the contrary is explicitly stated, we shall confine ourselves entirely to the consideration of "small strains," implying thereby that all these first derivatives, like the displacements themselves, are small quantities of the first order, or else zero.

Homogeneous Strain.

59. **Definition.** We shall now suppose the character of the strain restricted in such a manner that all the first derivatives, $\partial u/\partial x, \ldots\ldots\ldots \partial w/\partial z$, are independent of x, y, z.

This assumption involves a relation between the displacements and initial coördinates of the form

$$\left.\begin{aligned} u &= ex + \beta_1 y + \gamma_1 z \\ v &= \alpha_2 x + fy + \gamma_2 z \\ w &= \alpha_3 x + \beta_3 y + \chi_3 z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

γ

where the coefficients are all absolute constants, which for a finite strain are finite or zero, and for a small strain are all small quantities of the first order or else zero.

A strain of the character defined by this assumption is said to be a **Homogeneous Strain.** We shall now proceed to investigate its principal properties.

60.] The results which we have already obtained for any small strain take the following forms in the case of small homogeneous strain.

The component displacements of the point $Q(x+h, y+k, z+l)$ relative to the point $P(x, y, z)$ are, by (1),

$$\left.\begin{aligned} \delta h &= eh + \beta_1 k + \gamma_1 l \\ \delta k &= a_2 h + fk + \gamma_2 l \\ \delta l &= a_3 h + \beta_3 k + gl \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

The elongation of the line PQ (direction-cosines λ, μ, ν) is given by

$$\epsilon = e\lambda^2 + f\mu^2 + g\nu^2 + (\beta_3 + \gamma_2)\mu\nu + (\gamma_1 + a_3)\nu\lambda + (a_2 + \beta_1)\lambda\mu \ldots\ldots\ldots(10)$$

whence it is obvious that e, f, g are the elongations of elementary straight lines parallel to Ox, Oy, Oz respectively.

Lastly, the new direction-cosines of $P'Q'$ are, by (7),

$$\left.\begin{aligned} \lambda' &= (1 - \epsilon + e)\lambda + \beta_1\mu + \gamma_1\nu \\ \mu' &= a_2\lambda + (1 - \epsilon + f)\mu + \gamma_2\nu \\ \nu' &= a_3\lambda + \beta_3\mu + (1 - \epsilon + g)\nu \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

61.] **Parallel Straight Lines.** It is obvious from equations (10) and (11) that ϵ, λ', μ', ν' depend entirely on λ, μ, ν; whence ve deduce that, in any small homogeneous strain, all parallel traight lines in the body, of elementary length, remain parallel .nd are elongated in the same ratio.

But we may consider any straight line, finite or infinite, in the ınstrained body, as made up of consecutive elementary straight ines, all of which are parallel to one another and meet consecutive :lements. By equations (11) these will be strained into elemen-ary straight lines all parallel to one another, and by § 56 each of :hese will meet the consecutive elements.

Hence they must all lie in a straight line; so that *a straight 'ine in the body, of whatsoever length, will remain a straight line, 'hough in general its direction will be changed.*

In the same way, since two parallel straight lines of any length may be divided into elements, all of which are necessarily parallel, it follows that *all parallel straight lines in the body remain parallel straight lines, though in general their direction will be changed.*

Also, since by (10) all their elements will be elongated in the ·ame ratio, *parallel straight lines of any length are elongated in 'he same ratio;* and, in particular, *equal and parallel straight lines are strained into equal and parallel straight lines,* though in general their length, direction, and distance apart are all altered by the strain.

62.] **Parallel Planes.** Again, since (§§ 56, 61) intersecting straight lines remain intersecting straight lines, *a plane must*

remain a plane; and since any two parallel planes intercept equal lengths on any system of parallel straight lines which meet them both, and since these intercepts are strained into equal and parallel (§ 61) straight lines, terminated (§ 56) by the strained planes, it follows that *all parallel planes are strained into parallel planes,* though in general their direction and distance apart are altered by the strain.

63.] **Similar and similarly situated Geometrical Figures.** From the two last articles it follows directly that *every parallelogram is strained into a parallelogram,* and *every parallelepiped into a parallelepiped,* though both are in general distorted.

Since similar and similarly situated plane figures (in the same or parallel planes) have their homologous sides parallel, it follows that *all similar and similarly situated plane figures are strained into plane figures similar and similarly situated* to one another, though not necessarily to the former.

In fact, since all parallel chords are elongated in the same ratio, it is obvious that the strained form of any plane figure is an enlarged or diminished orthographic projection of its unstrained form upon some plane.

Hence, in particular, an *ellipse (including the circle) is always strained into an ellipse or circle;* and when a circle is strained into an ellipse *every pair of orthogonal radii of the circle is strained into a pair of conjugate radii of the ellipse.*

Again, since in similar and similarly situated solid figures all similarly situated sections are similar, it follows that *all similar and similarly situated solid figures are strained into solid figures similar and similarly situated to one another,* though not in general to their unstrained forms.

64.] **Strain Ellipsoid.** Since all the sections of an ellipsoid are ellipses (or circles), and since no other surface possesses this property, it follows from the last article that every ellipsoid (or sphere) is strained into an ellipsoid (or sphere); and when a sphere is strained into an ellipsoid, every set of three orthogonal radii of the sphere becomes a set of three conjugate radii of the ellipsoid.

The ellipsoid into which a sphere of unit radius, described about any point P of the unstrained body as centre, is altered by the strain is called the *Strain Ellipsoid* at the point P. Of course, in a homogeneous strain, the strain ellipsoids at all points of the body will be equal, similar and similarly situated.

65.] **Principal Axes of the Strain.** Every set of orthogonal radii of the unit sphere becomes, by § 64, a set of conjugate radii of the Strain Ellipsoid; and the ellipsoid has one—

and only one—set of orthogonal conjugate radii, namely, its principal axes.

Hence, in every homogeneous strain there is one—and only one—set of three orthogonal straight lines passing through each point of the body, which remain orthogonal after the strain, although their directions are generally altered.

These principal diameters of the Strain Ellipsoid are called the *Principal Axes of the Strain* at P.

66.] **Pure Strain.** When the strain is such that the Principal Axes retain their initial directions it is said to be a *Pure* or *Irrotational* Strain.

It is sufficiently obvious that the most general small homogeneous strain will consist of a small pure homogeneous strain, sufficient to produce the required *distortion*, together with a small *rotation of the body as a whole*, about a suitable axis, sufficient to bring the Principal Axes at each point into their new positions.

Analytical Investigation.

67.] We shall now proceed to prove these properties of Homogeneous Strain analytically.

Since, by equations (8), u, v, w are *linear* functions of x, y, z, their partial derivatives of the second and all higher orders will vanish. Hence, equations (1) or (9) will be absolutely true, independently of the magnitude of h, k, l, so that equations (10) and (11) will hold for straight lines of any length. From this § 61 follows immediately.

68.] **Initial and Final Coördinates.** The equations giving the final coördinates (x', y', z') of any point P in terms of the initial coördinates (x, y, z) are, by equations (8),

$$\left.\begin{aligned} x' &= x + u = (1+e)x + \beta_1 y + \gamma_1 z \\ y' &= y + v = \alpha_2 x + (1+f)y + \gamma_2 z \\ z' &= z + w = \alpha_3 x + \beta_2 y + (1+g)z \end{aligned}\right\} \qquad (12)$$

Hence, to the first order of small quantities,

$$\left.\begin{aligned} x &= (1-e)x' - \beta_1 y' - \gamma_1 z' \\ y &= -\alpha_2 x' + (1-f)y' - \gamma_2 z' \\ z &= -\alpha_3 x' - \beta_2 y' + (1-g)z' \end{aligned}\right\} \qquad (13)$$

and, to the same approximation,

$$\left.\begin{aligned} u &= ex' + \beta_1 y' + \gamma_1 z' \\ v &= \alpha_2 x' + fy' + \gamma_2 z' \\ w &= \alpha_3 x' + \beta_2 y' + gz' \end{aligned}\right\} \qquad (14)$$

Thus, in the equations of displacement for a small strain, the final coördinates (x', y', z') may be substituted for the initial coördinates (x, y, z) without introducing any error perceptible to the order of approximation adopted. This is a very useful principle in practice, and it is obviously not confined to homogeneous strain, since it depends solely on the smallness of the coefficients involved.

69.] **Linear Transformation of Equations.** It is a very important consequence of the last Article that the equations of surfaces in the unstrained body are only altered by a linear transformation of the coördinates, and, consequently, every such surface remains *of the same order* as before the strain.

For example, the surface in the unstrained body given by the equation $\phi(x, y, z)=0$, becomes after the strain the surface given by

$$\phi\{[(1-e)x'-\beta_1 y'-\gamma_1 z'],\ [(1-f)y'-\gamma_2 z'-a_2 x'],\\ [(1-g)z'-a_3 x'-\beta_3 y']\}=0,$$

which equation, the coefficients being constants, is clearly of the same order as the former.

Thus, planes are strained into planes, and quadrics into quadrics; and since a small (or even a finite) strain cannot possibly convert a finite line into one of infinite length, it is clear that a closed surface must remain a closed surface. Thus, an ellipsoid or a sphere is always strained into an ellipsoid or sphere.

The straight line being formed by the intersection of two planes; and the ellipse or circle, being formed by the intersection of a plane with an ellipsoid or sphere, must obviously retain their original characters.

Also, since equal and parallel straight lines are strained into equal and parallel straight lines, it follows that a plane bisecting a system of parallel straight lines is strained into a plane bisecting a system of parallel straight lines, so that any system of parallel chords of an ellipsoid, with their diametral plane, become a system of parallel chords and corresponding diametral plane of the strained ellipsoid.

Hence it follows at once that every set of three conjugate diameters becomes a set of three conjugate diameters.

70.] **Strain Ellipsoid.** The ellipsoid

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}=1$$

becomes the ellipsoid

$$\frac{1-2e}{A^2}\cdot x'^2+\frac{1-2f}{B^2}\cdot y'^2+\frac{1-2g}{C^2}\cdot z'^2-2\left(\frac{\beta_3}{C^2}+\frac{\gamma_2}{B^2}\right)\cdot y'z'-2\left(\frac{\gamma_1}{A^2}+\frac{a_3}{C^2}\right)\cdot z'x'$$
$$-2\left(\frac{a_2}{B^2}+\frac{\beta_1}{A^2}\right)\cdot x'y'=1\,;$$

and, in particular, the sphere

$$x^2+y^2+z^2=1$$

becomes

$$(1-2e)x'^2+(1-2f)y'^2+(1-2g)z'^2-2(\beta_3+\gamma_2)y'z'-2(\gamma_1+a_3)z'x'$$
$$-2(a_2+\beta_1)x'y'=1 \quad\ldots\ldots(15)$$

which is the Strain Ellipsoid at the origin (§ 64), referred to the fixed axes.

71.] **Change of Notation.** It will be observed in equations (10) and (15) that the coefficients β_3 and γ_2, γ_1 and a_3, a_2 and β_1 occur in pairs. This will frequently happen in future equations, and we shall considerably simplify our analysis, and make it much easier to interpret, by changing our notation as follows:—let us take

$$\left.\begin{aligned}2s_1&=\beta_3+\gamma_2\\2s_2&=\gamma_1+a_3\\2s_3&=a_2+\beta_1\\2\theta_1&=\beta_3-\gamma_2\\2\theta_2&=\gamma_1-a_3\\2\theta_3&=a_2-\beta_1\end{aligned}\right\}\quad\ldots\ldots(16)$$

, f, g being retained.

72.] The equations of displacement for small homogeneous train then take the form

$$\left.\begin{aligned}u&=ex+(s_3-\theta_3)y+(s_2+\theta_2)z\\v&=(s_3+\theta_3)x+fy+(s_1-\theta_1)z\\w&=(s_2-\theta_2)x+(s_1+\theta_1)y+gz\end{aligned}\right\}\quad\ldots\ldots(17)$$

he elongation becomes

$$\epsilon=e\lambda^2+f\mu^2+g\nu^2+2s_1\mu\nu+2s_2\nu\lambda+2s_3\lambda\mu\quad\ldots\ldots(18)$$

nd the final direction-cosines are

$$\left.\begin{aligned}\lambda'&=(1-\epsilon+e)\lambda+(s_3-\theta_3)\mu+(s_2+\theta_2)\nu\\\mu'&=(s_3+\theta_3)\lambda+(1-\epsilon+f)\mu+(s_1-\theta_1)\nu\\\nu'&=(s_2-\theta_2)\lambda+(s_1+\theta_1)\mu+(1-\epsilon+g)\nu\end{aligned}\right\}\quad\ldots\ldots(19)$$

while the equation of the Strain Ellipsoid, referred to the fixed xes, is

$$(1-2e)x'^2+(1-2f)y'^2+(1-2g)z'^2-4s_1y'z'-4s_2z'x'-4s_3x'y'=1\quad\ldots\ldots(20)$$

73.] The direction-cosines of the principal axes of this ellipsoid are given by the equations

$$\frac{(1-2e)\lambda' - 2s_3\mu' - 2s_2\nu'}{\lambda'} = \frac{-2s_3\lambda' + (1-2f)\mu' - 2s_1\nu'}{\mu'}$$
$$= \frac{-2s_2\lambda' - 2s_1\mu' + (1-2g)\nu'}{\nu'}$$

which may also be written in the form

$$\frac{e\lambda' + s_3\mu' + s_2\nu'}{\lambda'} = \frac{s_3\lambda' + f\mu' + s_1\nu'}{\mu'} = \frac{s_2\lambda' + s_1\mu' + g\nu'}{\nu'} \;\ldots\ldots\ldots\ldots(21)$$

These equations therefore give us the directions, *after the strain*, of the Principal Axes of the Strain.

Graphic Properties of the Strain.

74.] **The Elongation and Compression Quadrics.** If we describe about the origin a quadric surface of the form

$$ex^2 + fy^2 + gz^2 + 2s_1yz + 2s_2zx + 2s_3xy = B^2 \ldots\ldots\ldots\ldots(22)$$

(which we shall regard as fixed in space, like the axes of reference), and if r be the radius vector intercepted by the surface on a straight line in the body drawn from the origin in the direction (λ, μ, ν) we shall have

$$r^2(e\lambda^2 + f\mu^2 + g\nu^2 + 2s_1\mu\nu + 2s_2\nu\lambda + 2s_3\lambda\mu) = B^2.$$

Thus, by equation (18), if ϵ be the elongation of this radius vector, or of any straight line in the body parallel to it

$$\epsilon = B^2/r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

This surface is called the *Elongation Quadric* of the strain.

75.] It follows from equation (23), the right-hand side of which is essentially positive, that every radius which meets this surface suffers a positive elongation, and conversely that every radius drawn in a direction of positive elongation will meet the surface.

If therefore the strain be such that all lines in the body are elongated, the Elongation Quadric must be an Ellipsoid.

If however the strain consists of elongations in some directions and contractions in others, ϵ will be negative for some radii, which therefore cannot meet (22).

In fact, in this case the Elongation Quadric is an hyperboloid whose radii are the lines which suffer elongation, while those lines which suffer contraction are the radii of the conjugate hyperboloid represented by

$$ex^2 + fy^2 + gz^2 + 2s_1yz + 2s_2zx + 2s_3xy = -B^2 \ldots\ldots\ldots\ldots 24)$$

which is called the *Compression Quadric.*

76.] In the case in which all lines in the body undergo contraction, all radii from the origin must meet the Compression Quadric (24), which is therefore an ellipsoid; and in this case there is no Elongation Quadric.

77.] **Cone of no Elongation.** In the case of § 75, the hyperboloids of elongation and contraction are separated by their asymptotic cone, whose equation is

$$ex^2 + fy^2 + gz^2 + 2s_1yz + 2s_2zx + 2s_3xy = 0 \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

It appears from (23) that for all the generators of this cone, and of course for all parallel lines in the body, $\epsilon = 0$. It is therefore called the *Cone of no Elongation.*

78.] **Cones of Constant Elongation.** Lastly, the direction-cosines of all lines in the body suffering a *given* elongation ϵ (whether positive or negative) must satisfy (18), which may be written

$$e\lambda^2 + f\mu^2 + g\nu^2 + 2s_1\mu\nu + 2s_2\nu\lambda + 2s_3\lambda\mu = \epsilon(\lambda^2 + \mu^2 + \nu^2).$$

All such lines must therefore be parallel to one or other of the generators of the cone

$$(e-\epsilon)x^2 + (f-\epsilon)y^2 + (g-\epsilon)z^2 + 2s_1yz + 2s_2zx + 2s_3xy = 0 \ldots\ldots\ldots (26)$$

We thus obtain a series of *Cones of Constant Elongation.*

79.] It is to be observed that all the quadrics described in the last five articles form a concentric and coaxial system.

If $O\xi$, $O\eta$, $O\zeta$ be their principal axes, their equations when referred to them will respectively become

$$\left.\begin{aligned} \epsilon_1\xi^2 + \epsilon_2\eta^2 + \epsilon_3\zeta^2 &= B^2 \\ \epsilon_1\xi^2 + \epsilon_2\eta^2 + \epsilon_3\zeta^2 &= -B^2 \\ \epsilon_1\xi^2 + \epsilon_2\eta^2 + \epsilon_3\zeta^2 &= 0 \\ (\epsilon_1-\epsilon)\xi^2 + (\epsilon_2-\epsilon)\eta^2 + (\epsilon_3-\epsilon)\zeta^2 &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

where ϵ_1, ϵ_2, ϵ_3 are the roots (in descending order of magnitude, let us suppose) of the discriminating cubic

$$\begin{vmatrix} e-\phi, & s_3, & s_2 \\ s_3, & f-\phi, & s_1 \\ s_2, & s_1, & g-\phi \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

the direction-cosines of $O\xi$, $O\eta$, $O\zeta$, with reference to the original axes, being given by the equations

$$\frac{e\lambda + s_3\mu + s_2\nu}{\lambda} = \frac{s_3\lambda + f\mu + s_1\nu}{\mu} = \frac{s_2\lambda + s_1\mu + g\nu}{\nu} = \phi \ldots\ldots\ldots (29)$$

where for ϕ is to be substituted ϵ_1, ϵ_2, or ϵ_3, according as λ, μ, ν are the direction-cosines of $O\xi$, $O\eta$, or $O\zeta$.

80.] **Principal Axes of the Strain.** Since the elongation ϵ of any radius of the elongation quadric varies inversely as the square of the radius, and since the squares of the least and greatest radii of the quadric are B^2/ϵ_1 and B^2/ϵ_3, it is obvious that ϵ always lies between ϵ_1 and ϵ_3, and that the directions of maximum and minimum elongation (or of minimum and maximum contraction) are those of the least and greatest axes of the quadric.

But if we consider the deformation of the unit sphere into the Strain Ellipsoid, it is clear that those radii of the sphere which are drawn in the directions of maximum and minimum elongation must become the greatest and least axes of the Ellipsoid.

Thus the lines in the body which, before the strain, coincided with the principal axes of the elongation quadrics, become the principal axes of the Strain Ellipsoid.

Equations (29) therefore give the *initial* directions of the Principal Axes of the Strain.

Pure Strain.

81.] **Conditions for Pure Strain.** The strain is said to be pure (§ 66) when the Principal Axes retain their initial directions.

Now, comparing equations (29), which give the directions of the Principal Axes *before* the strain, with equations (21) which give the directions of the same lines *after* the strain, we see that they *appear* to be identical. We must not, however, infer from this that the Principal Axes necessarily retain their initial directions. From equations (19) it appears that the *differences* between the initial and final values of the direction-cosines of any line are of the same small order as the strain coefficients; now in equations (21) and (29) the direction-cosines all appear multiplied by these same coefficients; so that it is quite impossible, to the order of approximation adopted, that any distinction should be made in such formulæ between λ and λ', μ and μ', ν and ν'.

For instance,

$$\frac{e\lambda' + s_3\mu' + s_2\nu'}{\lambda'} = \{e\lambda + s_3\mu + s_2\nu + e(\lambda' - \lambda) + s_3(\mu' - \mu)$$
$$+ s_2(\nu' - \nu)\} \cdot \frac{1}{\lambda} \cdot \left\{1 - \frac{\lambda' - \lambda}{\lambda}\right\},$$

and substituting from equations (19), this expression is identical, to the first order of small quantities, with

$$\frac{e\lambda + s_3\mu + s_2\nu}{\lambda}.$$

82.] **Non-rotated Straight Lines.** The three straight lines through the origin which (together with all lines in the body parallel to them) really retain their initial directions in space are to be found by putting $\lambda' = \lambda$, $\mu' = \mu$, $\nu' = \nu$ in (19).

Thus we get

$$\frac{e\lambda + (s_3 - \theta_3)\mu + (s_2 + \theta_2)\nu}{\lambda} = \frac{(s_3 + \theta_3)\lambda + f\mu + (s_1 - \theta_1)\nu}{\mu}$$
$$= \frac{(s_2 - \theta_2)\lambda + (s_1 + \theta_1)\mu + g\nu}{\nu} \quad \text{.............................(30)}$$

for the direction-cosines of the non-rotated straight lines.

The conditions for *Pure Strain* are therefore simply the conditions that the equations (21), (29) and (30) may be identical; and these obviously are

$$\theta_1 = 0,\ \theta_2 = 0,\ \theta_3 = 0.$$

83.] **Equations of Displacement. Principal Elongations.** The equations of displacement (17) thus become, in the case of Pure Strain,

$$\left.\begin{aligned} u &= ex + s_3 y + s_2 z \\ v &= s_3 x + fy + s_1 z \\ w &= s_2 x + s_1 y + gz \end{aligned}\right\} \quad \text{.............................(31)}$$

It will be observed that they only involve *six* independent strain coefficients.

If now U, V, W be the displacements of any point P in the body, parallel to the Principal Axes $O\xi$, $O\eta$, $O\zeta$; if $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, $(\lambda_3, \mu_3, \nu_3)$ be the direction-cosines of these axes referred to the original arbitrary axes Ox, Oy, Oz; and if (x, y, z) and (ξ, η, ζ) be the coördinates of P referred to the two systems; we have

$$\left.\begin{aligned} \xi &= \lambda_1 x + \mu_1 y + \nu_1 z \\ \eta &= \lambda_2 x + \mu_2 y + \nu_2 z \\ \zeta &= \lambda_3 x + \mu_3 y + \nu_3 z \end{aligned}\right\},$$

$$\left.\begin{aligned} U &= \lambda_1 u + \mu_1 v + \nu_1 w \\ V &= \lambda_2 u + \mu_2 v + \nu_2 w \\ W &= \lambda_3 u + \mu_3 v + \nu_3 w \end{aligned}\right\}.$$

Thus, from (31)

$$\begin{aligned} U &= \lambda_1(ex + s_3 y + s_2 z) + \mu_1(s_3 x + fy + s_1 z) + \nu_1(s_2 x + s_1 y + gz) \\ &= (\lambda_1 e + \mu_1 s_3 + \nu_1 s_2)x + (\lambda_1 s_3 + \mu_1 f + \nu_1 s_1)y + (\lambda_1 s_2 + \mu_1 s_1 + \nu_1 g)z \\ &= \lambda_1 \epsilon_1 x + \mu_1 \epsilon_1 y + \nu_1 \epsilon_1 z, \text{ by (29).} \end{aligned}$$

Ultimately therefore we find

$$\left.\begin{aligned} U &= \epsilon_1 \xi \\ V &= \epsilon_2 \eta \\ W &= \epsilon_3 \zeta \end{aligned}\right\} \quad \text{.............................(32)}$$

The point initially at (ξ, η, ζ) is therefore displaced to

$$(1+\epsilon_1)\xi,\ (1+\epsilon_2)\eta,\ (1+\epsilon_3)\zeta,$$

and obviously the effect of any Pure Strain is simply an elongation (or contraction) of the whole body parallel to each of the Principal Axes.

The three Principal Elongations $\epsilon_1, \epsilon_2, \epsilon_3$ are the roots of the discriminating cubic (28) of the Elongation Quadrics.

By comparing equations (31) and (32) it is evident that equation (18), giving the elongation ϵ of any line in the body may be written in the form

$$\epsilon = \epsilon_1 l^2 + \epsilon_2 m^2 + \epsilon_3 n^2 \qquad \text{(18}a\text{)}$$

where l, m, n are the direction-cosines of the line referred to $O\xi, O\eta, O\zeta$.

By comparing equations (31) and (22), or (32) and (27), we see that, in a pure strain, the resultant displacement of any point P in the body is along the normal to the elongation quadric which passes through P, and that its amount is B/p, where P is the perpendicular from the centre (the origin) on the tangent plane at P.

84.] **Position-Ellipsoid.** Describe about the origin the fixed quadric

$$(1+e)x^2 + (1+f)y^2 + (1+g)z^2 + 2s_1yz + 2s_2zx + 2s_3xy = C^2 \qquad (33)$$

This is obviously coaxial with the elongation quadrics, and when referred to the Principal Axes takes the form

$$(1+\epsilon_1)\xi^2 + (1+\epsilon_2)\eta^2 + (1+\epsilon_3)\zeta^2 = C^2.$$

Since $\epsilon_1, \epsilon_2, \epsilon_3$ are small, it is necessarily an ellipsoid.

Let r be the radius vector drawn in the direction (λ, μ, ν) and let (l, m, n), as in the last Article, denote the direction-cosines of r referred to $O\xi, O\eta, O\zeta$. Let p be the perpendicular from the centre on the tangent plane at the extremity of r, and let (l', m', n') be the direction-cosines of p referred to $O\xi, O\eta, O\zeta$. Finally, let ϵ be the elongation suffered by r.

By the ordinary formulæ of Solid Geometry,

$$\left.\begin{aligned} l' &= prl(1+\epsilon_1)/C^2 \\ m' &= prm(1+\epsilon_2)/C^2 \\ n' &= prn(1+\epsilon_3)/C^2 \end{aligned}\right\}.$$

Hence, squaring and adding,

$$p^2r^2\{1 + 2(\epsilon_1 l^2 + \epsilon_2 m^2 + \epsilon_3 n^2)\} = C^4;$$

$$\therefore \qquad p^2r^2(1+2\epsilon) = C^4;$$

$$\therefore \qquad r(1+\epsilon) = C^2/p.$$

Thus the strained length of the line in the body initially coinciding with r varies inversely as p.

Again, since equations (19) refer to any arbitrary set of axes, we may suppose them to refer to $O\xi$, $O\eta$, $O\zeta$; hence the new direction-cosines, *referred to these axes*, of the line in the body initially coinciding with r will be

$$(1-\epsilon+\epsilon_1)l,\ (1-\epsilon+\epsilon_2)m,\ (1-\epsilon+\epsilon_3)n.$$

But, by what we have just shown,

$$1+\epsilon = C^2/pr.$$

Thus, taking reciprocals,

$$1-\epsilon = pr/C^2,$$

and therefore

$$(1+\epsilon_1)pr/C^2 = (1+\epsilon_1)(1-\epsilon) = 1-\epsilon+\epsilon_1.$$

Thus we see that

$$\left.\begin{aligned} l' &= (1-\epsilon+\epsilon_1)l \\ m' &= (1-\epsilon+\epsilon_2)m \\ n' &= (1-\epsilon+\epsilon_3)n \end{aligned}\right\},$$

or the line in the body which initially coincided with the radius vector r finally coincides with the perpendicular p, and its final length varies inversely as p.

This ellipsoid is called the Position Ellipsoid, from the fact that it gives us a graphic construction for the position and length, after pure strain, of any line in the body whose position and length before the strain are known.

Rotational Strain.

85.] Returning to our arbitrary axes, let us suppose that the body, after undergoing the small Pure Strain represented by equations (31), is further subjected to a small rotation *of the body as a whole*, of amount Ω, about any axis (λ, μ, ν) through the fixed origin O.

The coördinates (x', y', z') of a point P, initially at (x, y, z), will be after the Pure Strain

$$\left.\begin{aligned} x' &= (1+e)x + s_3 y + s_2 z \\ y' &= s_3 x + (1+f)y + s_1 z \\ z' &= s_2 x + s_1 y + (1+g)z \end{aligned}\right\},$$

and the final coördinates (x'', y'', z'') of the same point after the rotation will be

$$\left.\begin{aligned} x'' &= x' + \mu\Omega z' - \nu\Omega y' \\ y'' &= y' + \nu\Omega x' - \lambda\Omega z' \\ z'' &= z' + \lambda\Omega y' - \mu\Omega x' \end{aligned}\right\},$$

the square and higher powers of the small quantity Ω being neglected.

To the same approximation we shall have for u, v, w, the resultant displacements,

$$\left.\begin{aligned} u &= ex + (s_3 - \nu\Omega)y + (s_2 + \mu\Omega)z \\ v &= (s_3 + \nu\Omega)x + fy + (s_1 - \lambda\Omega)z \\ w &= (s_2 - \mu\Omega)x + (s_1 + \lambda\Omega)y + gz \end{aligned}\right\}.$$

86.] Comparing these equations with (17) we deduce that *the general Homogeneous Strain represented by* (17) *consists of the Pure Strain represented by* (31), *together with a small rotation of the body as a whole*, the components of which about the fixed axes are θ_1, θ_2, θ_3; so that the amount Ω of this rotation, and the direction-cosines of its axis, are given by

$$\theta_1/\lambda = \theta_2/\mu = \theta_3/\nu = \Omega = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}.$$

This is the result that was anticipated in § 66.

Principle of Superposition.

87.] Writing equations (17) in the form

$$\left.\begin{aligned} u &= [ex + s_3y + s_2z] + [\theta_2z - \theta_3y] \\ v &= [s_3x + fy + s_1z] + [\theta_3x - \theta_1z] \\ w &= [s_2x + s_1y + gz] + [\theta_1y - \theta_2x] \end{aligned}\right\},$$

it is evident that the displacements due to a small rotational strain are simply the algebraic sums of the displacements due severally to the pure strain and the accompanying rotation; and it is further evident, from § 85, that this result depends entirely on the supposition that all the coefficients involved in the successive displacements are small quantities whose squares and higher powers may be neglected.

Consequently the same principle ought to apply to all small strains and rotations, whether they be homogeneous or not; and it is easy to show that this is the case.

Suppose the body first subjected to a small strain whose displacement coefficients are $[e, f, g, s_1, s_2, s_3, \theta_1, \theta_2, \theta_3]$.

The coördinates of any point P in the body, after this strain, will be

$$x' = (1 + e)x + (s_3 - \theta_3)y + (s_2 + \theta_2)z,$$

etc., etc.

Now let the body be subjected to a second small strain $[e', f', g', s'_1, s'_2, s'_3, \theta'_1, \theta'_2, \theta'_3]$. The final coördinates of P will be given by

$$x'' = (1 + e')x' + (s'_3 - \theta'_3)y' + (s'_2 + \theta'_2)z',$$

etc., etc.

Thus the resultant displacements of P, due to the two successive strains, will be

$$\begin{aligned} u = (1+e')\{(1+e)x + (s_3-\theta_3)y + (s_2+\theta_2)z\} \\ + (s'_3-\theta'_3)\{(s_3+\theta_3)x + (1+f)y + (s_1-\theta_1)z\} \\ + (s'_2+\theta'_2)\{(s_2-\theta_2)x + (s_1+\theta_1)y + (1+g)z\} - x, \end{aligned}$$

etc., etc.,

and, to the first order of small quantities,

$$u = [(e+e')x + (s_3+s'_3)y + (s_2+s'_2)z] + [(\theta_2+\theta'_2)z - (\theta_3+\theta'_3)y],$$

etc., etc.

This result may be extended to any number of *small strains*, so that finally we have for the resultant displacements

$$\left.\begin{aligned} u &= [\Sigma(e)\,.\,x + \Sigma(s_3)\,.\,y + \Sigma(s_2)\,.\,z] + [\Sigma(\theta_2)\,.\,z - \Sigma(\theta_3)\,.\,y] \\ v &= [\Sigma(s_3)\,.\,x + \Sigma(f)\,.\,y + \Sigma(s_1)\,.\,z] + [\Sigma(\theta_3)\,.\,x - \Sigma(\theta_1)\,.\,z] \\ w &= [\Sigma(s_2)\,.\,x + \Sigma(s_1)\,.\,y + \Sigma(g)\,.\,z] + [\Sigma(\theta_1)\,.\,y - \Sigma(\theta_2)\,.\,x] \end{aligned}\right\} \quad \ldots\ldots(34)$$

88.] Thus the resultant of any number of small strains is a small strain in which the coefficients of pure strain and rotation are respectively the algebraic sums of the corresponding coefficients in the component strains.

And, conversely, any small strain may be arbitrarily resolved into any number of small component strains, subject only to the condition that the algebraic sums of the several coefficients must be equal to the corresponding coefficients in the original strain.

This result is called the Principle of Superposition of small strains, and is a particular case of a theorem of very general application in Mathematical Physics.

Components of Pure Strain.

89.] We are now in a position to analyse equations (31) which represent the most general Pure Strain. By the last Article it may be regarded as the resultant of the six component pure strains represented respectively by

$$\left.\begin{aligned} u &= ex \\ v &= 0 \\ w &= 0 \end{aligned}\right\}\text{(i.)} \qquad \left.\begin{aligned} u &= 0 \\ v &= fy \\ w &= 0 \end{aligned}\right\}\text{(ii.)} \qquad \left.\begin{aligned} u &= 0 \\ v &= 0 \\ w &= gz \end{aligned}\right\}\text{(iii.)}$$

$$\left.\begin{aligned} u &= 0 \\ v &= s_1 z \\ w &= s_1 y \end{aligned}\right\}\text{(iv.)} \qquad \left.\begin{aligned} u &= s_2 z \\ v &= 0 \\ w &= s_2 x \end{aligned}\right\}\text{(v.)} \qquad \left.\begin{aligned} u &= s_3 y \\ v &= s_3 x \\ w &= 0 \end{aligned}\right\}\text{(vi.)}$$

Each of these components only involves one strain coefficient, and they are in consequence called Simple Strains (*compare* 33); and since any one of the coefficients may be altered without affecting the others, these simple component strains are *independent*.

90.] **Simple Elongations.** Let us consider first the strain represented by (i.), assuming e to be positive. The discriminating cubic (28) becomes

$$\phi^2(e-\phi)=0.$$

Thus $\epsilon_1=e$, $\epsilon_2=0$, $\epsilon_3=0$; and (with the notation of § 83) equations (29) give $\lambda_1=1$, $\mu_1=0$, $\nu_1=0$.

The Elongation Quadric degenerates into the pair of parallel planes $ex^2=B^2$, the Principal Axis $O\xi$ coinciding with Ox, while $O\eta$, $O\zeta$ are indeterminate.

The cone of no elongation degenerates into the plane of yz, and the cones of given elongation ϵ are the cones of revolution

$$(e-\epsilon)x^2=\epsilon(y^2+z^2).$$

The strain evidently consists of a uniform elongation, of amount e, of all lines in the body parallel to Ox, all lines in perpendicular directions remaining unchanged in length, while the elongation of any other line depends only on its inclination to the axis of the strain, being given by $\epsilon=e\lambda^2$. In fact, the Position Ellipsoid (§ 84) becomes the prolate spheroid

$$(1+e)\xi^2+\eta^2+\zeta^2=C^2.$$

It is obvious that this strain increases the volume of the body, or of any portion of it, in the ratio $(1+e)$.

These results can easily be adapted, *mutatis mutandis*, to the case where e is negative (uniform contraction).

91.] Similarly (ii.) and (iii.) represent simple elongations of amounts f and g, respectively parallel to Oy and Oz. The elongations produced by them in the line (λ, μ, ν) are $f\mu^2$ and $g\nu^2$, and they increase the volume of all portions of the body in the ratios $(1+f)$ and $(1+g)$ respectively.

92.] **Simple Shears.** In the case represented by (iv.) the discriminating cubic is

$$\begin{vmatrix} -\phi, & 0, & 0 \\ 0, & -\phi, & s_1 \\ 0, & s_1, & -\phi \end{vmatrix}=0.$$

which reduces to

$$\phi(\phi^2-s_1^2)=0.$$

Thus $\epsilon_1=s_1$, $\epsilon_2=0$, $\epsilon_3=-s_1$, if s_1 be assumed positive. Substituting in equations (29) they give

$$\left.\begin{aligned} &\lambda_1=0,\ \mu_1=\frac{1}{\sqrt{2}},\ \nu_1=\frac{1}{\sqrt{2}} \\ &\lambda_2=1,\ \mu_2=0,\ \nu_2=0, \\ &\lambda_3=0,\ \mu_3=-\frac{1}{\sqrt{2}},\ \nu_3=\frac{1}{\sqrt{2}} \end{aligned}\right\}.$$

Thus $O\eta$ coincides with Ox, while $O\xi$ and $O\zeta$ lie in the plane of yz, and bisect internally and externally the angle between the positive directions of axes Oy, Oz; and the strain consists of a uniform elongation, of amount s_1, parallel to $O\xi$, together with a uniform contraction, *of equal amount*, parallel to $O\zeta$, all lines in the body parallel to $O\eta$ or Ox retaining their initial lengths unaltered.

The volume V of any portion of the body thus becomes

$$V(1+s_1)(1-s_1);$$

that is to say, it remains unchanged, and the strain produces *distortion only*.

93.] The Elongation and Compression Quadrics are cylinders, whose generators are parallel to $O\eta$ or Ox, and whose transverse sections are conjugate rectangular hyperbolas, their equations being

$$2s_1 yz = \pm B^2,$$

or

$$s_1(\xi^2-\zeta^2) = \pm B^2.$$

The Cone of no Elongation degenerates into the pair of asymptotic planes

$$yz=0,$$

or

$$\xi^2-\zeta^2=0.$$

Hence every line lying in either of the planes xy and zx retains its length and its inclination to Ox unaltered, and so of course does every line in the body parallel to either of these planes. [This may be shown directly by substitution in equations (18) and (19).]

Thus every geometrical figure described in any plane parallel to xy or zx will retain its initial form and dimensions unaltered.

94.] For this reason these two sets of parallel planes are called *Planes of no Distortion.*

Since each set of parallel planes remains a set of parallel planes, and their lines of intersection maintain their identity, the strain can only consist of a relative shifting of the two sets of planes of no distortion, after the manner of jointed wicker-work, so as slightly to diminish the right angle between the *positive* directions of Ox and Oy (which include between them the axis of elongation $O\xi$), and to increase the supplementary angle by the same small amount.

A strain of this nature is called a *Simple Shear of the two systems of planes* of no distortion; or simply a shear of the planes of xy and zx. Since the strain is really in two dimensions, all the effects produced in the plane of yz being exactly reproduced in all parallel planes, it is sometimes called a Simple Shear

in the plane of yz (in which case the positions of the axes $O\xi$, $O\eta$ must be specified); and this or any parallel plane may be termed the *Plane of the Shear.*

The Amount of the Shear is measured by the change in the right angles between the planes of no distortion, as described in the last Article.

Let Fig. 1 represent the section by the plane of yz of a prismatic portion of the body, bounded by planes of no distortion which cut the plane of the section in the square $ABCD$. Then

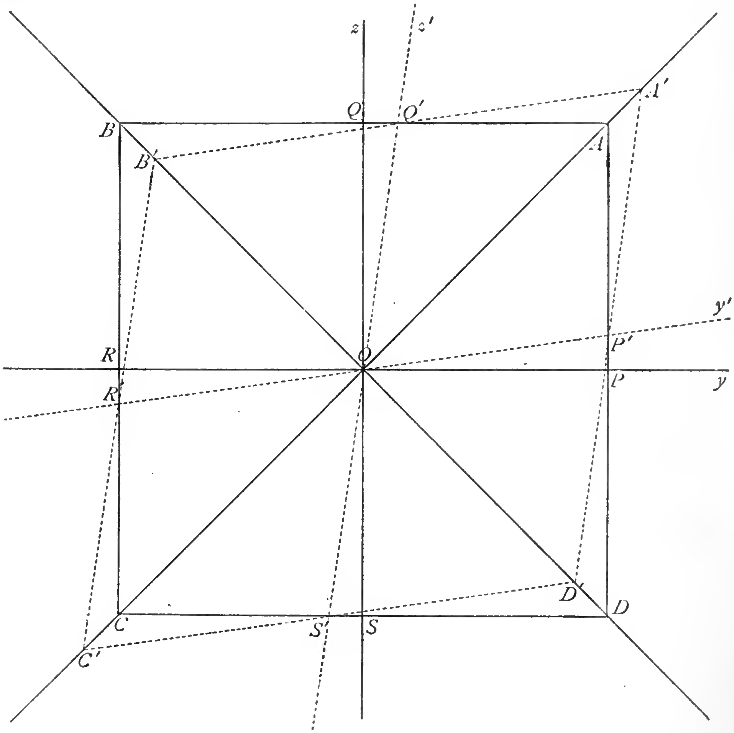


Fig. 1.

the axis of elongation $O\xi$ will coincide with the diagonal AOC, and the axis of contraction $O\zeta$ with the diagonal BOD.

The sole effect of the shear will be to change the square base of the prism into the rhombus $A'B'C'D'$, where $A'C' = (1 + s_1)AC$; $B'D' = (1 - s_1)BD$.

If the sides of the square meet the axes of reference in P, Q, R, S, and if PR, QS are strained into $P'R'$, $Q'S'$, the amount of the shear will be the sum of the angles QOQ' and POP', and since these angles are equal the amount is *twice* the angle POP'.

Now, by equations (19), we have for the strained position $P'R'$ of the line in the body PR, initially coinciding with Oy,

$$\nu' = s_1.$$

Hence $\cos P'Oz = s_1$, and to the first order of approximation

$$POP' = s_1,$$

and $2s_1$ is the amount of the shear.

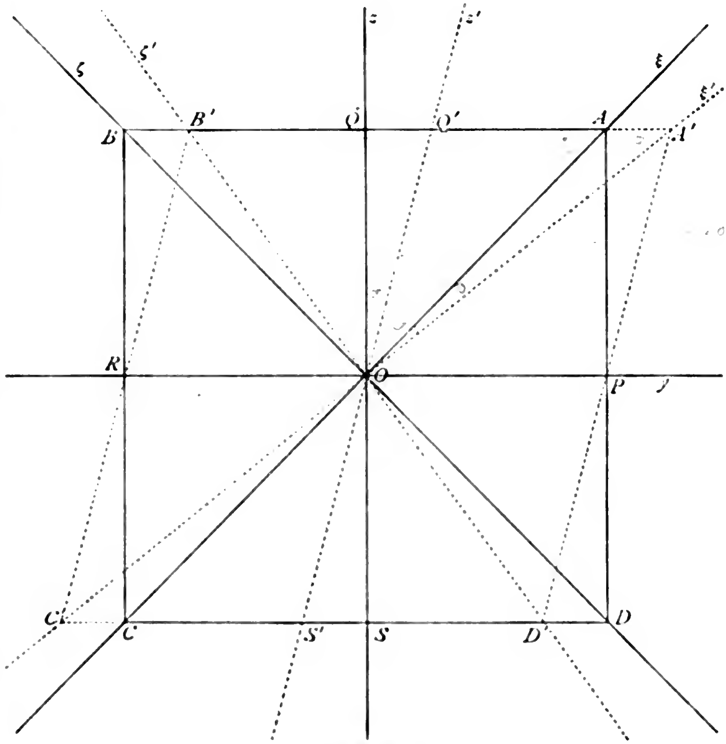


Fig. 2.

95.] **Shearing Motion.** There is another, slightly different, point of view from which we may regard the mode of operation of a small shear.

Let $ABCD$ in Fig. 2 represent the transverse section by the plane of yz of the same square prism as in Fig. 1; and let P, Q, R, S be as before the middle points of the sides.

Suppose that, keeping fixed the plane of xy, we give to every parallel plane in the body a motion parallel to the fixed plane, and proportional to its perpendicular distance from it, those

planes lying on the *positive* side of xy being shifted in the *positive* direction of Oy, and those on the *negative* side in the *negative* direction.

Since each point in OQ for (instance) moves perpendicularly to OQ through a space proportional to its distance from the fixed end O, it is obvious that OQ is strained into a straight line OQ'; and the displacements of points in QS at equal distances on opposite sides of O being equal and opposite, $Q'O$ and OS' will remain in one and the same straight line.

It follows that all planes in the body parallel to zx are simply turned through a constant angle of QOQ' about the lines in which they meet the plane of xy, while by hypothesis every plane in the body parallel to the latter undergoes a bodily translation in its own plane.

If the strain be of very small amount the lengths of the lines QS, etc., will not be appreciably altered, so that the result will be to strain the square $ABCD$ into the rhombus $A'B'C'D'$ without altering the lengths of its sides.

Thus it is obvious that the planes in the body parallel to xy and zx respectively form two systems of Planes of no Distortion.

96.] A strain of this kind is called a *Shearing Motion* of the planes parallel to xy in the positive direction of Oy.

Its *amount* is measured by the constant ratio between the distance traversed by any one plane and its perpendicular distance from the fixed plane: that is, by the tangent of the angle QOQ'.

The amount of a *small* shearing motion is therefore measured by the diminution or increase of the supplementary right angles between the planes of no distortion.

The change of direction AOA' of the diagonal plane AC is

$$\begin{aligned}\tan^{-1}\left(\frac{QA'}{OQ}\right) - \frac{\pi}{4} &= \tan^{-1}\left(\frac{QA' - QA}{QA' + QB}\right) \\ &= \tan^{-1}\left(\frac{QQ'}{2 . OQ}\right) \\ &= \frac{1}{2}QOQ' \text{ very nearly.}\end{aligned}$$

Similarly, the change of direction BOB' of the other diagonal plane is also approximately $\frac{1}{2}QOQ'$.

97.] Comparing these results with §§ 93, 94, it is obvious that a small *shearing motion* of amount $2s_1$ of planes perpendicular to Oz in the positive direction of Oy, is equivalent to a small *Shear* of amount $2s_1$ of planes perpendicular to Oy and Oz, together with a small rotation of the body as a whole through the angle s_1 about Ox in the negative direction (*i.e.*, from Oz towards Oy).

In like manner the student may satisfy himself, by drawing a suitable figure, that a small shearing motion of amount $2s_1$ of planes perpendicular to Oy in the positive direction of Oz is equivalent to *the same small shear*, together with a rotation of amount s_1 of the body as a whole about Ox in the *positive* direction.

98.] A shearing motion is therefore a *rotational* strain, the shear or *pure* strain being the same to whichever set of planes we give the shearing motion, while the accompanying rotations are in opposite directions in the two cases.

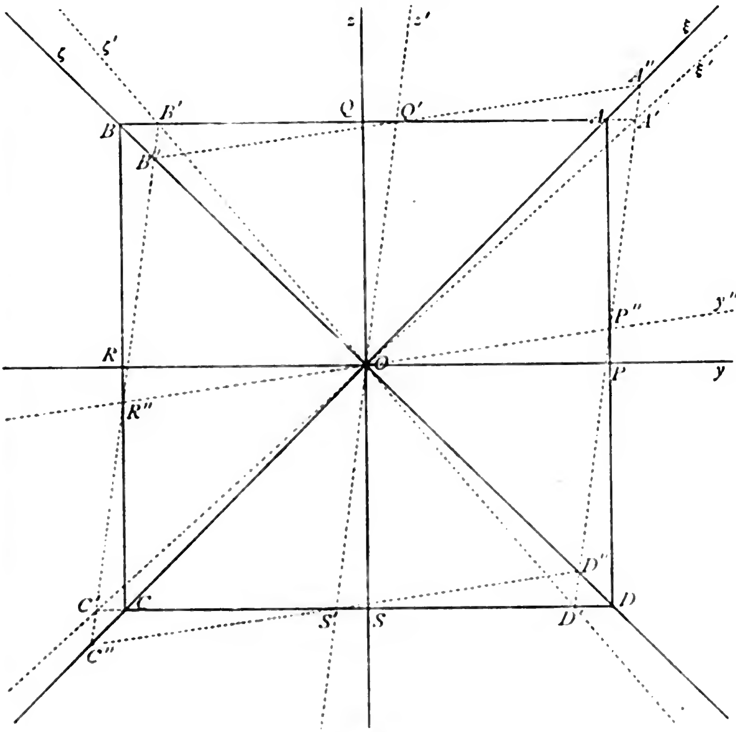


Fig. 3.

This fact suggests a method of producing (as in Fig. 3) a *non-rotational* shear of amount $2s_1$, by means of two shearing motions each of amount s_1, applied successively (or simultaneously) to the two sets of planes.

In Fig 3, the first shearing motion takes place parallel to Oy, so as to change the square $ABCD$ into the rhombus $A'B'C'D'$, at the same time producing the rotation $\xi O \xi'$. The second equal shearing motion, parallel to Oz', produces the equal and opposite

rotation $\xi O \xi'$, thus bringing the principal axes back to their initial positions, and at the same time shearing the rhombus $A'B'C'D'$ into the rhombus $A''B''C''D''$, which will be seen to be identical with the $A'B'C'D'$ of Fig. 1.

99.] All these results can, of course, be shown analytically. The equations of displacement for the shearing motion represented in Fig. 2 are manifestly

$$\left.\begin{aligned} u &= 0 \\ v &= 2s_1 z \\ w &= 0 \end{aligned}\right\},$$

which may be written

$$\left.\begin{aligned} u &= 0 \\ v &= (s_1 + s_1)z \\ w &= (s_1 - s_1)y \end{aligned}\right\}.$$

Comparing these with equations (17), we see that they represent a shear of amount $2s_1$ accompanied by a rotation $-s_1$ about Ox.

Similarly a shearing motion of amount $2s_1$ parallel to Oz is represented by

$$\left.\begin{aligned} u &= 0 \\ v &= (s_1 - s_1)z \\ w &= (s_1 + s_1)y \end{aligned}\right\},$$

and is therefore equivalent to the same shear, together with a rotation $+s_1$ about Ox.

Finally, the case of the last article is to be represented by superposing the two shearing motions

$$\left.\begin{aligned} u &= 0 \\ v &= s_1 z \\ w &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} u &= 0 \\ v &= 0 \\ w &= s_1 y \end{aligned}\right\},$$

the resultant of which is obviously the simple irrotational shear

$$\left.\begin{aligned} u &= 0 \\ v &= s_1 z \\ w &= s_1 y \end{aligned}\right\}.$$

100.] **Notation for Shears.** Similarly, equations (v.) and (vi.) of § 89 represent small simple shears of amounts $2s_2$ and $2s_3$, of planes perpendicular to Oz and Ox, and of planes perpendicular to Ox and Oy respectively.

We shall generally find it more convenient to use new symbols a, b, c for the *amounts* of these small shears, reserving s_1, s_2, s_3 for their component elongations and contractions.

Thus we shall have

$$a = 2s_1, \; b = 2s_2, \; c = 2s_3.$$

101.] **Finite Shear.** The properties of small shear which have been discussed in the preceding Articles are only the *limiting* forms assumed by the properties of Shear in general, when its amount is indefinitely diminished. Consequently, although they may be accepted as rigorously true for the purposes of our analysis of small strains (§ 58), it is impossible to draw figures which shall answer with perfect accuracy to the descriptions given.

The student will find in Appendix II., at the end of this Chapter, a short account of the corresponding properties of *Finite Shear*, which however have for us only a kinematic interest.

102.] **Cubical Dilatation.** Of the six component strains we have seen that (i.), (ii.) and (iii.) increase the volume of the body, or of any part of it in the ratios $(1+e)$, $(1+f)$, $(1+g)$ respectively, while (iv.), (v.) and (vi.) consist of pure distortions without change of volume.

If the volume V of any portion of the body be increased by the strain to V', the ratio $(V'-V)/V$ is called the *Cubical Dilatation* of the body. This may be either positive or negative: in the latter case, the positive ratio $(V-V')/V$ is sometimes called the *Cubical Compression*.

We shall always use the symbol Δ to denote cubical dilatation.

103.] It appears from the last Article that

$$\begin{aligned} V'/V &= (1+e)(1+f)(1+g) \\ &= (1+e+f+g). \end{aligned}$$

Hence

$$\Delta = e+f+g \qquad\qquad (35)$$

Since rotation cannot affect the volume, this relation holds equally for the general Homogeneous Strain.

It is obvious that the expression for the dilatation should be independent of the directions of the arbitrary axes of reference, and we see by expanding equation (28) that this is the case, $-\Delta$ being the coefficient of ϕ^2 in that equation.

Hence we may write

$$\Delta = \epsilon_1 + \epsilon_2 + \epsilon_3 \qquad\qquad (36)$$

104.] **Uniform Dilatation.** Dilatation is generally accompanied by distortion: for (putting shear aside as contributing nothing to dilatation) if e, f, g be different for any system of axes, a sphere in the body will be strained into an ellipsoid, and so on.

It is however possible to produce dilatation without dis-

tortion; for suppose the strain such that the three principal elongations are all equal, so that

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = \tfrac{1}{3}\Delta.$$

Any cubical portion of the body with its edges parallel to the principal axes will then have each edge elongated in the ratio $(1+\frac{1}{3}\Delta)$, and will remain a cube, the effect of the strain being simply to increase its volume in the ratio $(1+\Delta)$.

105.] In this case it is obvious from equations (27) that the Elongation Quadric becomes a sphere, and in order that (22) may reduce to the proper form, we must have

$$\left.\begin{array}{c} e = f = g = \frac{1}{3}\Delta \\ s_1 = s_2 = s_3 = 0 \end{array}\right\},$$

whatever be the axes of reference.

This strain is called a *Uniform Cubical Dilatation* of amount Δ, and, as we have seen, is equivalent to three equal elongations, each of amount $\frac{1}{3}\Delta$, in *any* three orthogonal directions.

The equations of displacement are

$$\left.\begin{array}{l} u = \frac{1}{3}\Delta x \\ v = \frac{1}{3}\Delta y \\ w = \frac{1}{3}\Delta z \end{array}\right\} \qquad \text{(vii.)}$$

Thus Uniform Dilatation, being expressed by a single coefficient, is to be (§ 89) regarded as a Simple Strain.

Types of Reference.

106.] **Summary of Results.** We have now shown that the simplest of strains—the Uniform Elongation—is the basis of all the more complex strains: that, in fact, the most general Pure Strain is the resultant of three orthogonal elongations parallel to its principal axes.

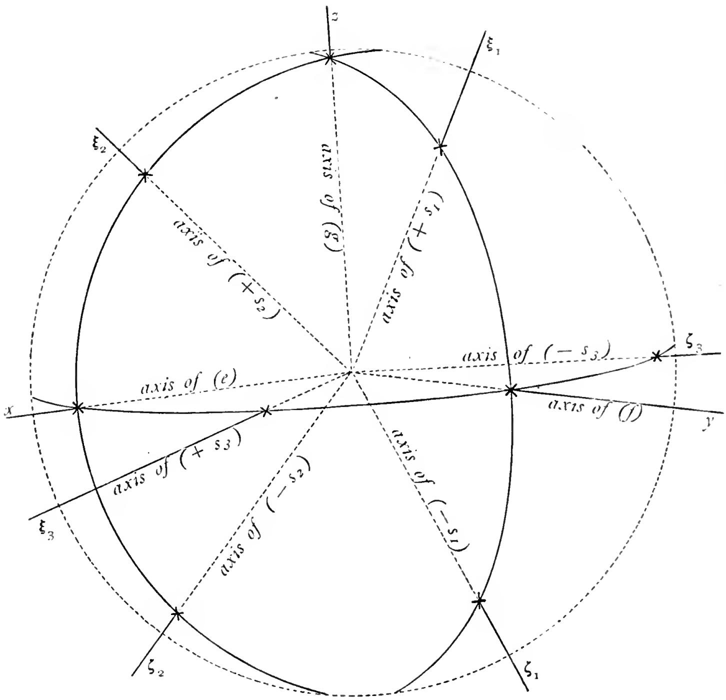
Further, we have shown that equal elongations (of like or unlike sign) may be so combined as to produce two more kinds of simple strain: namely, a distortion without dilatation or a dilatation without distortion.

107.] Again, it has been proved that the most general equations (31) of Pure Strain may be regarded as expressing it as the resultant of the following *six independent simple pure strains*:—

(I.) An elongation of amount e parallel to Ox.

(II.) An elongation of amount f parallel to Oy.

PLATE I.



Distribution of the

STANDARD COMPONENT STRAINS.

(*Page 47.*)

(III.) An elongation of amount g parallel to Oz.

(IV.) A shear of amount a, of planes perpendicular to Oy and Oz.

(V.) A shear of amount b, of planes perpendicular to Oz and Ox.

(VI.) A shear of amount c, of planes perpendicular to Ox and Oy.

The completeness with which these components express the most general pure strain will be realised when it is remembered that, since every set of parallel planes in the body must remain a set of parallel planes, the strain will be completely specified when we can express every possible relative motion of any set of parallel planes.

Now, the axes of reference are perfectly arbitrary, and from the preceding articles we can construct the following schedule:—

The Symbol	denotes Relative Motion Parallel to Axis of	of Planes Perpendicular to Axis of
e	x	x
c	x	y
b	x	z
c	y	x
f	y	y
a	y	z
b	z	x
a	z	y
g	z	z

so that any small pure strain can be represented by a proper combination of these six quantities.

108.] Thus the most general equations of Pure Strain really refer it to an arbitrarily chosen system of six orthogonal standard types: namely, three elongations parallel to three arbitrary orthogonal axes of reference, and three simple shears of the planes perpendicular to them, the axes of the shears bisecting the angles between the axes of elongation.

The most general equations of Rotational Strain (17) refer it to the same six standard types of strain, with the addition of three component rotations about the axes of reference.

Plate I. represents the positions of the principal axes of the

component strains; Ox, Oy, Oz being the axes of elongation, $O\xi_1$ and $O\zeta_1$ the axes of the shear a, and so on.

109.] Referring to §§ 32 and 46, we see that these six standard strains satisfy all the requirements of the system of "strain-coördinates" which we set out to seek; they may be chosen arbitrarily, they are perfectly independent, and any small strain can be expressed in terms of them, while they possess the great advantage—in point of simplicity—of vanishing in the natural state of the body.

We therefore adopt them as our standard types of simple strain, and, in order to completely specify any given small strain, we have only to enumerate its six orthogonal components in terms of the corresponding standard units.

110.] **Type of Strain.** When the six standard components of any two strains are to one another, each to each, in the same ratio, the strains are said to be of the same type, or of exactly opposite types, according as this ratio is positive or negative. (See § 33.)

The ratio of their components is called the Ratio of the Strains, and when this ratio is ± 1, the Strains are said to be equal.

Strains of the same and of opposite types are also called "concurrent" and "contrary."

Any number of small strains belonging to two opposite types compound into a strain belonging to one of these types.

Two equal and contrary strains exactly annul one another.

Specification of Strains.

111.] By equation (34) any number of Pure or Rotational small homogeneous strains can be compounded into one, if we are able to enumerate the standard components of each.

Now, every pure strain consists of a uniform cubical dilatation, a uniform elongation in some given direction, a simple shear with given axes; or is compounded of any or all of these (§ 89).

We shall therefore be able to form the equations of motion for the most complex combination of pure strains, when we know how to specify each of these simple strains in terms of its standard components.

The more general combination of homogeneous rotational strains may then be deduced by compounding the rotations separately, as in equations (34).

We shall now therefore proceed to show how the specifications of the various simple strains may be separately obtained.

The simplest method is by consideration of the *Invariants* of

the Elongation Quadric, which are the coefficients of the discriminating cubic (28). Expanding that equation it becomes

$$\phi^3 - \phi^2(e+f+g) + \phi(fg - s_1^2 + ge - s_2^2 + ef - s_3^2) - \begin{vmatrix} e, & s_3, & s_2 \\ s_3, & f, & s_1 \\ s_2, & s_1, & g \end{vmatrix} = 0 \quad \text{.......(37)}$$

or $$\phi^3 - \phi^2(\epsilon_1 + \epsilon_2 + \epsilon_3) + \phi(\epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2) - \epsilon_1\epsilon_2\epsilon_3 = 0 \quad \text{.......(38)}$$

Denoting these coefficients by D, J, K respectively, we have

$$\left.\begin{aligned} D &= e+f+g = \epsilon_1 + \epsilon_2 + \epsilon_3 \\ J &= fg - s_1^2 + ge - s_2^2 + ef - s_3^2 = \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2 \\ K &= \begin{vmatrix} e, & s_3, & s_2 \\ s_3, & f, & s_1 \\ s_2, & s_1, & g \end{vmatrix} = \epsilon_1\epsilon_2\epsilon_3 \end{aligned}\right\} \quad \text{.......(39)}$$

112.] **Uniform Cubical Dilatation of amount Δ.** This case has been discussed in § 105. The Quadric is a sphere, and the three roots of the cubic (37) are equal. The requisite conditions are

$$\left.\begin{aligned} e = f = g &= \frac{\Delta}{3} \\ s_1 = s_2 = s_3 &= 0 \end{aligned}\right\} \quad \text{.......(40)}$$

nd the equations of displacement are

$$\left.\begin{aligned} u &= \tfrac{1}{3}\Delta x \\ v &= \tfrac{1}{3}\Delta y \\ w &= \tfrac{1}{3}\Delta z \end{aligned}\right\} \quad \text{.......(41)}$$

Conversely, any strain or combination of strains whose components satisfy (40) amounts to a uniform cubical dilatation of mount Δ.

113.] **Simple Elongation of amount ϵ in direction** l, m, n). In this case the roots of the cubic are respectively , 0, 0. Hence it must reduce to $\phi^2(\phi - \epsilon) = 0$.

Thus,

$$\left.\begin{aligned} D &= \epsilon \\ J &= 0 \\ K &= 0 \end{aligned}\right\} \quad \text{.......(42)}$$

The two last conditions in combination are easily shown (Aldis' *Solid Geometry*, § 91) to be equivalent to either of the sets of three

$$\left.\begin{aligned} fg - s_1^2 &= 0 \\ ge - s_2^2 &= 0 \\ ef - s_3^2 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} es_1 - s_2s_3 &= 0 \\ fs_2 - s_3s_1 &= 0 \\ gs_3 - s_1s_2 &= 0 \end{aligned}\right\} \quad \text{.......(43)}$$

while the first gives us

$$e+f+g=\epsilon \qquad (44)$$

or

$$\frac{1}{s_1^2}+\frac{1}{s_2^2}+\frac{1}{s_3^2}=\frac{\epsilon}{s_1s_2s_3} \qquad (45)$$

by virtue of (43).

Again, l, m, n are the direction-cosines of the only determinate axis (§ 90) of the strain.

Hence, by equations (29),

$$\frac{el+s_3m+s_2n}{l}=\frac{s_3l+fm+s_1n}{m}=\frac{s_2l+s_1m+gn}{n}=\epsilon \qquad (46)$$

Eliminating e, f, g from these equations by means of (43) we get

$$\frac{s_1s_2s_3}{\epsilon}\left(\frac{l}{s_1}+\frac{m}{s_2}+\frac{n}{s_3}\right)=ls_1=ms_2=ns_3.$$

Thus,

$$\left.\begin{aligned} s_1&=\epsilon mn\\ s_2&=\epsilon nl\\ s_3&=\epsilon lm\end{aligned}\right\} \qquad (47)$$

whence, by (43),

$$\left.\begin{aligned} e&=\epsilon l^2\\ f&=\epsilon m^2\\ g&=\epsilon n^2\end{aligned}\right\} \qquad (48)$$

and the equations of displacement are

$$\left.\begin{aligned} u&=\epsilon l^2x+\epsilon lmy+\epsilon nlz\\ v&=\epsilon lmx+\epsilon m^2y+\epsilon mnz\\ w&=\epsilon nlx+\epsilon mny+\epsilon n^2z\end{aligned}\right\} \qquad (49)$$

Conversely, if the components of a given strain satisfy (43) it amounts to a simple elongation.

Its amount is then given by (44) or (45), and its direction by

$$ls_1=ms_2=ns_3=(e+f+g)lmn \qquad (50)$$

114.] **A Simple Shear of amount 2σ whose axes of elongation and contraction are in the directions (l_1, m_1, n_1) (l_2, m_2, n_2).**

In this case (§ 92) we have $\epsilon_1=\sigma$, $\epsilon_2=0$, $\epsilon_3=-\sigma$, and the cubic reduces to $\phi(\phi^2-\sigma^2)=0$.

Hence

$$\left.\begin{array}{l} D=0 \\ K=0 \end{array}\right\} \quad \text{.....(51)}$$

and

$$J = -\sigma^2 \quad \text{.....(52)}$$

Also, by equations (29)

$$\frac{el_1+s_3m_1+s_2n_1}{l_1}=\frac{s_3l_1+fm_1+s_1n_1}{m_1}=\frac{s_2l_1+s_1m_1+gn_1}{n_1}=\sigma,$$

$$\frac{el_2+s_3m_2+s_2n_2}{l_2}=\frac{s_3l_2+fm_2+s_1n_2}{m_2}=\frac{s_2l_2+s_1m_2+gn_2}{n_2}=-\sigma:$$

whence we find

$$\left.\begin{array}{ll} m_1n_1=\dfrac{s_2s_3-s_1(e-\sigma)}{2\sigma^2}; & m_2n_2=\dfrac{s_2s_3-s_1(e+\sigma)}{2\sigma^2} \\ n_1l_1=\dfrac{s_3s_1-s_2(f-\sigma)}{2\sigma^2}; & n_2l_2=\dfrac{s_3s_1-s_2(f+\sigma)}{2\sigma^2} \\ l_1m_1=\dfrac{s_1s_2-s_3(g-\sigma)}{2\sigma^2}; & l_2m_2=\dfrac{s_1s_2-s_3(g+\sigma)}{2\sigma^2} \end{array}\right\} \quad \text{.....(53)}$$

Hence we easily deduce

$$\left.\begin{array}{ll} e=\sigma(l_1^2-l_2^2); & s_1=\sigma(m_1n_1-m_2n_2) \\ f=\sigma(m_1^2-m_2^2); & s_2=\sigma(n_1l_1-n_2l_2) \\ g=\sigma(n_1^2-n_2^2); & s_3=\sigma(l_1m_1-l_2m_2) \end{array}\right\} \quad \text{.....(54)}$$

and the equations of displacement are

$$\left.\begin{array}{l} u=\sigma(l_1^2-l_2^2)x+\sigma(l_1m_1-l_2m_2)y+\sigma(n_1l_1-n_2l_2)z \\ v=\sigma(l_1m_1-l_2m_2)x+\sigma(m_1^2-m_2^2)y+\sigma(m_1n_1-m_2n_2)z \\ w=\sigma(n_1l_1-n_2l_2)x+\sigma(m_1n_1-m_2n_2)y+\sigma(n_1^2-n_2^2)z \end{array}\right\} \quad \text{.....(55)}$$

Equations (54) and (55) might of course have been deduced by superposition from equations (48) and (49).

Conversely, if the components of a given strain satisfy (51) it amounts to a simple shear whose amount 2σ is given by (52), while the directions of its axes are given by (53). In these last equations σ must be taken as the positive root of (51).

115.] **Resultant of any number of simple strains.** We can now form, with the greatest ease, the equations of displacement for the most complex combinations of small strains, either pure or rotational. Retaining the notation of the last three articles for pure components, and remembering (§§ 85, 86, 87) that any rotation Ω of the body as a whole about an axis (λ, μ, ν) can be resolved into component rotations $\Omega\lambda$, $\Omega\mu$, $\Omega\nu$ about the axis of reference, and that these rotations are to be compounded separately, the principle of superposition gives us at once for the

standard components of strain and rotation in the single equivalent strain

$$\left.\begin{aligned}
e &= \Sigma(\tfrac{1}{3}\Delta) + \Sigma(\epsilon l^2) + \Sigma(\sigma l_1^2 - \sigma l_2^2)\\
f &= \Sigma(\tfrac{1}{3}\Delta) + \Sigma(\epsilon m^2) + \Sigma(\sigma m_1^2 - \sigma m_2^2)\\
g &= \Sigma(\tfrac{1}{3}\Delta) + \Sigma(\epsilon n^2) + \Sigma(\sigma n_1^2 - \sigma n_2^2)\\
s_1 &= \Sigma(\epsilon mn) + \Sigma(\sigma m_1 n_1 - \sigma m_2 n_2)\\
s_2 &= \Sigma(\epsilon nl) + \Sigma(\sigma n_1 l_1 - \sigma n_2 l_2)\\
s_3 &= \Sigma(\epsilon lm) + \Sigma(\sigma l_1 m_1 - \sigma l_2 m_2)\\
\theta_1 &= \Sigma(\Omega\lambda)\\
\theta_2 &= \Sigma(\Omega\mu)\\
\theta_3 &= \Sigma(\Omega\nu)
\end{aligned}\right\} \quad \text{.................(56)}$$

116.] **Resolution into arbitrarily chosen simple components.** We stated in § 88, as the converse of the Principle of Superposition, that a pure strain might be arbitrarily resolved into any number of pure strains, subject only to the condition that the algebraic sums of their components must be severally equal to the corresponding components of the original strain.

It is an interesting problem to investigate the different ways, beside the standard way, in which a pure strain may be resolved into simple strains without in any way limiting its generality:—that is, without imposing any restrictions upon its standard components.

117.] Since the number of these standard components is *six*, the number of *independent* elements involved in any such equivalent system of simple strains must also be *exactly six*, in order that the solution may be at once perfectly general and completely determinate. These six independent elements will then be given by equations (56), in which e, f, g, s_1, s_2, s_3, must be taken to represent the standard components of the pure strain to be resolved.

If the number m of independent elements involved in any proposed system be greater than six, we must introduce $m-6$ relations between them, which may be quite arbitrarily chosen (with a few obvious restrictions to be presently pointed out).

If m be less than six we assume $6-m$ identical relations between the standard components, and thereby limit the general character of the strain; or, geometrically speaking, determine to a greater or less extent the type of the Elongation Quadric by introducing relations between its invariants.

118.] Now a uniform cubical dilatation involves only *one* element, its amount Δ.

A uniform elongation involves four elements (ϵ, l, m, n), of which however only *three* are independent, in virtue of the relation

$$l^2 + m^2 + n^2 = 1.$$

A simple shear involves seven elements (σ, l_1, m_1, n_1, l_2, m_2, n_2) of which only *four* are independent, in virtue of the relations

$$\left.\begin{array}{r} l_1^2+m_1^2+n_1^2=1 \\ l_2^2+m_2^2+n_2^2=1 \\ l_1l_2+m_1m_2+n_1n_2=0 \end{array}\right\}.$$

119.] If then we wish to represent the most general pure strain as the resultant of a dilatation, an elongation and a shear, we may subject these to any *two* arbitrary conditions, and the problem is then completely determinate.

For example, we may assign arbitrarily *either*—

(*i.*) The direction of the elongation.

(*ii.*) The plane of the shear.

(*iii.*) The inclinations of the axis of elongation to the axes of the shear: *e.g.*, we may take the elongation perpendicular to the plane of the shear.

(*iv.*) The *ratios* of the amounts of the three simple strains.

120.] As examples of assumptions which restrict the type of the strain, we may take the following:—

(*i.*) If we assume the strain to be compounded of any number of shears alone, we assume the volume of every portion of the body to remain unaltered. This involves the relation

$$e+f+g=0, \text{ or } D=0,$$

and the Elongation and Compression Quadrics are either conjugate hyperboloids, or cylinders whose transverse sections are conjugate rectangular hyperbolas.

(*ii.*) If we assume the strain to consist of a dilatation and a shear without independent elongation, it is evident from considerations of symmetry that the axes of the shear will coincide with the principal axes of the strain, and the Elongation Quadrics, referred to these axes, will take the form

$$\left(\frac{\Delta}{3}+\sigma\right)\xi^2+\frac{\Delta}{3}\eta^2+\left(\frac{\Delta}{3}-\sigma\right)\zeta^2=1,$$

the circular sections of which are

$$\xi\pm\zeta=0.$$

We thus constrain the Elongation Quadrics to have orthogonal circular sections.

The identical relation involved between the invariants is easily found, for

$$\epsilon_1=\tfrac{1}{3}\Delta+\sigma,\ \epsilon_2=\tfrac{1}{3}\Delta,\ \epsilon_3=\tfrac{1}{3}\Delta-\sigma,$$

$$\therefore \quad \left.\begin{array}{l} D=\Delta \\ J=\tfrac{1}{3}\Delta^2-\sigma^2 \\ K=\tfrac{1}{3}\Delta(\tfrac{1}{9}\Delta^2-\sigma^2) \end{array}\right\},$$

$$\therefore \quad D(2D^2-9J)+27K=0.$$

(*iii.*) If we assume the strain to consist of a uniform dilatation and an elongation only, the Quadric becomes

$$\left(\frac{\Delta}{3}+\epsilon\right)\xi^2+\frac{\Delta}{3}(\eta^2+\zeta^2)=1,$$

which is a surface of revolution.

The relations assumed in this case between the standard components (Frost. *Solid Geometry*, § 373) are known to be

$$e-\frac{s_2s_3}{s_1}=f-\frac{s_3s_1}{s_2}=g-\frac{s_1s_2}{s_3},$$

or, since the cubic has two equal roots (Todhunter. *Theory of Equations*, § 173)

$$J^2(D^2-4J)-DK(4D^2-18J)-27K^2=0.$$

Change of Axes of Reference.

121.] It is often convenient to change the directions of our axes of reference, and it then becomes necessary to obtain the specification of the strain referred to the new axes in terms of its original specification.

Let Ox', Oy', Oz' be the new system of axes, their direction-cosines referred to the old system being given by the annexed schedule, and let the two sets of equations

	x	y	z
x'	λ_1	μ_1	ν_1
y'	λ_2	μ_2	ν_2
z'	λ_3	μ_3	ν_3

$$\left.\begin{aligned} u&=ex+(s_3-\theta_3)y+(s_2+\theta_2)z\\ v&=(s_3+\theta_3)x+fy+(s_1-\theta_1)z\\ w&=(s_2-\theta_2)x+(s_1+\theta_1)y+gz\end{aligned}\right\}\dots\dots\dots(17)$$

$$\left.\begin{aligned} u'&=e'x'+(s_3'-\theta_3')y'+(s_3'+\theta_2')z'\\ v'&=(s_3'+\theta_3')x'+f'y'+(s_1'-\theta_1')z'\\ w'&=(s_2'-\theta_2')x'+(s_1'+\theta_1')y'+g'z'\end{aligned}\right\}\dots\dots\dots(57)$$

represent the same strains referred to the two systems. We have

$$\left.\begin{aligned} x&=\lambda_1x'+\lambda_2y'+\lambda_3z'\\ y&=\mu_1x'+\mu_2y'+\mu_3z'\\ z&=\nu_1x'+\nu_2y'+\nu_3z'\end{aligned}\right\}\dots\dots\dots(C.)$$

$$\left.\begin{aligned} u'&=\lambda_1u+\mu_1v+\nu_1w\\ v'&=\lambda_2u+\mu_2v+\nu_2w\\ w'&=\lambda_3u+\mu_3v+\nu_3w\end{aligned}\right\}\dots\dots\dots(D.)$$

In equations (D.) substitute for u, v, w their values in terms of x, y, z from (17); then substitute for x, y, z their values in terms of x', y', z' from (C). We thus obtain u', v', w' in terms of x', y', z' and comparing these results with (57) we easily find

$$\left.\begin{aligned}
e' &= \lambda_1^2 e + \mu_1^2 f + \nu_1^2 g + 2\mu_1\nu_1 s_1 + 2\nu_1\lambda_1 s_2 + 2\lambda_1\mu_1 s_3\\
f' &= \lambda_2^2 e + \mu_2^2 f + \nu_2^2 g + 2\mu_2\nu_2 s_1 + 2\nu_2\lambda_2 s_2 + 2\lambda_2\mu_2 s_3\\
g' &= \lambda_3^2 e + \mu_3^2 f + \nu_3^2 g + 2\mu_3\nu_3 s_1 + 2\nu_3\lambda_3 s_2 + 2\lambda_3\mu_3 s_3\\
s_1' &= \lambda_2\lambda_3 e + \mu_2\mu_3 f + \nu_2\nu_3 g + (\mu_2\nu_3 + \mu_3\nu_2)s_1 + (\nu_2\lambda_3 + \nu_3\lambda_2)s_2\\
&\qquad + (\lambda_2\mu_3 + \lambda_3\mu_2)s_3\\
s_2' &= \lambda_3\lambda_1 e + \mu_3\mu_1 f + \nu_3\nu_1 g + (\mu_3\nu_1 + \mu_1\nu_3)s_1 + (\nu_3\lambda_1 + \nu_1\lambda_3)s_2\\
&\qquad + (\lambda_3\mu_1 + \lambda_1\mu_3)s_3\\
s_3' &= \lambda_1\lambda_2 e + \mu_1\mu_2 f + \nu_1\nu_2 g + (\mu_1\nu_2 + \mu_2\nu_1)s_1 + (\nu_1\lambda_2 + \nu_2\lambda_1)s_2\\
&\qquad + (\lambda_1\mu_2 + \lambda_2\mu_1)s_3\\
\theta_1' &= (\mu_2\nu_3 - \mu_3\nu_2)\theta_1 + (\nu_2\lambda_3 - \nu_3\lambda_2)\theta_2 + (\lambda_2\mu_3 - \lambda_3\mu_2)\theta_3\\
\theta_2' &= (\mu_3\nu_1 - \mu_1\nu_3)\theta_1 + (\nu_3\lambda_1 - \nu_1\lambda_3)\theta_2 + (\lambda_3\mu_1 - \lambda_1\mu_3)\theta_3\\
\theta_3' &= (\mu_1\nu_2 - \mu_2\nu_1)\theta_1 + (\nu_1\lambda_2 - \nu_2\lambda_1)\theta_2 + (\lambda_1\mu_2 - \lambda_2\mu_1)\theta_3
\end{aligned}\right\}\ldots\ldots(58)$$

Heterogeneous Strain.

122.] From § 59 up to this point we have always assumed the small strain under discussion to be Homogeneous, and its components in consequence to be constant throughout the body.

We shall now realise the analytical advantages of our conception of continuous matter; for it is obvious that, if the constitution of a body be infinitely fine in comparison with the refinement of our analytical machinery (§ 42), it is possible to conceive of a portion of the body, differing in no physical quality from the body taken as a whole, and yet so minute that any algebraical function of position, *varying continuously*, shall be sensibly constant throughout it.

Thus it appears from §§ 51-58 that all the properties which we have proved to belong, *throughout the body*, to a small homogeneous strain will also hold good, *throughout any infinitely small element of the body*, for the (sensibly homogeneous) strain of that element due to a small Heterogeneous Strain.

123.] **Strain-Components.** The standard components of the strain will, of course, vary from point to point of the body. We shall retain our former notation for them, and by comparing equations (1), (9), (16), we see that we must make

$$\left.\begin{aligned}
e=\frac{\partial u}{\partial x},\ f=\frac{\partial v}{\partial y},\ g&=\frac{\partial w}{\partial z}\\
a=2s_1&=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\\
b=2s_2&=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\\
c=2s_3&=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\\
2\theta_1&=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\\
2\theta_2&=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\\
2\theta_3&=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\\
\text{and, by § 103,}\qquad\qquad&\\
\Delta&=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}
\end{aligned}\right\}\quad\text{.....................(59)}$$

If we give these components their proper values at any point P (x, y, z), the strain of an element of the body described about P will possess all the properties discussed in §§ 59-121, the various surfaces involved being, of course, referred to axes drawn through P parallel to the fixed axes of reference Ox, Oy, Oz.

The directions of the principal axes (§ 65) and the form and dimensions of the Strain Ellipsoid will of course vary from point to point of the body. The Strain Ellipsoid must now be defined as the ellipsoid into which a sphere of unit radius and centre P would be strained, if the strain-components had throughout the sphere their actual values at P.

Irrotational Strain.

124.] The conditions that the strain may be irrotational, *i.e.*, that every element may suffer pure strain without rotation of its principal axes, are, as before, $\theta_1=0$, $\theta_2=0$, $\theta_3=0$, at every point of the body.

Thus, by equations (59),

$$\left.\begin{aligned}
\frac{\partial w}{\partial y}&=\frac{\partial v}{\partial z}\\
\frac{\partial u}{\partial z}&=\frac{\partial w}{\partial x}\\
\frac{\partial v}{\partial x}&=\frac{\partial u}{\partial y}
\end{aligned}\right\}\quad\text{..............................(60)}$$

These are the well-known conditions that

$$u \,.\, dx + v \,.\, dy + w \,.\, dz$$

may be a perfect differential of some function of x, y, z.

Denoting this function by ϕ we have

$$udx + vdy + wdz = d\phi,$$

and therefore

$$u = \frac{\partial\phi}{\partial x},\ v = \frac{\partial\phi}{\partial y},\ w = \frac{\partial\phi}{\partial z}.$$

The function ϕ may be called by analogy the *Displacement-Potential* of Irrotational Strain. It may be any continuous single-valued function of the coördinates, except that (since the origin is supposed fixed) it must not contain any terms of the first degree.

Equations (59) may now be written

$$\left.\begin{array}{c} e = \dfrac{\partial^2\phi}{\partial x^2},\ f = \dfrac{\partial^2\phi}{\partial y^2},\ g = \dfrac{\partial^2\phi}{\partial z^2} \\ s_1 = \dfrac{\partial^2\phi}{\partial y\partial z},\ s_2 = \dfrac{\partial^2\phi}{\partial z\partial x},\ s_3 = \dfrac{\partial^2\phi}{\partial x\partial y} \\ \Delta = \nabla^2\phi \end{array}\right\} \quad\text{.........................(61)}$$

where, as usual, the symbol ∇^2 denotes the operator

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right).$$

The condition that the dilatation may everywhere vanish, or that the strain may consist of distortions (shears) only, without change in the volume of any element, is therefore

$$\nabla^2\phi = 0 \quad\text{....................................(62)}$$

125] **Resultant Displacement.** If we write

$$U^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2$$
$$= u^2 + v^2 + w^2,$$

then U is the resultant displacement of the point $P\,(x, y, z)$. The direction-cosines of this displacement are u/U, v/U, w/U.

But if we describe in the body the system of surfaces whose equations are formed by equating ϕ to different constants, and which are consequently called *Equipotential Surfaces*, the direction-cosines of the normal at P to the equipotential surface passing through P are also u/U, v/U, w/U.

Hence *each point of the body is displaced along the normal to the equipotential surface passing through the point.*

Again, if through P we draw an elementary straight line $d\nu$

normal to the equipotential surface through P, and if the coördinates of its extremity be $x+dx$, $y+dy$, $z+dz$, we have

$$dv = \frac{u}{U} \cdot dx + \frac{v}{U} \cdot dy + \frac{w}{U} \cdot dz;$$

$$\therefore \qquad U dv = u dx + v dy + w dz$$
$$= d\phi.$$

Hence the *amount* of the resultant displacement at P is

$$U = \frac{d\phi}{dv} \qquad\qquad (63)$$

126.] If ϕ is a homogeneous quadratic function of (x, y, z) it is obvious from equations (61) that the strain is homogeneous throughout the body.

The equipotential surfaces for Homogeneous Strain are therefore concentric Quadrics.

By Euler's theorem on homogeneous functions we have in this case

$$2\phi = x^2\frac{\partial^2\phi}{\partial x^2} + y^2\frac{\partial^2\phi}{\partial y^2} + z^2\frac{\partial^2\phi}{\partial z^2} + 2yz\frac{\partial^2\phi}{\partial y\partial z} + 2zx\frac{\partial^2\phi}{\partial z\partial x} + 2xy\frac{\partial^2\phi}{\partial x\partial y}$$
$$= ex^2 + fy^2 + gz^2 + 2s_1yz + 2s_2zx + 2s_3xy.$$

Thus (§ 22) in pure homogeneous strain the equipotential surfaces and elongation quadrics are identical.

It has already been pointed out (§ 84) that in this case the resultant displacement is normal to the elongation quadric, and this agrees with the result of the last Article.

127.] **Lines of Displacement.** Since in every irrotational strain the displacement of each point is normal to the equipotential surface through the point, it follows that, if we draw a system of equipotential surfaces throughout the body, the displacements of all points in the body will take place along a system of curves which cut these surfaces everywhere orthogonally. These curves are called the Lines of Displacement.

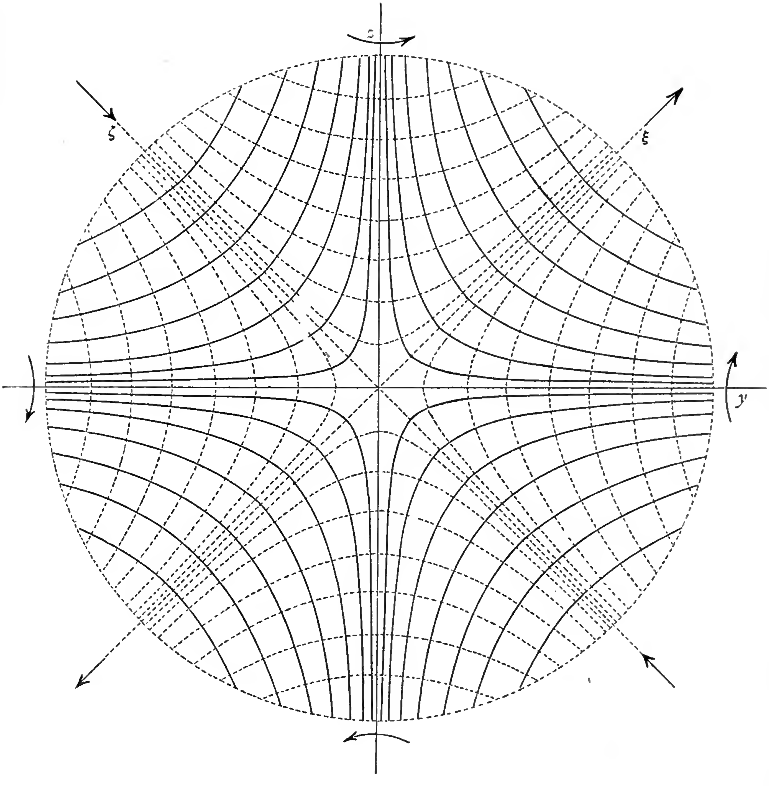
If ds be the element of arc (drawn in the positive direction of the axes) of the displacement-curve through P, we evidently have

$$\frac{1}{u} \cdot \frac{dx}{ds} = \frac{1}{v} \cdot \frac{dy}{ds} = \frac{1}{w} \cdot \frac{dz}{ds},$$

or

$$\frac{dx}{\frac{\partial\phi}{\partial x}} = \frac{dy}{\frac{\partial\phi}{\partial y}} = \frac{dz}{\frac{\partial\phi}{\partial z}}.$$

The function ϕ must therefore always be such that it is possible to draw a system of continuous curves cutting orthogonally the system of continuous surfaces defined by $\phi = constant$.



Equipotential Cylinders and Curves of Displacement in SIMPLE SHEAR.

(*Page 59.*)

128.] As a simple example, take the case of a shear in the plane of yz. This is a strain in two dimensions, and the equipotential surfaces are the rectangular-hyperbolic cylinders

$$yz = \text{constant}.$$

Thus the differential equation of the line of displacement through P is

$$\frac{dx}{0} = \frac{dy}{z} = \frac{dz}{y}.$$

They are therefore the orthogonal rectangular hyperbolas given by

$$\left.\begin{array}{r} x = \text{constant} \\ y^2 \sim z^2 = \text{constant} \end{array}\right\}.$$

See Plate II., in which the dotted lines represent the curves of displacement, and the entire lines the sections of the equipotential cylinders by the plane of the shear.

The directions in which displacement takes place along the curves in the four quadrants are shown by the arrows.

Strain in two Dimensions.

129.] It will be useful to collect here the forms assumed by our various results when one, and one only, of the roots of the discriminating cubic (28) is zero. One of the principal elongations (which we will suppose always ϵ_3) will then vanish, and the displacement of every point in the body (if the strain be homogeneous) or of every point in a given element of the body (if it be heterogeneous) will be parallel to the plane containing ϵ_1 and ϵ_2. The elongation quadrics become cylinders, having this plane for a normal section, and the strain may be said to be wholly in two dimensions.

We shall, as before, use the notation of Homogeneous Strain.

Taking Oz perpendicular to the plane of the strain, the equations of displacement take the form

$$\left.\begin{array}{r} u = ex + (s - \theta)y \\ v = (s + \theta)x + fy \end{array}\right\} \quad\quad (17')$$

or, if the strain be pure,

$$\left.\begin{array}{r} u = ex + sy \\ v = sx + fy \end{array}\right\} \quad\quad (31')$$

The elongation of the line OP lying in the plane of xy, and making an angle ψ with Ox, is given by

$$\epsilon = e\cos^2\psi + f\sin^2\psi + 2s\sin\psi\cos\psi \quad\quad (18')$$

and the angle ψ', into which ψ is altered by the strain, by

$$\left.\begin{aligned}\cos\psi' &= (1-\epsilon+e)\cos\psi + (s-\theta)\sin\psi\\ \sin\psi' &= (s+\theta)\cos\psi + (1-\epsilon+f)\sin\psi\end{aligned}\right\}\quad\text{(19')}$$

The circle $x^2+y^2=1$ becomes the Strain Ellipse

$$(1-2e)x'^2+(1-2f)y'^2-4sx'y'=1\quad\text{(20')}$$

ϵ_1 and ϵ_2 are the (greater and less) roots of the discriminating quadratic

$$\begin{vmatrix} e-\phi, & s \\ s, & f-\phi \end{vmatrix}=0\quad\text{(28')}$$

and the angles made with Ox by the corresponding Principal Axes are given by

$$\left.\begin{aligned}\tan\psi_1 &= \frac{\epsilon_1-e}{s}=\frac{s}{\epsilon_1-f}\\ \tan\psi_2 &= \frac{\epsilon_2-e}{s}=\frac{s}{\epsilon_2-f}\end{aligned}\right\}\quad\text{(29')}$$

ψ_1, ψ_2 being the roots of the equation

$$\tan 2\psi=\frac{2s}{e-f}\quad\text{(64)}$$

The graphic properties of the strain depend upon the Elongation and Compression Conics and the Position Ellipse, which are the normal sections by the plane of the strain of the cylinders into which the respective quadrics degenerate.

If ϵ_1 and ϵ_2 be both positive, we have the elongation ellipse

$$\left.\begin{aligned} & ex^2+fy^2+2sxy=B^2\\ \text{or}\quad & \epsilon_1\xi^2+\epsilon_2\eta^2=B^2\end{aligned}\right\}\quad\text{(22')}$$

If both negative, the compression ellipse

$$\left.\begin{aligned} & ex^2+fy^2+2sxy=-B^2\\ \text{or}\quad & \epsilon_1\xi^2+\epsilon_2\eta^2=-B^2\end{aligned}\right\}\quad\text{(24')}$$

If of opposite signs, the conjugate elongation and compression hyperbolas

$$\left.\begin{aligned} & ex^2+fy^2+2sxy=\pm B^2\\ \text{or}\quad & \epsilon_1\xi^2+\epsilon_2\eta^2=\pm B^2\end{aligned}\right\}.$$

In the latter case, we have two planes of no elongation through Oz, cutting the plane of xy in the lines

$$\left.\begin{aligned} & ex^2+fy^2+2sxy=0\\ \text{or}\quad & \epsilon_1\xi^2+\epsilon_2\eta^2=0\end{aligned}\right\}\quad\text{(25')}$$

which are the asymptotes of the above hyperbolas.

The Position-Ellipse is

$$\left.\begin{array}{r}(1+e)x^2+(1+f)y^2+2sxy=C^2\\ \text{or}\quad (1+\epsilon_1)\xi^2+(1+\epsilon_2)\eta^2=C^2\end{array}\right\}\quad\text{(33')}$$

If then r be any radius vector of an Elongation or Compression Conic, and p the perpendicular from the centre on the tangent at the extremity P of r, the elongation ϵ of OP is given by

$$\epsilon = B^2/r^2 \quad\text{(23')}$$

and the resultant displacement of P will be along the normal at P, and its amount will be B/p.

On the other hand, if P be on the position-ellipse, and r and p have similar meanings, OP will be strained into the position of p, and its new length will be C^2/p.

In other words, the displacement of the extremity of any radius of the elongation conic is perpendicular and proportional to the conjugate radius; while any radius of the position-ellipse is, after the strain, perpendicular and proportional to the conjugate radius.

Since there is no elongation parallel to Oz, the cubical dilatation of the body is equal to the "*areal dilatation*" of any plane area parallel to the plane of the strain. Thus,

$$\Delta = e + f = \epsilon_1 + \epsilon_2 \quad\text{(35')}$$

The conditions that the strain may be an areal dilatation, uniform in every direction, are

$$\left.\begin{array}{r}e=f=\tfrac{1}{2}\Delta\\ s=0,\ \theta=0\end{array}\right\}\quad\text{(40')}$$

The conditions that it may be a simple elongation are

$$\left.\begin{array}{r}ef-s^2=0\\ \theta=0\end{array}\right\}\quad\text{(42')}$$

The conditions that it may be a simple shear are

$$\left.\begin{array}{r}e+f=0\\ \theta=0\end{array}\right\}\quad\text{(51')}$$

If the strain be heterogeneous

$$\left.\begin{array}{r}e=\dfrac{\partial u}{\partial x},\quad f=\dfrac{\partial v}{\partial y}\\[2ex] 2s=\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y},\quad 2\theta=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\end{array}\right\}.$$

If the strain be everywhere irrotational

$$udx + vdy = d\phi,$$

where ϕ is the displacement-potential.

The equipotential curves are given by ϕ=constant, and the curves of displacement are the orthogonal system.

If there be no dilatation anywhere, ϕ satisfies

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0\ldots\ldots\ldots\ldots\ldots\ldots(61')$$

EXAMPLES.

N.B.—*The factor* a *is introduced to denote a small quantity whose square and higher powers may be neglected. The expression*

$$\{e, f, g, s_1, s_2, s_3\}$$

is used to denote the specification of a strain (§§ **111** *et seqq.*)

1. Refer to its principal axes the Elongation Quadric of the strain

$$\{3a, \ -a, \ -a, \ 0, \ 0, \ 2a\},$$

and hence show that it consists of a simple shear of amount $6a$, together with a uniform elongation of amount a perpendicular to the plane of the shear.

2. Show that the strain $\{0, 0, 0, a, a, a\}$ consists of a uniform cubical compression and a uniform linear elongation, each of amount $3a$.

3. Show that the strain $\{a, a, 0, a, a, a\}$ consists of a shear of amount $2a\sqrt{3}$, a linear contraction of amount a perpendicular to its plane, and a uniform cubical dilatation of amount $3a$.

4. Show that the strain $\{a, 0, 0, a, a, a\}$ is equivalent to a uniform cubical dilatation of amount a, together with three shears in orthogonal planes of amounts $2a\sqrt{2}$, $+\frac{4}{3}a$, $-\frac{4}{3}a$; the shears having $O\xi$ and $O\eta$, $O\eta$ and $O\zeta$, $O\zeta$ and $O\xi$ for their respective axes.

5. Prove that the strain $\{\sigma\cos2\theta, \ -\sigma\cos2\theta, \ 0, \ 0, \ \sigma\sin2\theta\}$ is a simple shear in the plane of xy, the axis of elongation making an angle θ with Ox.

6. Hence show that the strain $\{e, f, g, s_1, s_2, s_3\}$ may be resolved into the following components:—a uniform cubical dilatation of amount $(e+f+g)$; a simple shear of amount $\sqrt{s_1^2+\frac{1}{9}(f-g)^2}$ in the plane of yz, the axis of elongation making an angle $\tan^{-1}[3s_1/(f-g)]$ with Oy; a shear of amount $\sqrt{s_2^2+\frac{1}{9}(g-e)^2}$ in the plane of zx, the axis of elongation making

an angle $\tan^{-1}[3s_2/(g-e)]$ with Oz; and a shear of amount $\sqrt{s_3{}^2+\frac{1}{9}(e-f)^2}$ in the plane of xy, the axis of elongation making an angle $\tan^{-1}[3s_3/(e-f)]$ with Ox.

7. Defining the term "areal dilatation" in analogy with linear elongation and cubical dilatation, show that in a homogeneous strain a system of quadrics can be described with the origin as centre such that the areal dilatation of any section varies inversely as the square of the perpendicular radius vector.

8. Prove that all planes in the body suffering a given areal dilatation δ have for their normals the generators of the cone

$$(\delta-f-g)x^2+(\delta-g-e)y^2+(\delta-e-f)z^2+2s_1yz+2s_2zx+2s_3xy=0.$$

9. Prove that in the case of § 119 (*iii.*), the elongation being perpendicular to the plane of the shear, and all three principal elongations being supposed positive, the direction of the elongation must coincide with the greatest axis of the Strain Ellipsoid if $\epsilon_1+\epsilon_3>2\epsilon_2$; and show that in this case

$$\left.\begin{aligned}\Delta&=\tfrac{3}{2}(\epsilon_2+\epsilon_3)\\ \sigma&=\tfrac{1}{2}(\epsilon_2-\epsilon_3)\\ \epsilon&=\tfrac{1}{2}(2\epsilon_1-\epsilon_2-\epsilon_3)\end{aligned}\right\}.$$

10. Show that a simple elongation e parallel to Ox may be replaced by a cubical dilatation e together with two shears, each of amount $\frac{2}{3}e$, having Ox and Oy, Ox and Oz respectively for their axes.

11. What is the nature of the strain represented by the equations of displacement

$$u=-\omega yz;\ v=\omega zx;\ w=0\ ?$$

12. Find the volume and the moments and products of inertia of a sphere of radius R, originally homogeneous, after undergoing the strain represented by

$$\left.\begin{aligned}u&=ax+a'x^2\\ v&=\beta y+\beta' y^2\\ w&=\gamma z+\gamma' z^2\end{aligned}\right\}.$$

13. Prove by combining equations (29) and (61) that if one of the principal axes at each point is normal to the equipotential surface through the point, then either

$$\left.\begin{aligned}\phi&=F(ax+\beta x+\gamma z+\delta)\\ \text{or}\quad \phi&=F(r)\end{aligned}\right\},$$

where $r^2=x^2+y^2+z^2$, a, β, γ, and δ are constants, and F is any function which makes $\partial\phi/\partial x$, $\partial\phi/\partial y$, $\partial\phi/\partial z$ vanish at the origin. What strains do these forms of ϕ represent?

[The equations may be written

$$\frac{1}{2u} \cdot \frac{\partial U^2}{\partial x} = \frac{1}{2v} \cdot \frac{\partial U^2}{\partial y} = \frac{1}{2w} \cdot \frac{\partial U^2}{\partial z} = \lambda \text{ (say)}.$$

Thus $$U \,.\, dU = \lambda \,.\, d\phi.$$

Assume $$A \,.\, dU = \lambda \,.\, d\omega.$$

Then $$U \,.\, d\omega = A d\phi, \text{ where } A \text{ is a constant.}$$

$\therefore$ $$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 = A^2 \,.\, \left(\frac{\partial \phi}{\partial \omega}\right)^2.$$

$\therefore$ when $\frac{\partial \phi}{\partial \omega} = 0$ we also have

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial z} = 0. \quad \text{Thus } \phi = F(\omega).$$

Now $$\frac{\partial \omega}{\partial x} \cdot \frac{d\phi}{d\omega} = \frac{\partial \phi}{\partial x}, \text{ etc.}$$

Squaring and adding

$$\left(\frac{d\phi}{d\omega}\right)^2 \left\{ \left(\frac{\partial \omega}{\partial x}\right)^2 + \left(\frac{\partial \omega}{\partial y}\right)^2 + \left(\frac{\partial \omega}{\partial z}\right)^2 \right\} = U^2$$

$$= A^2 \left(\frac{d\phi}{d\omega}\right)^2.$$

Hence $$\left(\frac{\partial \omega}{\partial x}\right)^2 + \left(\frac{\partial \omega}{\partial y}\right)^2 + \left(\frac{\partial \omega}{\partial z}\right)^2 = A^2.$$

The only real solutions of this are

$$\left.\begin{array}{l} \omega = \alpha x + \beta y + \gamma z + \delta \\ \omega = \alpha r + \delta \end{array}\right\}$$

whence, etc.].

14. Prove that in any strain which consists of a combination of any number of shears (homogeneous or not) in the plane of xy the displacement curves are given by

$$\left.\begin{array}{l} z = constant \\ \chi = constant \end{array}\right\}$$

where ϕ and χ are conjugate functions of x and y (§ 245).

15. Prove by equating the values of λ', μ', ν' given by equations (19) to λ, μ, ν that, in *any* homogeneous strain, there is always *one* and may be *three* straight lines through every point of the body which retain their initial directions.

Show that the elongations in these directions are the roots of the cubic

$$\begin{vmatrix} e - \phi, & s_3 - \theta_3, & s_2 + \theta_2 \\ s_3 + \theta_3, & f - \phi, & s_1 - \theta_1 \\ s_2 - \theta_2, & s_1 + \theta_1, & g - \phi \end{vmatrix} = 0.$$

Hence show that, when all the roots of this cubic are real, these three directions are orthogonal, and $\theta_1 = \theta_2 = \theta_3 = 0$.

16. Show that the integral

$$\int (udx + vdy + wdz)$$

taken round any closed curve in the body is zero if the strain be irrotational, and is otherwise equal to

$$2\iint (\lambda\theta_1 + \mu\theta_2 + \nu\theta_3) dS,$$

where dS is an element of *any* surface drawn within the body and having for its edge the given closed curve; λ, μ, ν being the direction-cosines of the normal to the element.

17. Show from equations (59) of § 123 that the integral

$$\iint(\lambda\theta_1 + \mu\theta_2 + \nu\theta_3)dS,$$

taken over any *closed* surface drawn within the body, is identically zero.

APPENDIX I.

On the Geometry of Strains.

All physical quantities may be broadly classified into two categories called respectively **Scalar** and **Vector**. A scalar quantity involves no conception but that of magnitude, but the characteristic property of all Vectors is that they involve the idea of *direction* as well as that of magnitude.

This broad distinction includes under the head of Vectors several classes of quantities which differ from one another in their degree of definition, as we shall presently explain. They may all be assigned to one of two divisions:—*linear* and *angular* vectors —which we shall discuss separately.

Linear Vectors.

(Displacement, Velocity, Elongation, Force, &c.)

The most perfectly defined linear vector, which may be called a **motor,** involves the specification of *five* characteristic elements.

(*i.*) Its **magnitude**, which it has in common with scalars, and which is expressed by a scalar or numerical factor multiplying its purely vector or directed factor, and denoting its ratio to an arbitrarily chosen unit vector with which it is in all its other properties identical. This factor is called its *Tensor.*

(*ii.*) Its **direction**, or that of a family of parallel straight lines in space, along any one of which it may be supposed to act.

(*iii.*) Its **way** of acting along these lines, which is analytically expressed by an arbitrary convention as to its *algebraical sign*, so that, if a vector acting in one way is considered positive, a vector acting in the directly opposite way is considered negative, the two vectors being otherwise identical.

(*iv.*) Its **position** in space, or the particular line of the family along which it may be supposed to act.

(*v.*) Its **origin**, or the particular point in this line from which it is to be reckoned, or at which it is to be applied.

The following are good examples of motors:—

(1) A given displacement of a given point in a given direction.

(2) A force of given magnitude and in a given direction acting at a given point of a body.

The component displacements, parallel to arbitrary rectangular axes, of each point of a strained body are of course vector quantities, but if the body be left free in space they are highly imperfect vectors; the reason being that such a strain does not specify the absolute displacements of points in the body, but only their relative displacements in given directions.

Consequently we are only given (*ii.*), (*iv.*), and (*v.*), while (*i.*) and (*iii.*) are quite indeterminate.

Vectors of this nature, which can be taken in either way so as to satisfy the specified conditions, are called *Dipolar*.

If, however, we determine in any abritrary way the absolute displacements of any *one* point in the body, it is obvious that we thereby raise the component displacements of *all* points in the body to the rank of perfect motors. The simplest condition to impose is of course that one point in the body shall remain fixed, and since this assumption cannot affect the strain, while the analytical advantages of increased simplicity and definition are so obvious, we shall always avail ourselves of it.

As an analytical example let us take the simple case of a uniform elongation of all lines in the body in the direction Ox.

If e be the amount of the elongation, and $x_1, x_2, x_3 \ldots, x_1', x_2', x_3' \ldots$ the initial and final abscissæ of any number of points in the body, the *only* condition to be satisfied is that the projections $(x_2 - x_1)$, $(x_3 - x_2) \ldots$ upon Ox of the distances between these points are to be increased in the constant ratio $(1+e)$.

We thus obtain a group of equations of the form

$$x_2' - x_1' = (1+e)(x_2 - x_1)$$

or

$$u_2 - u_1 = e(x_2 - x_1).$$

The solution of this group is of course

$$u - ex = \text{constant},$$

or

$$u = ex - C,$$

where the constant C may be of either sign and of any magnitude whatever.

Let x', x'' be the abscissæ of those points of the body which are nearest to and farthest from the plane of yz.

(*i.*) If we take $C < ex'$, u will be positive for every point in the body.

(*ii.*) If we take $ex' < C < ex''$, u will be negative for all points of the body between the planes $x = x'$ and $x = C/e$, and positive for all points between the planes $x = C/e$ and $x = x''$.

(*iii.*) If we take $C > ex''$, u will be negative for every point in the body.

All these solutions obviously satisfy the conditions of the strain.

It is clear that (*ii.*) amounts to regarding a plane in the body as fixed in space—namely, that for which $x = C/e$. If we take this for the plane of yz, $C = 0$, and the equation of displacement becomes

$$u = ex,$$

and u is now a perfectly defined motor.

The simplicity of this solution points to the advantage (much greater of course in more complex strains) of regarding one point in the body as absolutely fixed, and taking that point as the origin of our arbitrary axes of reference.

Angular Vectors.

(*Rotation, Angular Velocity, Couple, &c.*).

As an example of the most perfectly defined class of angular vector, which may be called a **Rotor**, we shall consider a simple rotation about a given axis. Its specification includes

(*i.*) Its **magnitude**.

(*ii.*) The **direction** of its axis, or the direction normal to the system of parallel planes in which the displacements take place which constitute the rotation.

(*iii.*) The **way** of the rotation, which is expressed by an arbitrary convention as to algebraical sign (*see below*).

(*iv.*) The **position** of the axis, or the particular line in the body, drawn in the direction defined by (*ii.*) which remains at rest.

(*v.*) Its **origin**, or the initial position of any plane in the body through the axis of rotation, from which we measure the angular displacement.

An ordinary Couple is a good example of an imperfect Angular Vector, for it may be moved about in any manner in its wn or any parallel plane without altering its effect. In fact we can only specify its magnitude, the direction of its axis, and its way.

The convention as to the *way* of angular vectors (*e.g.*, rotations) is as follows:—

Taking the coördinate axes always in their cyclical order—xy, yz, zx—a rotation about any one of these axes in the direction

from that axis which comes next *towards* that which comes last in the cyclical order is reckoned *positive,* a rotation in the reverse direction being reckoned negative. A positive couple is one which tends to produce a positive rotation, and so on.

In all branches of Physics but one the directions in which we suppose the axes drawn, with reference to their cyclical order, is quite indifferent; but, in order to secure uniformity of notation, it is desirable to adopt in all cases that already employed in Electromagnetism, in which the positive direction of rotation *about* either axis bears the same relation to the positive direction of translation *along it* as does the rotation to the translation in the case of an ordinary "right-handed" screw (Fig. 4). This is also sometimes called a "counter-clockwise" rotation, from the fact that if one of the coördinate axes be drawn *outwards* from the centre of the clock-face, the positive direction of rotation is contrary to that of the hands.

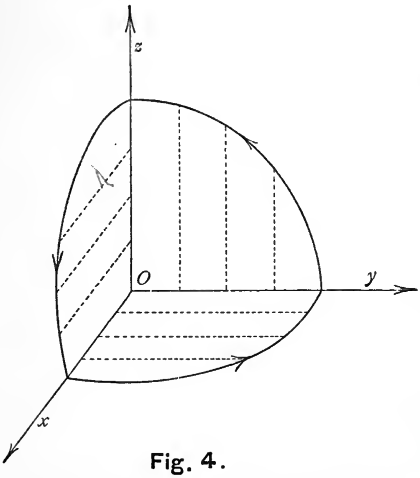


Fig. 4.

Now if a body left free in space is subjected to a strain accompanied by a rotation of given small amount Ω and with its axis in a given direction (λ, μ, ν) it follows from the purely relative character of the displacements specified in the strain that those portions of them due to the rotation will be given (like those previously discussed) by a group of equations of the form

$$\left.\begin{aligned} u_2 - u_1 &= \mu\Omega(z_2 - z_1) - \nu\Omega(y_2 - y_1) \\ v_2 - v_1 &= \nu\Omega(x_2 - x_1) - \lambda\Omega(z_2 - z_1) \\ w_2 - w_1 &= \lambda\Omega(y_2 - y_1) - \mu\Omega(x_2 - x_1) \end{aligned}\right\}$$

the general solution of which is

$$\left.\begin{aligned} u &= \mu\Omega z - \nu\Omega y + A \\ v &= \nu\Omega x - \lambda\Omega z + B \\ w &= \lambda\Omega y - \mu\Omega x + C \end{aligned}\right\}$$

where A, B, C are purely arbitrary constants.

In other words, a small rotation of the body as a whole about any axis may be reduced to a small rotation about any parallel axis, by the superposition of a suitable linear displacement of the body as a whole.

Such a displacement does not affect the strain, and therefore, so far as the conditions of the strain go, the *position* (*iv.*) of the axis of the rotation is completely unspecified, and with it the *amounts* of the component displacements.

Hence, as before, in order to transform the strain-rotation into a complete Rotor, we assume the point of the body coinciding with the coördinate origin to remain at rest; an assumption which clearly amounts to determining that all axes about which the body can rotate must pass through the origin.

APPENDIX II.

On Finite Shears.

A simple finite shear consists of a uniform elongation of all lines in the body parallel to a given axis $O\xi$, accompanied by a contraction in the reciprocal ratio of all lines in a perpendicular direction $O\eta$, lines parallel to $O\zeta$ retaining their initial lengths unaltered.

Thus lines of unit length parallel to $O\xi$, $O\eta$, $O\zeta$ respectively become lines of lengths a, 1, $1/a$, where a is a finite quantity greater than unity which is called the **Ratio of the Shear**.

The displacements parallel to the principal axes are given by

$$\left.\begin{aligned} \xi + U &= a\xi \\ \eta + V &= \eta \\ \zeta + W &= \zeta/a \end{aligned}\right\}$$

Hence if the point (ξ, η, ζ) be displaced to (ξ', η', ζ')

$$\xi'/a\xi = \eta'/\eta = a\zeta'/\zeta = 1.$$

Thus the equation of the **Strain Ellipsoid** is

$$\frac{\xi'^2}{a^2} + \eta'^2 + a^2\zeta'^2 = 1,$$

and its semi-axes are a, 1, a^{-1}. Fig. 5 represents the principal section in the plane of $\zeta\xi$ of the ellipsoid, and of the unit sphere from which it is derived; the mean axis (which retains its unit length) being perpendicular to the plane of the paper.

Since the radius of the sphere and the mean semi-axis of the ellipsoid are both of unit length, the common sections of the two surfaces are the **Circular Sections** of the ellipsoid.

These are the two planes through $O\eta$, whose lines of intersection with the plane of the paper are the common radii $A'OC'$, $B'OD'$.

Since these sections remain great circles of the sphere—and therefore retain their original form and dimensions—it follows from the properties of Homogeneous Strain that the two systems

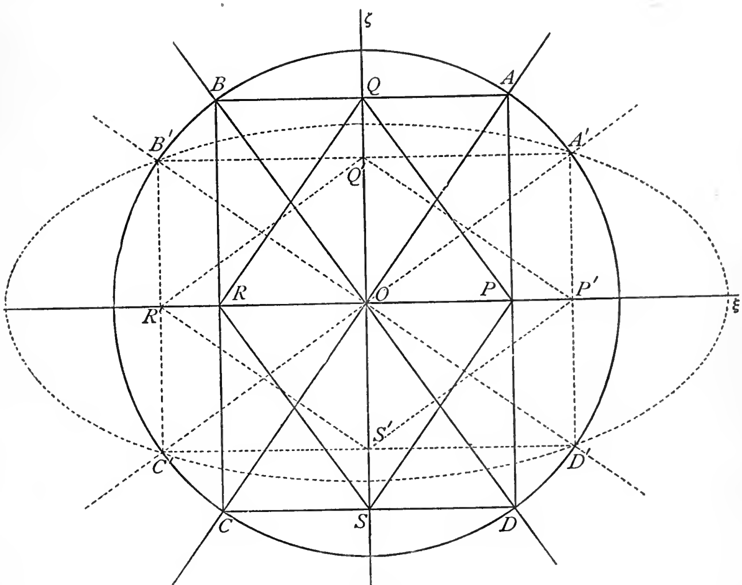


Fig. 5.

of planes in the body parallel to the circular sections of the Strain Ellipsoid are **Planes of no Distortion.**

The equations of these planes in their strained positions are given by

$$\frac{\xi'^2}{a^2} + \eta'^2 + a^2\zeta'^2 = \xi'^2 + \eta'^2 + \zeta^2,$$

or by $$\xi' \pm a\zeta' = 0.$$

Consequently their positions in the unstrained body are given by

$$a\xi \pm \zeta = 0.$$

Let these cut the plane of the section in AOC, BOD. Then

$$AO\xi = DO\xi = \tan^{-1}a,$$
$$A'O\xi = D'O\xi = \tan^{-1}(1/a).$$

The effect of a simple finite shear is therefore to change that angle between the two systems of undistorted planes which is bisected by the plane of $\xi\eta$ from $2\tan^{-1} a$ to $2\tan^{-1}(1/a)$.

The angle AOA' through which any one of these planes is turned is obviously

$$\tan^{-1}\tfrac{1}{2}(a-a^{-1}).$$

It is clear that any rhomboidal prism, such as $PQRS$, bounded by undistorted planes, is strained into an equal and reciprocally-similar rhomboidal prism $P'Q'R'S'$, by a simple interchange of the angles and diagonals of its transverse section.

To represent the effect of a finite shear by a **Finite Shearing Motion** we must therefore take any such rhomboidal prism, and—holding fixed one of its mesial planes BOD—cause all the undistorted planes of the same system to move parallel to it, each through a distance proportional to its perpendicular distance from the fixed plane, until each angle of the rhombus has been changed into the supplementary angle.

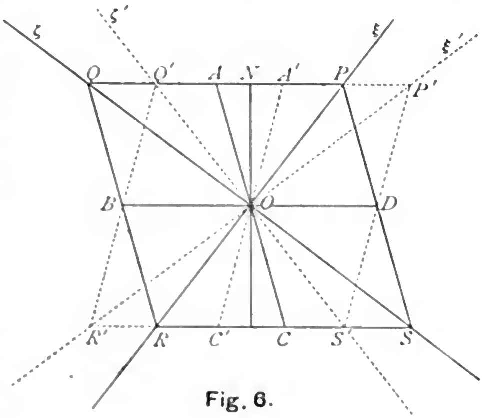


Fig. 6.

Let $PQRS$, $P'Q'R'S'$ (Fig. 6) be the initial and final forms of the rhombus, and let AOC, $A'OC'$ be the initial and final positions of its other mesial plane; ON being perpendicular to PQ.

We have

$$\text{angle } AOB = 2\tan^{-1}(a^{-1}),$$

$$\therefore \quad AON = A'ON = \tan^{-1}\tfrac{1}{2}(a-a^{-1}).$$

Now if A be the **Amount** of the shearing motion (or the ratio of the displacement of any sheared plane to its perpendicular distance from the fixed plane),

$$A = AA'/ON = 2\ AN/ON.$$

Thus $$A = a - a^{-1}.$$

Again,

$$\begin{aligned}\text{angle } POP' &= \text{angle } QOQ' \\ &= \tan^{-1}(a) - \tan^{-1}(a^{-1}) \\ &= \tan^{-1}\tfrac{1}{2}(a-a^{-1}).\end{aligned}$$

Thus, finally, we see that a simple irrotational shear of ratio a may be replaced by a shearing motion of amount $A = a - a^{-1}$, together with a backward rotation of the body as a whole through an angle

$$\tan^{-1}(\tfrac{1}{2}A) = \tan^{-1}\tfrac{1}{2}(a-a^{-1}).$$

To apply these results to the limiting case of an infinitely small shear (§§ 95-98) we have only to write $a=1+s$, so that $a^{-1}=(1+s)^{-1}=1-s$.

Thus, $A=2s$, and the rest follows.

The Analytical Equations of a finite shear in the plane of xy whose axis of elongation $O\xi$ makes an angle θ with Ox may be found as follows.

All lines parallel to $O\xi$ are lengthened in the ratio $a:1$, and all lines parallel to $O\zeta$ are contracted in the ratio $1:a$. Hence the initial and final coördinates of any point are connected by the relations

$$\xi'=a\xi\,;\ \ \zeta'=\zeta/a,$$

or

$$\left.\begin{aligned} y'\sin\theta+x'\cos\theta&=a(y\sin\theta+x\cos\theta)\\ y'\cos\theta-x'\sin\theta&=a^{-1}(y\cos\theta-x\sin\theta)\end{aligned}\right\},$$

or, if

$$\left.\begin{aligned} 2s&=a-a^{-1}\\ 2\sigma&=a+a^{-1}\end{aligned}\right\},$$

$$\left.\begin{aligned} x'&=x\,.\,(\sigma+s\cos 2\theta)+y\,.\,s\sin 2\theta\\ y'&=x\,.\,s\sin 2\theta+y\,.\,(\sigma-s\cos 2\theta)\end{aligned}\right\}.$$

If we put $s=s_3$, $\sigma=1$, $\theta=\frac{1}{4}\pi$, these reduce to equations (vi.) of § 89.

Composition of Finite Shears. It is a curious fact that although a single shear of any magnitude does not cause any rotation of the body as a whole, and although (§ 88) any number of infinitely small irrotational shears produce as their resultant an irrotational strain, yet if two or more shears *of finite amount*, each of them irrotational, be applied in succession, their resultant effect will in general be a rotational strain.

To prove this we will consider two finite shears in the same plane, whose axes do not coincide.

Retaining the notation just explained, let their elements be

$$(a,\ s,\ \sigma,\ \theta),\ (\beta,\ s',\ \sigma',\ \theta').$$

The coördinates $(x',\ y')$ after the first shear of the point initially at $(x,\ y)$ will be given by the above equations, and its final coördinates $x'',\ y'')$ by

$$\left.\begin{aligned} x''&=x'(\sigma'+s'\cos 2\theta')+y's'\sin 2\theta'\\ y''&=x's'\sin 2\theta'+y'(\sigma'-s'\cos 2\theta')\end{aligned}\right\}$$

Hence, finally,

$$\left.\begin{aligned} x''&=x[\sigma\sigma'+ss'\cos 2(\theta'-\theta)+(\sigma s'\cos 2\theta'+\sigma's\cos 2\theta)]\\ &\quad-y[ss'\sin 2(\theta'-\theta)-(\sigma s'\sin 2\theta'+\sigma's\sin 2\theta)]\\ y''&=x[ss'\sin 2(\theta'-\theta)+(\sigma s'\sin 2\theta'+\sigma's\sin 2\theta)]\\ &\quad+y[\sigma\sigma'+ss'\cos 2(\theta'-\theta)-(\sigma s'\cos 2\theta'+\sigma's\cos 2\theta)]\end{aligned}\right\}.$$

To interpret these equations let us suppose the point brought back to (x, y), and displaced to (x''', y''') by an irrotational shear (S, Σ, ϕ), and then *if possible* brought to (x'', y'') by a simple rotation of the body through an angle δ in the positive direction about Oz.

We shall then have

$$\left.\begin{aligned} x''' &= x(\Sigma + S\cos 2\phi) + yS\sin 2\phi \\ y''' &= xS\sin 2\phi + y(\Sigma - S\cos 2\phi) \end{aligned}\right\},$$

$$\left.\begin{aligned} x'' &= x'''\cos\delta - y'''\sin\delta \\ y'' &= x'''\sin\delta + y'''\cos\delta \end{aligned}\right\};$$

And, finally,

$$\left.\begin{aligned} x'' &= x[\Sigma\cos\delta + S\cos(2\phi+\delta)] - y[\Sigma\sin\delta - S\sin(2\phi+\delta)] \\ y'' &= x[\Sigma\sin\delta + S\sin(2\phi+\delta)] + y[\Sigma\cos\delta - S\cos(2\phi+\delta)] \end{aligned}\right\}.$$

In order that these two values for (x'', y'') may be *identical* for all values of x and y we must have

$$\left.\begin{aligned} \Sigma\cos\delta &= \sigma\sigma' + ss'\cos 2(\theta' - \theta) \\ \Sigma\sin\delta &= ss'\sin 2(\theta' - \theta) \\ S\cos(2\phi+\delta) &= \sigma s'\cos 2\theta' + \sigma' s\cos 2\theta \\ S\sin(2\phi+\delta) &= \sigma s'\sin 2\theta' + \sigma' s\sin 2\theta \end{aligned}\right\}.$$

Squaring and adding the first two of these equations

$$\Sigma^2 = \sigma^2\sigma'^2 + s^2s'^2 + 2\sigma\sigma' ss'\cos 2(\theta' - \theta).$$

Squaring and adding the last two

$$S^2 = \sigma^2 s'^2 + \sigma'^2 s^2 + 2\sigma\sigma' ss'\cos 2(\theta' - \theta).$$

Thus
$$\begin{aligned} \Sigma^2 - S^2 &= (\sigma^2 - s^2)(\sigma'^2 - s'^2) \\ &= 1. \end{aligned}$$

Hence these assumed identities give compatible values for S and Σ.

It follows that two finite shears in the same plane, whose axes do not coincide, are together equivalent to a finite shear in the same plane and a finite rotation about an axis perpendicular to it.

The same property can easily be shown geometrically in the case where the two shears have one system of undistorted planes in common.

Let AB, CD be any two planes of this system, and let AB be held fixed. Let OP_1, OP_2 be the elongation-axes of the two shears, and let ON be perpendicular to AB.

The first shear will bring P_1 to P_1' where $P_1P_1' = 2s \,.\, ON$, and angle $P_1OP_1' = \tan^{-1}s$.

The second shear will bring P_2 to P_2' and P_1' to P_1'', where $P_2P_2' = P_1'P_1'' = 2s' \,.\, ON$, and angle $P_2OP_2' = \tan^{-1}s'$.

The resultant of the two *irrotational* shears will therefore be equivalent to a shearing motion of amount $2(s+s')$ towards the

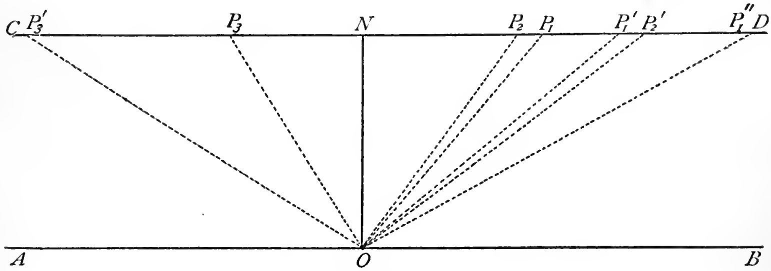


Fig. 7.

right hand, together with a counter-clockwise rotation through an angle $\tan^{-1}s + \tan^{-1}s'$.

$$= \tan^{-1}\frac{s+s'}{1-ss'}.$$

Now if OP_3 be the elongation-axis of a single shear which will restore P_1'' to P_1, the same shear will bring P_3 to P_3' where $P_3P_3' = P_1P_1'' = 2(s+s') \,.\, ON$, and angle $P_3OP_3' = \tan^{-1}(s+s')$. To make this an *irrotational* shear we must therefore give the body a clockwise rotation through an angle $\tan^{-1}(s+s')$.

Hence we see finally that the resultant of the two *irrotational* shears of amounts $2s$ and $2s'$ is compounded of a single *irrotational* shear of amount $2(s+s')$ together with a counter-clockwise rotation through an angle

$$P_1OP_1' + P_2OP_2' - P_3OP_3'$$
$$= \tan^{-1}s + \tan^{-1}s' - \tan^{-1}(s+s')$$
$$\therefore \qquad \delta = \tan^{-1}\left[\frac{ss'(s+s')}{1+ss'+s^2+s'^2}\right].$$

CHAPTER III.

ANALYSIS OF STRESSES.

130.] We originally defined Stress (§§ 28-30) as the elastic force called into play between the molecules of the body to resist the Strain or relative displacement of the molecules produced by the application of external forces. We have now to substitute a new definition of Stress (§ 46) adapted to our conception of continuous matter.

It has already been stated (§ 3) that there is reason to believe that the forces exerted upon one another by the molecules are only sensible when the distances between them are exceedingly minute. A body strained by external forces must therefore be supposed held in equilibrium by stresses between contiguous sets of molecules, the force being passed on, as it were, in the form of stress, from each layer of molecules to the next following.

For example, an elastic bar of uniform section, stretched by forces in the direction of its length, uniformly distributed over its ends, may be regarded as made up of extremely thin layers of molecules in planes perpendicular to its length. The tension is then passed on in the form of stress from layer to layer along the bar, so that by this means the equal forces applied at the two ends are ultimately placed in opposition to one another.

131.] **Definition of Stress.** Thus, returning to our ideal continuous matter, we are led to conceive of Stress as *the mutual action exerted across any surface drawn in the body by the two layers of matter, of elementary thickness, immediately separated by it.*

This action is reckoned as **positive** when it is of the nature of a tension, and **negative** when a thrust.

The **intensity** of the stress across any surface is measured, when uniform, by the tension *per unit area* exerted upon one another by the two portions of matter immediately in contact with it on either side.

If this action varies from point to point of the surface, the

intensity of the stress at any point P is measured by the tension which would be exerted across a unit area described about P in the surface, if the stress had at every point of that area the same value as at P. In other words, it is in all cases measured by the ratio which the tension across any small area described in the surface about P bears to that area, in the limit when the latter is indefinitely diminished.

In accordance with the usage of Hydrostatics, we shall reserve the term **Stress** for intensity of stress (force per unit area), and employ **Total Stress** to denote the algebraical sum of the tensions across all portions of a given area.

A positive stress (tension per unit area) is called a **Traction**: a negative stress (thrust per unit area) is called a **Pressure**.

132.] **Normal and Tangential Components.** The stress across a surface may at each point be normal, tangential, or oblique; and since in the latter case the stress (being merely a force per unit area) can always be resolved into a normal and two orthogonal tangential components, we need only consider the former two.

A positive **normal stress** across a surface is then a normal traction between the portions of matter separated by it. The function of such a stress is obviously to resist normal separation of these portions, or, in other words, to resist **elongation** of the neighbouring portion of the body in the direction of the normal.

Similarly, a negative normal stress or pressure tends to resist **contraction**, or negative elongation, in its own direction.

These are also sometimes called *longitudinal* stresses.

A **tangential stress,** or the component in the tangent plane of the stress across a surface at any point, clearly resists any tendency of the matter on one side of the surface at that point to *slide* relatively to the matter on the other side, in the direction of the tangent plane. The function of the tangential stress is therefore to resist **shearing motion,** and for this reason it is very often called *Shearing Stress.*

133.] **Total Stress.** Stress being a purely mutual reaction between two portions of matter (compare §§ 28-30), it follows that the stress exerted by any portion A of the body on a contiguous portion B, across the surface which separates them, is precisely equal and opposite to that exerted by B on A.

The sum of the two is therefore always identically zero, and similarly, if we suppose the given portion A divided by any number of surfaces drawn within it into smaller portions, the mutual stresses between these must have an identically null action upon A taken as a whole.

The *Total Stress* exerted on or by any given portion of the body is therefore simply the total action exerted *across its*

bounding surface, between its outer layer and the matter immediately in contact with it.

The same result of course holds for the entire body, so that the Total Stress on the body is simply the total action exerted across its bounding surface by matter in contact with that surface, whether homogeneous with the body or not.

Stresses applied by external agents at the bounding surface of a body are called **Surface Tractions**; they are measured, like all other stresses, by the force applied per unit area, and may be positive (*tractions*) or negative (*pressures*); and either normal, tangential, or oblique at each point.

For instance, the stress on a solid body immersed in a quiescent fluid at uniform hydrostatic pressure p is a uniform normal Surface Traction $(-p)$ per unit area: and the Total Stress on the body is $(-pS)$ where S is the area of its surface.

134.] **The two Aspects of Stress.** We have hitherto regarded Stress simply as offering resistance to Strain: it is obvious however, from its reciprocal character, that it may also be regarded from another point of view—namely, as producing and maintaining strain.

This simply amounts to stating that the stress exerted by the portion A of a body on the contiguous portion B may be considered with reference to its effect on A or to its effect on B. In the former aspect it resists further strain of A, and tends to restore A to its natural state; while in the latter it tends to increase the strain of B and to prevent it from returning to its natural state. Similarly the equal and opposite stress exerted on A by B tends to increase the strain of A, and diminish that of B.

This will become quite clear if we consider a simple example; for instance, the uniform bar longitudinally stretched of § 130. Consider three consecutive layers of matter, of elementary thickness, bounded by normal sections: call them A, B, C. The stress-action across the plane surfaces separating B from A and C is a mutual tension, while the strain consists of an increase in the natural thickness of each layer, due to the uniform elongation of the bar. Now the tensions exerted *by* B on A and C clearly *tend* to bring them closer together, which can only be done by diminishing the thickness of B; this action therefore tends to *diminish* the strain of B itself. On the other hand, the tensions exerted *on* B by A and C in the two opposite directions tend to increase its thickness, and consequently to *increase* the strain of B.

135.] **Interpretation of this Distinction.** These two functions of Stress correspond to the two points of view from which we may approach the subject.

If our object be, as in Chapter I., to investigate theoretically the physical effects of Strain, especially with reference to the increase of energy of the strained body, our most obvious method is to imagine a given state of strain produced, to calculate the stresses called into play to resist it, and hence the work required to be done by external force in order to overcome these resistances to any required extent. Our attention is fixed upon the fact that stresses are only aroused by departure from the natural state, and hence Strain and Stress always appear to us in the relation of cause and effect. This is the *physical* point of view.

In practice, however, when we deal with actual bodies, our only method of producing Strain is by the application of external forces, and principally of Surface Tractions or Pressures, which (§ 133) are simply *boundary Stresses.* Engineers in particular, who are chiefly concerned with the capacity of materials for supporting shocks or continuous burdens without permanent set or rupture, necessarily obtain all their working data by this experimental method, the weight of the load being continually increased until the limit of resistance is reached.

The object of our theory being to afford a guide for practical work, we naturally adopt the same point of view: we therefore regard the applied forces and Surface Tractions (which are under our control) as the subjects of observation, and we require to be able to calculate the system of stresses which must exist throughout the body to balance these opposing forces, and hence to deduce the strain produced by them.

This may be described as looking at all the phenomena of Strain from an *outside* point of view. Each body, or portion of a body, is regarded, not as an *agent* opposing strain of its own substance by the exertion of stress, but as passively yielding to the stresses exerted *on it* from without. This is the *mechanical* point of view.

We shall therefore make a distinction between the **Stress on the portion** A, being the action exerted on it by the surrounding matter which together with the applied forces on A produces and maintains the state of strain, and the equal and opposite **Resistance to Stress offered by** A, which balances the stress so long as equilibrium is maintained.

136.] **Applied Forces.** Besides Surface Tractions or Pressures bodies may (§ 4) be strained by forces—such as gravity—which act directly on every portion of the matter of which it is composed.

These are variously known as *Impressed Forces, Applied Forces,* or *Bodily Forces,* to distinguish them from Surface Tractions. Their **intensity** at any point of the body is measured by the *force per unit mass* on an indefinitely small portion of

the body having the given point for centre; and, when not constant or zero throughout the body, they are assumed, as in nature, to vary *continuously* from point to point, and to be everywhere *finite*.

137.] **Continuity of Stress.** In a body under continuous and finite (or zero) applied forces, the components of the stress across a small plane area drawn in a given direction through various points of the body must also vary continuously from point to point (unless they are constant).

For if we consider a small plate of matter in the interior of a body in equilibrium under applied forces of finite intensity, bounded by parallel plane faces separated by an indefinitely small distance δ, the components of the applied force on the plate will be ultimately of the same dimensions as δ, and so therefore must the differences between the equilibrating stress components across its opposite faces.

For an infinitely small change of position, therefore, of a small plane area drawn in a given direction, the components of the stress across it must vary by infinitely small quantities of the same dimensions—*i.e.*, they must be continuous functions of the position of the area. It also follows that the stresses across two opposite parallel faces of an element of the body must both be tractions or both pressures, unless the stress is zero across a parallel plane within the element.

General Equations of Equilibrium.

138.] **Equilibrium of an elementary rectangular parallelepiped.** Let P be any point in the substance of the body, whose coördinates referred to arbitrary rectangular axes are (x, y, z). Through P draw Px', Py', Pz' parallel to these axes.

Describe the elementary parallelepiped $EFGHJKLM$, having its centre at P, and its edges, of lengths dx, dy, dz, respectively, parallel to Px', Py', Pz'.

Let the planes of $y'z'$, $z'x'$, $x'y'$ cut the faces of the parallelepiped in the rectangles $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$ respectively (Fig. 8).

Since the volume of the parallelepiped and the areas of its faces are elementary, its density and the intensity of the applied force upon it (if any) may be supposed to have throughout its volume their actual values at its centre P, and the components of the applied force may be supposed to act at P: similarly the intensity of the stress across each face may be supposed uniform all over it, and the total stress across each face may be replaced by a single force acting at its centre.

139.] Let us consider first the stress across the small plane area $A_1B_1C_1D_1$, drawn through P perpendicular to the axis of x. Let us assume that it is a traction—which we must in general suppose oblique—of intensity R_1, sensibly constant over the area.

The total stress exerted across the area between the two portions into which it divides the parallelepiped may then be taken to be $R_1 . dy\,dz$ which may be regarded as a single force acting at its centre P, and of the nature of a tension, so that the force due to stress on that portion lying on the *positive* side of the area acts in the *negative* direction of the axis, and *vice versâ*.

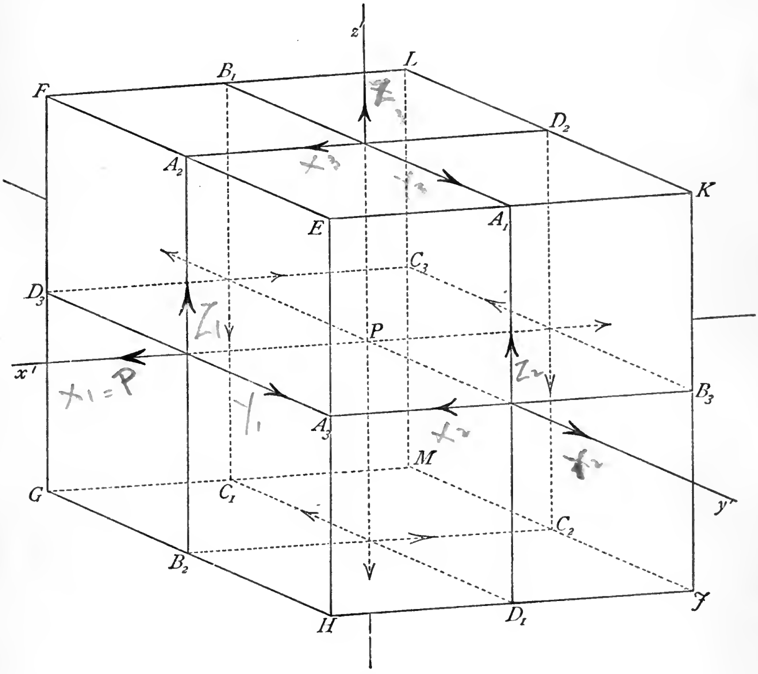


Fig. 8.

Let the components of R_1 along Px', Py', Pz' be X_1, Y_1, Z_1 respectively, and let us assume that they are all of the same sign as R_1. Then the component forces due to stress exerted by the matter on the positive side of the area on that on the negative side will be $X_1 . dydz$, $Y_1 . dydz$, $Z_1 . dydz$ all acting in the *positive* directions of the axes; while the matter on the negative side exerts upon that on the positive side exactly equal component forces in the *negative* directions of the axes.

Now, by § 137, these stress-components X_1, Y_1, Z_1 vary continuously (if not constant) for different small plane areas drawn

in the body parallel to $A_1B_1C_1D_1$—that is, they are continuous functions of x, y, z.

Hence, since the perpendicular distance of $A_1B_1C_1D_1$ from either of the parallel faces—*EFGH* and *JKLM*—of the element is $\frac{1}{2}dx$, it follows that the total stress exerted on the element across the face *EFGH* by the matter on the positive side of it may be represented by a force acting at the middle point of the face, whose components are

$$\left.\begin{array}{l} \left(X_1 + \frac{1}{2}dx \,.\, \dfrac{\partial X_1}{\partial x}\right)dydz \\ \left(Y_1 + \frac{1}{2}dx \,.\, \dfrac{\partial Y_1}{\partial x}\right)dydz \\ \left(Z_1 + \frac{1}{2}dx \,.\, \dfrac{\partial Z_1}{\partial x}\right)dydz \end{array}\right\}$$

all acting in the *positive* directions of the axes (§ 137).

Similarly the components of the force which may be supposed to act on the element at the centre of the face *JKLM*, due to stress exerted by matter on the negative side, are

$$\left.\begin{array}{l} \left(X_1 - \frac{1}{2}dx \,.\, \dfrac{\partial X_1}{\partial x}\right)dydz \\ \left(Y_1 - \frac{1}{2}dx \,.\, \dfrac{\partial Y_1}{\partial x}\right)dydz \\ \left(Z_1 - \frac{1}{2}dx \,.\, \dfrac{\partial Z_1}{\partial x}\right)dydz \end{array}\right\}$$

all acting in the *negative* directions of the axes (§ 137). The arrowheads in Fig. 8 denote the directions of the component forces at the centre of each face.

These force-components on the two opposite faces perpendicular to Px' or Ox together amount to component forces

$$\left.\begin{array}{l} \dfrac{\partial X_1}{\partial x} \,.\, dxdydz \\ \dfrac{\partial Y_1}{\partial x} \,.\, dxdydz \\ \dfrac{\partial Z_1}{\partial x} \,.\, dxdydz \end{array}\right\}$$

on the element in the *positive* directions of the axes, and component couples

$$Z_1 \,.\, dxdydz$$

in the *negative* direction about Py', and

$$Y_1 \,.\, dxdydz$$

in the *positive* direction about Pz'.

140.] Similarly, if the components of the stress across the small plane area $A_2B_2C_2D_2$, drawn through P perpendicular to the axis of y, be X_2, Y_2, Z_2, the total stresses across the pair of opposite faces $EHJK$ and $FGML$ together amount to **component forces**

$$\left.\begin{array}{l}\frac{\partial X_2}{\partial y} \,.\, dxdydz \\ \frac{\partial Y_2}{\partial y} \,.\, dxdydz \\ \frac{\partial Z_2}{\partial y} \,.\, dxdydz\end{array}\right\}$$

in the positive directions of the axes, and **component couples**

$$X_2 \,.\, dxdydz$$

in the *negative* direction about Pz', and

$$Z_2 \,.\, dxdydz$$

in the *positive* direction about Px'.

Lastly, if the components of the stress across the small plane area $A_3B_3C_3D_3$, drawn through P perpendicular to Oz or Px', be X_3, Y_3, Z_3, the total stresses across the pair of opposite faces $EKLF$ and $HJMG$ together amount to the **component forces**

$$\left.\begin{array}{l}\frac{\partial X_3}{\partial z} \,.\, dxdydz \\ \frac{\partial Y_3}{\partial z} \,.\, dxdydz \\ \frac{\partial Z_3}{\partial z} \,.\, dxdydz\end{array}\right\}$$

in the positive directions of the axes and the **component couples**

$$Y_3 \,.\, dxdydz$$

in the *negative* direction about Px', and

$$X_3 \,.\, dxdydz$$

in the *positive* direction about Py'.

141.] **Conditions for Equilibrium of the Element.** It is sufficiently obvious that when the body is in equilibrium in any given state of strain, any portion of it may be supposed to become *rigid* in that state [*compare* § 30 (*i.*)] without affecting its own equilibrium, or that of any other portion of the body.

Thus the conditions for equilibrium of the element under consideration must be precisely the same as if it were a rigid body at rest under the actual stresses and applied forces.

Now if ρ be the density of the body at P, and X, Y, Z the intensities at P of the component applied forces per unit mass, it follows from § 138 that the components of the applied force on the element may be taken to be

$$\left.\begin{array}{l}\rho X \,.\, dxdydz\\ \rho Y \,.\, dxdydz\\ \rho Z \,.\, dxdydz\end{array}\right\}$$

and that they may be supposed to act at its centre P.

Collecting the results of the last three Articles we see that the element is subject to component forces

$$\left.\begin{array}{l}\left[\dfrac{\partial X_1}{\partial x}+\dfrac{\partial X_2}{\partial y}+\dfrac{\partial X_3}{\partial z}+\rho X\right]dxdydz\\ \left[\dfrac{\partial Y_1}{\partial x}+\dfrac{\partial Y_2}{\partial y}+\dfrac{\partial Y_3}{\partial z}+\rho Y\right]dxdydz\\ \left[\dfrac{\partial Z_1}{\partial x}+\dfrac{\partial Z_2}{\partial y}+\dfrac{\partial Z_3}{\partial z}+\rho Z\right]dxdydz\end{array}\right\}$$

parallel to the coördinate axes, and to component couples

$$\left.\begin{array}{l}[Z_2-Y_3]dxdydz\\ [X_3-Z_1]dxdydz\\ [Y_1-X_2]dxdydz\end{array}\right\}$$

about these axes, respectively.

The conditions of equilibrium of the element are therefore expressed by the six equations

$$\left.\begin{array}{l}\dfrac{\partial X_1}{\partial x}+\dfrac{\partial X_2}{\partial y}+\dfrac{\partial X_3}{\partial z}+\rho X=0\\ \dfrac{\partial Y_1}{\partial x}+\dfrac{\partial Y_2}{\partial y}+\dfrac{\partial Y_3}{\partial z}+\rho Y=0\\ \dfrac{\partial Z_1}{\partial x}+\dfrac{\partial Z_2}{\partial y}+\dfrac{\partial Z_3}{\partial z}+\rho Z=0\end{array}\right\}\quad\ldots\ldots\ldots\ldots(1)$$

$$\left.\begin{array}{l}Z_2-Y_3=0\\ X_3-Z_1=0\\ Y_1-X_2=0\end{array}\right\}\quad\ldots\ldots\ldots\ldots(2)$$

142.] **Simplification of Notation.** Equations (2) will be satisfied, and our analysis much simplified, if we adopt the new notation formed by writing

$$X_1=P,\ \ Y_2=Q,\ \ Z_3=R,$$
$$Z_2=Y_3=S,\ \ X_3=Z_1=T,\ \ Y_1=X_2=U.$$

The general equations (1) of equilibrium then become

$$\left.\begin{aligned}\frac{\partial P}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial T}{\partial z}+\rho X=0\\ \frac{\partial U}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial S}{\partial z}+\rho Y=0\\ \frac{\partial T}{\partial x}+\frac{\partial S}{\partial y}+\frac{\partial R}{\partial z}+\rho Z=0\end{aligned}\right\}\quad\text{.....................}\quad(3)$$

where X, Y, Z are the components of the applied force per unit mass at (x, y, z), ρ is the density at the same point, and the other symbols are best explained by the following schedule:—

The Symbol	denotes the Stress-Component Parallel to Axis of	across a small Plane Area drawn through (x, y, z) Perpendicular to Axis of
P	x	x
Q	y	y
R	z	z
S	$\begin{cases} y \\ z \end{cases}$	$\left.\begin{matrix} z \\ y \end{matrix}\right\}$
T	$\begin{cases} z \\ x \end{cases}$	$\left.\begin{matrix} x \\ z \end{matrix}\right\}$
U	$\begin{cases} x \\ y \end{cases}$	$\left.\begin{matrix} y \\ x \end{matrix}\right\}$

143.] **Equations of Motion.** If the body, instead of being in equilibrium in a given state of strain, be *in process of straining*—*i.e.*, if any relative motion of its parts is taking place, the component forces of § 141, instead of vanishing, must be equal to the components of the "effective" force on the element, which, if $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ be the component accelerations of P, are

$$\left.\begin{aligned}\rho\ddot{x}\,.\,dxdydz\\ \rho\ddot{y}\,.\,dxdydz\\ \rho\ddot{z}\,.\,dxdydz\end{aligned}\right\}.$$

Since the effective *couples* involve the Moments of Inertia in place of the mass of the element, they are always indefinitely small in comparison with the effective *forces*.

Hence equations (2) are still very approximately true, and the equations of motion are

$$\left.\begin{aligned} \frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z} + \rho(X - \ddot{x}) = 0 \\ \frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial S}{\partial z} + \rho(Y - \ddot{y}) = 0 \\ \frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial R}{\partial z} + \rho(Z - \ddot{z}) = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

In these equations, since u, v, w are the variable portions of the coördinates of any point, we may obviously write $\ddot{u}$, $\ddot{v}$, $\ddot{w}$ instead of $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, whenever the former will be preferable.

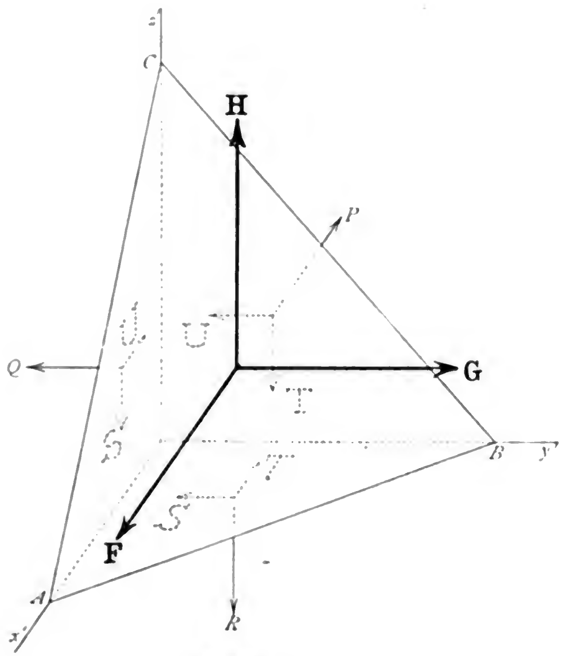


Fig. 9.

144.] **Resolution and Composition of Stresses.** The six quantities, P, Q, R, S, T, U are the normal and tangential components of the stresses across the three small orthogonal plane areas drawn through any point P (x, y, z) of the body perpendicular to Ox, Oy, Oz respectively. The fact that these six quantities are the only stresses involved in the equations of equilibrium and of motion suggests that we may be able to adopt

them as our standard system of stress-components, and to express in terms of them the stress across a small plane area drawn through P in any direction whatever.

From P draw, as in § 138, Px', Py', Pz' parallel to Ox, Oy, Oz, and cut off from the body an elementary right-angled tetrahedron $PABC$, having for its base any oblique plane which will cut Px', Py', Pz' in points A, B, C, such that the edges PA, PB, PC are all positive in direction.

Let λ, μ, ν be the direction-cosines of the normal to the base, directed *outwards*, or away from P; then λ, μ, ν are all positive quantities.

Let Δ be the area of the base ABC, and p its perpendicular distance from P. Then $\frac{1}{6}p\Delta$ is the volume of the tetrahedron, and $\lambda\Delta$, $\mu\Delta$, $\nu\Delta$ are the areas of the faces PBC, PCA, PAB.

Let F, G, H be the components of the stress across the base, in the positive directions of the axes. Since the other three faces are all turned towards the negative directions of the axes, the components of the stress across them must also be taken in the negative directions (§ 139); these components are respectively:— on the face PBC, P, U, T; on the face PCA, U, Q, S; on the face PAB, T, S, R.

If X, Y, Z, ρ denote the same quantities as in § 140, the component forces on the tetrahedron will be

$$\left.\begin{array}{l} F\,.\,\Delta + \rho X\,.\,\frac{1}{6}p\Delta - P\,.\,\lambda\Delta - U\,.\,\mu\Delta - T\,.\,\nu\Delta \\ G\,.\,\Delta + \rho Y\,.\,\frac{1}{6}p\Delta - U\,.\,\lambda\Delta - Q\,.\,\mu\Delta - S\,.\,\nu\Delta \\ H\,.\,\Delta + \rho Z\,.\,\frac{1}{6}p\Delta - T\,.\,\lambda\Delta - S\,.\,\mu\Delta - R\,.\,\nu\Delta \end{array}\right\}$$

and the conditions of equilibrium

$$\left.\begin{array}{r} F + \frac{1}{6}p\,.\,\rho X = P\lambda + U\mu + T\nu \\ G + \frac{1}{6}p\,.\,\rho Y = U\lambda + Q\mu + S\nu \\ H + \frac{1}{6}p\,.\,\rho Z = T\lambda + S\mu + R\nu \end{array}\right\}$$

Similarly, if the body be in process of straining (§ 143) the equations of motion are

$$\left.\begin{array}{r} F + \frac{1}{6}p\rho(X - \ddot{x}) = P\lambda + U\mu + T\nu \\ G + \frac{1}{6}p\rho(Y - \ddot{y}) = U\lambda + Q\mu + S\nu \\ H + \frac{1}{6}p\rho(Z - \ddot{z}) = T\lambda + S\mu + R\nu \end{array}\right\}.$$

These equations must hold, however much the size of the element may be reduced, by causing the plane ABC to move up parallel to itself towards P. Since then they hold up to the limit, we may assume that they are also true at the limit, when p vanishes, and the plane ABC passes through P.

We have then, whether the point P be at rest or in motion, the three relations

$$\left.\begin{aligned} F &= P\lambda + U\mu + T\nu \\ G &= U\lambda + Q\mu + S\nu \\ H &= T\lambda + S\mu + R\nu \end{aligned}\right\} \quad\text{.........................(5)}$$

between the components F, G, H of the traction exerted across a small plane area drawn through (x, y, z) in the direction (λ, μ, ν), and the six orthogonal components P, Q, R, S, T, U.

145.] **Boundary Conditions.** Similarly, if λ, μ, ν be taken to denote the direction-cosines of the outward drawn normal at any point (x, y, z) of the bounding surface of the body, we may describe an elementary tetrahedron whose inner faces are parallel to the coördinate planes, while its outer face ABC is formed by a triangular element of the bounding surface about the point (x, y, z).

If F, G, H be now taken to be the components of the Surface Traction at the point, the conditions of equilibrium of the tetrahedron must be the same as those just investigated, and (proceeding to the limit in which the vertex of the tetrahedron moves up to the surface) equations (5) will represent the relations which must exist between the components of Surface Traction and the orthogonal Stress-components at each point of the surface.

The general problem in the Mathematical Theory of Elasticity (see § 135) is to find solutions of equations (3) or (4), for the stress-components throughout the body, which will also satisfy equations (5) at every point of the bounding surface.

The solution will not be complete until we know the relations between Strain and Stress, so that we can find the alteration of form and volume of any element of the body. These relations will be investigated in the next chapter.

146.] **Equilibrium of the body as a whole.** It may be observed that the solution of equations (3) and (5) converse to that just proposed—namely, given the distribution of Stress throughout the body, to find the distribution of Applied Force and Surface Traction required to maintain it—is always obtainable. For when P, Q, R, S, T, U, ρ are known as functions of x, y, z, these equations give *explicitly* the appropriate values of X, Y, Z, F, G, H.

Now we have shown (*see* §§ 133, 134, *and compare* § 29) that the applied Forces and Surface Tractions (considered together in Chapter I., under the head of "External Forces") are the only forces which can be considered as acting upon the body as a whole, and it follows from the considerations of § 141 that, when the body is in equilibrium in a given state of strain, these two systems of forces must be so connected as to satisfy the ordinary conditions of equilibrium of a rigid body.

We ought then to be able to show that the values of these forces given by (3) and (5) satisfy the analytical conditions

$$\left.\begin{aligned}\iiint\rho X dxdydz + \iint F dS = 0\\ \iiint\rho Y dxdydz + \iint G dS = 0\\ \iiint\rho Z dxdydz + \iint H dS = 0\end{aligned}\right\} \qquad (6)$$

$$\left.\begin{aligned}\iiint\rho(yZ - zY)dxdydz + \iint(yH - zG)dS = 0\\ \iiint\rho(zX - xZ)dxdydz + \iint(zF - xH)dS = 0\\ \iiint\rho(xY - yX)dxdydz + \iint(xG - yF)dS = 0\end{aligned}\right\} \qquad (7)$$

obtained by equating to zero the component forces and couples acting on the body as a whole; the triple integrals being taken throughout the volume of the strained body, and the double integrals over the whole of its bounding surface.

Now we have by equations (3)

$$\rho X + \frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z} = 0,$$

$$\therefore \qquad \iiint\rho X dxdydz = -\iiint\left(\frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z}\right)dxdydz;$$

or, integrating by parts,

$$= -\iint[P]dydz - \iint[U]dzdx - \iint[T]dxdy,$$

where the square brackets [ ] denote that the enclosed term is to be taken within proper limits.

Hence if λ, μ, ν be the direction-cosines of the outward normal to the element dS

$$\iiint\rho X dxdydz = -\iint(P\lambda + U\mu + T\nu)dS,$$

and therefore by equations (5),

$$\iiint\rho X dxdydz + \iint F dS = 0.$$

In the same manner it may be shown that the second and third of equations (6) are satisfied.

Again, by (3),

$$\begin{aligned}&\iiint\rho(yZ - zY)dxdydz\\ &\quad = \iiint\left\{z\left(\frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial S}{\partial z}\right) - y\left(\frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial R}{\partial z}\right)\right\}dxdydz\\ &\quad = \iint[zU - yT]dydz + \iint[zQ - yS]dzdx + \iint[zS - yR]dxdy\\ &\quad = \iint\{\lambda(zU - yT) + \mu(zQ - yS) + \nu(zS - yR)\}dS\\ &\quad = \iint\{z(\lambda U + \mu Q + \nu S) - y(\lambda T + \mu S + \nu R)\}dS.\end{aligned}$$

Hence, by equations (5),

$$\iiint \rho(yZ - zY)\,dx\,dy\,dz + \iint (yH - zG)\,dS = 0.$$

Similarly we can prove the remaining two of equations (7).

Thus the values of the components of Applied Force and Surface Traction given by equations (3) and (5) satisfy identically the conditions of equilibrium of the body as a whole.

This result verifies the statement of § 130 that force is passed on through the body from layer to layer by appropriate stresses, so that the external forces are ultimately brought into opposition with one another as if the body were rigid.

147.] If the body, instead of being in equilibrium throughout, be in process of straining, then taking the values of the components given by (4) and (5) it is easy to show by a similar method that the expressions on the left-hand side of equations (6) and (7), instead of vanishing, are equal to the effective forces and couples.

$$\left.\begin{array}{l}\iiint \rho\ddot{x}\,dx\,dy\,dz\\ \iiint \rho\ddot{y}\,dx\,dy\,dz\\ \iiint \rho\ddot{z}\,dx\,dy\,dz\end{array}\right\}.$$

$$\left.\begin{array}{l}\iiint \rho(y\ddot{z} - z\ddot{y})\,dx\,dy\,dz\\ \iiint \rho(z\ddot{x} - x\ddot{z})\,dx\,dy\,dz\\ \iiint \rho(x\ddot{y} - y\ddot{x})\,dx\,dy\,dz\end{array}\right\}.$$

Types of Reference.

148.] **The Three Normal Stress Components.** Reasoning as in § 132, with the modification introduced in § 135, we see that

(*i.*) The normal traction P, across the small plane area drawn perpendicular to Ox, tends to produce an elongation in the direction of Ox of the neighbouring portion of the body. Thus (§ 33) the function of the Simple Stress P is to produce and maintain the Simple Strain e.

(*ii.*) The normal traction Q, across the small plane area drawn perpendicular to Oy, tends to produce an elongation in the direction of Oy. Thus the function of the Simple Stress Q is to produce and maintain the Simple Strain f.

(*iii.*) Similarly the function of the Simple Stress R is to produce and maintain the Simple Strain g.

These statements must only be taken as pointing out the

analogies between the components of Strain and Stress, and the primary and most obvious functions of the latter. We are not at present concerned with the exact relations of Stress to Strain, and we must not take it for granted that any one stress-component can produce the analogous component strain *alone* without producing a simultaneous change in the other components.

149.] **The Tangential Stress Components.** The tangential traction S, in the positive direction of Oy across the small plane area drawn perpendicular to Oz, clearly tends to drag in that direction the layer of matter immediately in contact with the negative side of the area relatively to the layer in contact with this one on its negative side. And if we consider that this action takes place across every small plane area drawn perpendicular to Oz in the neighbourhood of the point P it is evident that the tendency of this tangential traction is to produce and maintain the Shearing Motion described in § 95 and represented in Fig. 2—that is, a positive shearing motion parallel to Oy of planes perpendicular to Oz.

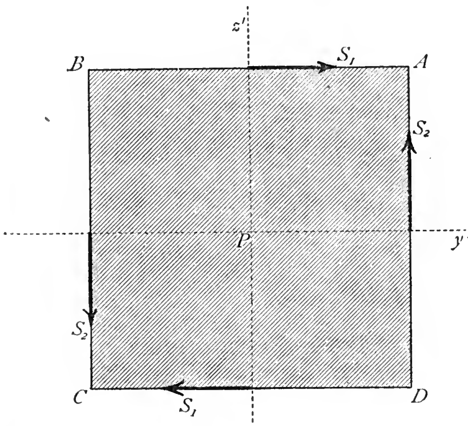


Fig. 10.

This will be perhaps more obvious on consulting Figure 10, which represents the action in the plane of $y'z'$ on the elementary cube having P for centre. For the sake of distinctness only the *traction - couples* about Px' are inserted, the normal components and those portions of the tangential tractions which combine to form a *force* on the element (§§ 139, 140) being omitted. The couple due to the traction we have just considered is marked S_1.

Similarly, the equal tangential traction S in the positive direction of Oz across small plane areas perpendicular to Oy, gives rise to the equal and opposite traction-couple marked S_2; the tendency of which is to produce a positive shearing motion parallel to Oz of planes perpendicular to Oy.

Now (§ 98) these shearing motions are rotational strains, compounded of *identical* shears and *opposite* rotations. The tendency of the two traction-couples in combination is to produce these two shearing motions simultaneously, and therefore (*see*

§ 98 and Fig. 3) to produce a simple irrotational shear of planes perpendicular to Py' and Pz', or to Oy and Oz.

(*iv.*) The two tangential stress-components S at the point P are therefore, when considered in combination, called a **Shearing Stress of amount** S in the plane of $y'z'$.

(*v.*) In like manner, the two tangential tractions T combine to form a Shearing Stress of amount T in the plane of $z'x'$; and

(*vi.*) The two tangential tractions U combine to form a Shearing Stress of amount U in the plane of $x'y'$.

The primary function of these three component stresses is then to produce and maintain the three component shears a, b, c respectively. (*See remarks at end of* § 148).

150.] **Resolution of Shearing Stress.** Just as we proved that all the simple strains—and therefore any strain whatever—can be resolved into simple elongations and contractions, so we may show that every stress may be regarded as the resultant of normal or longitudinal tractions or pressures.

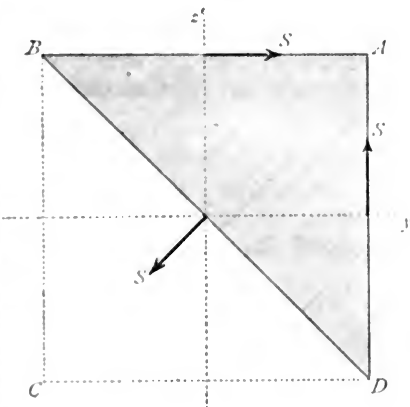


Fig. 11.

This will follow from the preceding Articles if we can prove it for a single shearing stress.

Let us then suppose the body held in a given state of strain, such that all the standard stress-components at the point P vanish, except the shearing stress S in the plane of $y'z'$. Equations (5) then give us for the stress-components across a small plane area drawn through P in any direction (λ, μ, ν)

$$\left.\begin{aligned} F &= 0 \\ G &= S\nu \\ H &= S\mu \end{aligned}\right\}.$$

Now if the stress across any such area be wholly normal we must have $F/\lambda = G/\mu = H/\nu$.

Substituting, we see that *two* planes can be drawn through P, such that the stress across them is wholly normal, their direction-cosines being respectively

$$(0,\ 1/\sqrt{2},\ 1/\sqrt{2}) \text{ and } (0,\ -1/\sqrt{2},\ 1/\sqrt{2});$$

and for both these planes

$$F = 0,\quad G = S\mu,\quad H = S\nu.$$

Thus the shearing stress of amount S in the plane of $y'z'$ is equivalent to a *normal traction* of amount S across the plane bisecting the *positive* angle between those of $x'y'$ and $z'x'$, together with a numerically equal *normal pressure* across the plane bisecting the *negative* angle between the same planes.

151.] The same result may be obtained by considering the equilibrium of the prism ABD (Fig. 11) one of the halves into which the cube of Figure 10 is divided by the diagonal plane BD. It is obvious that we may regard this prism as isolated if we suppose the existing stresses still to act across its faces.

Now, if a be the elementary length of either edge of the cube, the areas of the faces AB and AD are each a^2, while that of the face BD is $a^2\sqrt{2}$. Also the forces on the prism due to the tangential tractions on the former faces are each $S\,.\,a^2$ in the positive directions of Py' and Pz'.

The force across the face BD must therefore have equal components in the negative directions of the axes. That is to say, it must be a normal traction of intensity

$$S\,.\,a^2\,.\,\sqrt{2}/a^2\sqrt{2}$$

or S. Similarly by dividing the cube along the plane AC we can show that the stress across this plane is a normal pressure of intensity S (Fig. 12).

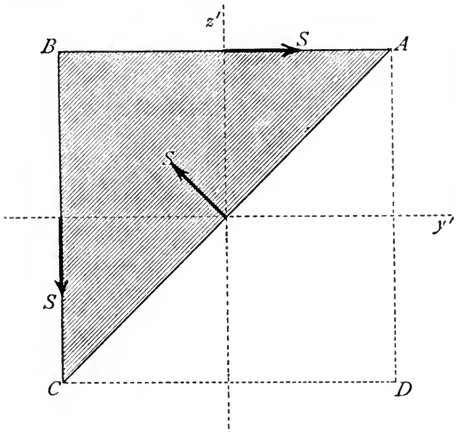


Fig. 12.

152.] **Discrepancy in the measurement of Shear and Shearing Stress.** Although the methods of resolution of shear and shearing stress are thus completely analogous, there is a discrepancy between the measurement of shear in terms of its component elongation and contraction, and that of shearing stress in terms of its component normal traction and pressure which should be carefully noted.

Thus the amount of a shearing stress compounded of a normal traction and pressure S is taken to be S, while the amount of a shear compounded of an elongation and contraction s_1 is taken to be $2s_1$ or a (*see* § 100).

The discrepancy exists solely in the *nomenclature* adopted

and, though on some accounts to be regretted, is not a serious defect.

The results of §§ 148, 149 are collected in the subjoined schedule for comparison with that of § 106.

The Component	tends to produce Relative Motion Parallel to Axis of	of Planes Perpendicular to Axis of
P	x	x
U	x	y
T	x	z
U	y	x
Q	y	y
S	y	z
T	z	x
S	z	y
R	z	z

153.] **Small Stresses.** All the properties of stress hitherto proved are equally true, whatever be the magnitude of the stress. Our analysis is however to be applied to the theory of bodies suffering infinitely small strains, and we shall therefore from this point restrict it to such stresses as are required to produce and maintain these strains.

Now we cannot suppose anything short of an absolutely rigid body to offer infinite resistance to a finite strain; and since it is a matter of experimental observation that the straining of the more perfectly elastic natural solids increases continuously with the stress applied, within the limits of their elasticity, we may safely assume that, if we adopt a finite unit of force per unit area, the numerical measures of the stress-components will always be of the same order of dimensions as those of the components of the strain which they serve to maintain.

For the purposes of our theory we may therefore always assume the components of stress to be small quantities of the first order (§ 58) whose squares and higher powers, together with their products with each other *or with the strain-components,* may be neglected in comparison with their first powers.

Now, strictly speaking, equations (3) of § 142, (4) of § 143, and (5) of § 144 express the relations between the forces across the faces of an element of the body which, *in the given state of strain,* is either a rectangular parallelepiped with its edges parallel to the fixed coördinate axes, or a tetrahedron with three edges parallel to the same axes. But, from what has just been said,

it follows that if the strain be a "small strain," and consequently the stress a "small stress," these relations will, to our order of approximation, take precisely the same form if we suppose the stress-components holding in equilibrium, in a given state of strain, elements of the body which have these shapes *in their natural state.*

154.] For instance the element of the body which, *in the natural state,* is the rectangular parallelepiped of § 138, becomes when strained an oblique parallelepiped, the areas of whose faces are

$$(1+f+g)dydz,\ (1+g+e)dzdx,\ (1+e+f)dxdy,$$

while its volume is $(1+\Delta)dxdydz$, and its density is $(1-\Delta)\rho$.

Hence it is easily shown that the first of equations (3) § 142 would in this case become

$$\frac{\partial P}{\partial x}(1+f+g)+\frac{\partial U}{\partial y}(1+g+e)+\frac{\partial T}{\partial z}(1+e+f)+\rho X=0$$

and by the last article this reduces to

$$\frac{\partial P}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial T}{\partial z}+\rho X=0$$

as before.

Again, the element of the body which in the natural state is the tetrahedron of § 144 becomes when strained an oblique-angled tetrahedron, the areas of whose faces are

$$\left.\begin{array}{l} \Delta(1+e+f+g-\epsilon) \\ \lambda\Delta(1+f+g) \\ \mu\Delta(1+g+e) \\ \nu\Delta(1+e+f) \end{array}\right\}$$

where Δ here denotes the unstrained area of the face ABC, and ϵ is the elongation in the initial direction (λ, μ, ν) of the normal to this face, given by equation 18 of § 72.

The first of equations (5) § 144 ought therefore, on this assumption, strictly to be

$$F(1+e+f+g-\epsilon)=P\lambda(1+f+g)+U\mu(1+g+e)+T\nu(1+e+f),$$

which, to our order of approximation, is identical with

$$F=P\lambda+U\mu+T\nu.$$

We may therefore, in all cases, take the system of standard stress-components which we have adopted as acting normally and tangentially across small plane areas through the point (x, y, z) which, *in the natural state of the body,* are perpendicular to the

fixed rectangular axes of coördinates; and though they are always taken parallel to the fixed axes, yet the strain they are capable of producing is so small that they may be considered to be normal or tangential throughout the process.

155.] **Principle of Superposition.** Since the stress-components are simply the components, resolved in fixed directions, of force per unit area, it is at once obvious that any number of stresses, applied *simultaneously*, must have for their resultant a single stress whose components are the algebraic sums of the corresponding components of the constituent stresses.

Moreover, it follows from the last Article that the application of any *small stress* will have the same effect, whether the body is in its natural state or has already been strained by another *small stress*.

This principle may be extended to any finite number of small stresses, the strain produced being still small (§ 87), and finally we see that the resultant of any number of small stresses, applied simultaneously or successively, is a single small stress whose components are the algebraic sums of the corresponding components.

And conversely (*compare* § 88) any small stress may be arbitrarily resolved into any number of small stresses, subject only to the above condition as to the sums of their components.

This result is called the Principle of Superposition of Small Stresses.

156.] **Type of Stress.** When the six components of any two stresses are to one another, each to each, in the same ratio, the stresses are said to be of the same type, or of exactly opposite types, according as this ratio is positive or negative. (*Compare* § 1 0).

The ratio of their components is called the Ratio of the Stresses, and when this ratio is ± 1 the stresses are said to be equal.

Stresses of the same and of opposite types are also called "concurrent" and "contrary."

Any number of small stresses belonging to two opposite types compound into a stress belonging to one of these types.

Two equal and contrary stresses annul one another.

157.] **Homogeneous Stress.** When the components of the stress have the same values at all points of the body, it is said to be homogeneous; and the body is said to be homogeneously stressed.

It is obvious from equations (3) that a body cannot be in equilibrium under homogeneous stress if there are any Applied Forces.

Homogeneous stress must therefore be produced by Surface Tractions only, the components of which must satisfy a certain condition at every point of the body.

For if we write

$$\mathfrak{K} = \begin{vmatrix} P, & U, & T \\ U, & Q, & S \\ T, & S, & R \end{vmatrix} \quad \text{.............................. (8)}$$

and denote by $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$, $\mathfrak{t}$, $\mathfrak{u}$, the minors of the determinant $\mathfrak{K}$ corresponding to P, Q, R, S, T, U, we get from equations (5) the relations

$$\left.\begin{aligned} \lambda &= (\mathfrak{p}F + \mathfrak{u}G + \mathfrak{t}H)/\mathfrak{K} \\ \mu &= (\mathfrak{u}F + \mathfrak{q}G + \mathfrak{s}H)/\mathfrak{K} \\ \nu &= (\mathfrak{t}F + \mathfrak{s}G + \mathfrak{r}H)/\mathfrak{K} \end{aligned}\right\} \quad \text{.......................... (9)}$$

to be satisfied by the direction-cosines of the normal at each point of the surface. Squaring and adding, we eliminate λ, μ, ν, and obtain

$$(\mathfrak{p}F + \mathfrak{u}G + \mathfrak{t}H)^2 + (\mathfrak{u}F + \mathfrak{q}G + \mathfrak{s}H)^2 + (\mathfrak{t}F + \mathfrak{s}G + \mathfrak{r}H)^2 = \mathfrak{K}^2 \text{......(10)}$$

Since in the case of homogeneous stress all the coefficients are absolute constants, equation (10) represents a definite relation existing between the components F, G, H of the Surface Traction at each point of the bounding surface.

158.] **Stress to be treated as Homogeneous.** In the following investigation into the graphic and other properties of Stress, we shall for the sake of simplicity treat it as if it were homogeneous; because then, its character being identical throughout the body, we can confine ourselves to the consideration of its properties at the origin.

Of course it will be understood that, as in the case of Strain (§ 122), the results obtained will be equally true for an elementary portion of a body under heterogeneous stress, described about any point P, if that point be taken as the origin of relative coördinates, and the stress-components be given their proper values at P.

These applications will be pointed out as occasion requires.

Graphic Properties of Stress.

159.] **Change of Axes of Reference.** Let P, Q, R, S, T, U be the components at the origin of a given stress, referred to the arbitrary system of rectangular axes Ox, Oy, Oz: required, in terms of these, the corresponding components P', Q', R', S', T', U' of the same stress referred to any other arbitrary system of rectangular axes Ox', Oy', Oz'.

Let the direction-cosines of the new axes be given by the schedule:—

	x'	y'	z'
x	λ_1	λ_2	λ_3
y	μ_1	μ_2	μ_3
z	ν_1	ν_2	ν_3

Then P', U', T' are the components parallel to Ox', Oy', Oz' of the stress across the small plane area drawn through the origin perpendicular to Ox'. But by equations (5) the components parallel to Ox, Oy, Oz of this stress are

$$\left.\begin{array}{l} P\lambda_1 + U\mu_1 + T\nu_1 \\ U\lambda_1 + Q\mu_1 + S\nu_1 \\ T\lambda_1 + S\mu_1 + R\nu_1 \end{array}\right\} \qquad (11)$$

Thus

$$P' = \lambda_1(P\lambda_1 + U\mu_1 + T\nu_1) + \mu_1(U\lambda_1 + Q\mu_1 + S\nu_1) + \nu_1(T\lambda_1 + S\mu_1 + R\nu_1)$$
$$U' = \lambda_2(P\lambda_1 + U\mu_1 + T\nu_1) + \mu_2(U\lambda_1 + Q\mu_1 + S\nu_1) + \nu_2(T\lambda_1 + S\mu_1 + R\nu_1)$$

etc., etc.

Thus we finally obtain, by rearrangement of terms

$$\left.\begin{array}{l} P' = P\lambda_1^2 + Q\mu_1^2 + R\nu_1^2 + 2S\mu_1\nu_1 + 2T\nu_1\lambda_1 + 2U\lambda_1\mu_1 \\ Q' = P\lambda_2^2 + Q\mu_2^2 + R\nu_2^2 + 2S\mu_2\nu_2 + 2T\nu_2\lambda_2 + 2U\lambda_2\mu_2 \\ R' = P\lambda_3^2 + Q\mu_3^2 + R\nu_3^2 + 2S\mu_3\nu_3 + 2T\nu_3\lambda_3 + 2U\lambda_3\mu_3 \\ S' = P\lambda_2\lambda_3 + Q\mu_2\mu_3 + R\nu_2\nu_3 + S(\mu_2\nu_3 + \mu_3\nu_2) \\ \qquad + T(\nu_2\lambda_3 + \nu_3\lambda_2) + U(\lambda_2\mu_3 + \lambda_3\mu_2) \\ T' = P\lambda_3\lambda_1 + Q\mu_3\mu_1 + R\nu_3\nu_1 + S(\mu_3\nu_1 + \mu_1\nu_3) \\ \qquad + T(\nu_3\lambda_1 + \nu_1\lambda_3) + U(\lambda_3\mu_1 + \lambda_1\mu_3) \\ U' = P\lambda_1\lambda_2 + Q\mu_1\mu_2 + R\nu_1\nu_2 + S(\mu_1\nu_2 + \mu_2\nu_1) \\ \qquad + T(\nu_1\lambda_2 + \nu_2\lambda_1) + U(\lambda_1\mu_2 + \lambda_2\mu_1) \end{array}\right\} \qquad (12)$$

Adding together the first three of these equations, we get

$$P' + Q' + R' = P + Q + R \qquad (13)$$

and since both systems of axes are completely arbitrary, this proves the perfectly general theorem that

The sum of the normal components of stress across any three small orthogonal plane areas drawn through a given point of the body is absolutely constant for that point, however the planes be turned about it; and in the case of homogeneous stress, this constant has the same value at every point of the body.

160.] **Resultant Stress.** Let A, B, C denote the resultant stresses across the small plane areas drawn through the origin perpendicular to Ox, Oy, Oz; and A', B', C' the resultant stresses across those perpendicular to Ox', Oy', Oz'.

The components of A, B, C parallel to Ox, Oy, Oz are of

course P, U, T; U, Q, S; T, S, R, respectively; while the components of A' parallel to the same axes are given by equations (11), and those of B' and C' by similar formulæ.

Squaring and adding, we get

$$\left.\begin{aligned} A^2 &= P^2 + U^2 + T^2 \\ B^2 &= U^2 + Q^2 + S^2 \\ C^2 &= T^2 + S^2 + R^2 \end{aligned}\right\} \dots\dots\dots(14)$$

$$\left.\begin{aligned} A'^2 &= (\lambda_1 P + \mu_1 U + \nu_1 T)^2 + (\lambda_1 U + \mu_1 Q + \nu_1 S)^2 + (\lambda_1 T + \mu_1 S + \nu_1 R)^2 \\ B'^2 &= (\lambda_2 P + \mu_2 U + \nu_2 T)^2 + (\lambda_2 U + \mu_2 Q + \nu_2 S)^2 + (\lambda_2 T + \mu_2 S + \nu_2 R)^2 \\ C'^2 &= (\lambda_3 P + \mu_3 U + \nu_3 T)^2 + (\lambda_3 U + \mu_3 Q + \nu_3 S)^2 + (\lambda_3 T + \mu_3 S + \nu_3 R)^2 \end{aligned}\right\} \dots(15)$$

Expanding (15) and adding them together, we get

$$A'^2 + B'^2 + C'^2 = P^2 + Q^2 + R^2 + 2(S^2 + T^2 + U^2).$$

Hence, by (14),

$$A'^2 + B'^2 + C'^2 = A^2 + B^2 + C^2. \dots\dots\dots(16)$$

from which we deduce that

The sum of the squares of the resultant stresses across any three orthogonal plane areas drawn through a given point of the body is constant, however they be turned about the point; and when the stress is homogeneous, this constant has the same value at every point of the body.

161.] **Reciprocal relation between Stress-components.** Since $(\lambda_1, \mu_1, \nu_1)$ are the direction-cosines of Ox' referred to Ox, Oy, Oz, and since P, U, T are the components of A parallel to these axes, it follows that the component of A parallel to Ox' is

$$P\lambda_1 + U\mu_1 + T\nu_1.$$

But we have already seen (11) that this is the component of A' parallel to Ox. Hence, since the directions of Ox, Ox' are quite arbitrary, we deduce that

If any two small plane areas be drawn through any given point of the body, the component perpendicular to the first area of the stress across the second is always equal to the component perpendicular to the second of the stress across the first.

162.] **First Stress Quadric.** We now proceed to give these theorems geometrical significance. Describe the quadric

$$Px^2 + Qy^2 + Rz^2 + 2Syz + 2Tzx + 2Uxy = 1 \dots\dots\dots(17)$$

Let r be the length of the radius vector in the direction (λ, μ, ν) and let p be the perpendicular from the centre on the tangent plane at the extremity of r, (l, m, n) being the direction cosines of p.

Then, if F, G, H be the components, parallel to the axes of reference, of the stress across the central section perpendicular to r, F, G, H will be given by equations (5). But we have

$$\frac{P\lambda + U\mu + T\nu}{l} = \frac{U\lambda + Q\mu + S\nu}{m} = \frac{T\lambda + S\mu + R\nu}{n} = \frac{1}{pr} \quad \text{.........(18)}$$

Thus $$F/l = G/m = H/n = 1/pr \quad \text{.........................(19)}$$

The resultant stress across the central section perpendicular to r therefore acts in the direction of p; its amount is $1/pr$; and the amount of its normal component is $1/r^2$.

163.] **Principal Axes of the Stress.** It is obvious from the last Article that if the section coincide with any one of the principal sections of the quadric, the stress across it will be wholly normal.

It is thus always possible to draw through each point of the body three orthogonal plane areas across which the stress is wholly normal. These are called the *Principal Planes* of the stress at the point, and their normals are called the *Principal Axes*.

The normal tractions across the principal planes are called the *Principal Normal Stresses*. We shall denote them by N_1, N_2, N_3.

Let $O\xi$, $O\eta$, $O\zeta$ be the Principal Axes of the stress at the origin; then they are also the principal axes of the quadric (17). It also appears from § 162 that the squared reciprocals of its principal semi-diameters are N_1, N_2, N_3.

Hence the equation of the quadric referred to the principal axes is

$$N_1\xi^2 + N_2\eta^2 + N_3\zeta^2 = 1 \quad \text{...........................(20)}$$

Of course N_1, N_2, N_3 are the roots (in descending order of magnitude, let us say) of the discriminating cubic

$$\begin{vmatrix} P-\phi, & U, & T \\ U, & Q-\phi, & S \\ T, & S, & R-\phi \end{vmatrix} = 0 \quad \text{.......................(21)}$$

while the direction-cosines of $O\xi, O\eta, O\zeta$ are given by the equations

$$\frac{P\lambda + U\mu + T\nu}{\lambda} = \frac{U\lambda + Q\mu + S\nu}{\mu} = \frac{T\lambda + S\mu + R\nu}{\nu} = N \quad \text{........(22)}$$

where N is to receive successively the suffixes 1, 2, 3.

These equations (22) might of course have been deduced directly from equations (5).

164.] **Invariants of the Stress.** Expanding the cubic (21), and employing the notation of § 157, it becomes

$$\phi^3 - \phi^2(P + Q + R) + \phi(\mathfrak{p} + \mathfrak{q} + \mathfrak{r}) - \mathfrak{K} = 0 \quad\ldots\ldots\ldots\ldots(23)$$

If we write this

$$\phi^3 - \phi^2 . \mathfrak{D} + \phi . \mathfrak{J} - \mathfrak{K} = 0 \quad\ldots\ldots\ldots\ldots(24)$$

then $\mathfrak{D}$, $\mathfrak{J}$, $\mathfrak{K}$ are the *Invariants* of the quadric, and are given by

$$\mathfrak{D} = P + Q + R = N_1 + N_2 + N_3 \quad\ldots\ldots\ldots\ldots(25)$$

$$\begin{aligned}\mathfrak{J} &= \mathfrak{p} + \mathfrak{q} + \mathfrak{r} = QR - S^2 + RP - T^2 + PQ - U^2 \\ &= N_2N_3 + N_3N_1 + N_1N_2 \quad\ldots\ldots\ldots\ldots(26)\end{aligned}$$

$$\mathfrak{K} = \begin{vmatrix} P, & U, & T \\ U, & Q, & S \\ T, & S, & R \end{vmatrix} = N_1N_2N_3 \quad\ldots\ldots\ldots\ldots(27)$$

[*Compare these with equations* (39) *of* § 111.]

It is now obvious that the theorem of § 159 simply states that $\mathfrak{D}$ is an invariant.

Again, we see that

$$\mathfrak{D}^2 - 2 . \mathfrak{J} = P^2 + Q^2 + R^2 + 2(S^2 + T^2 + U^2) = A^2 + B^2 + C^2 \ldots\ldots(28)$$

Thus the theorem of § 160 simply states that $\mathfrak{D}^2 - 2 . \mathfrak{J}$ is an invariant.

165.] **Traction and Pressure.** We saw in § 162 that the normal component of the stress across the plane perpendicular to the radius vector r was $1/r^2$. Hence if the stress be such as to produce a *traction* across every small plane area drawn through the origin, the quadric (17) is an ellipsoid, and N_1, N_2, N_3 are all positive, and so therefore is $\mathfrak{K}$.

If the stress across every plane be a *pressure*, the quadric represented by equations (17) and (20) will be imaginary; N_1, N_2, N_3, $\mathfrak{K}$ will all be negative, and the pressures will be given by the ellipsoid

$$Px^2 + Qy^2 + Rz^2 + 2Syz + 2Tzx + 2Uxy = -1 \quad\ldots\ldots\ldots\ldots(29)$$

or

$$N_1\xi^2 + N_2\eta^2 + N_3\zeta^2 = -1 \quad\ldots\ldots\ldots\ldots(30)$$

166.] **Normal Cone of Shearing Stress.** If the stress at the origin be a traction or a pressure according to the direction of the plane across which it is measured, equations (17) and (29), or (20) and (30), will represent two real conjugate hyperboloids, radii which meet the first being normals to planes across which there is a traction, while radii which meet the second are normals to planes across which there is pressure.

These two hyperboloids are separated by their asymptotic cone

$$Px^2 + Qy^2 + Rz^2 + 2Syz + 2Tzx + 2Uxy = 0 \dots\dots\dots\dots (31)$$

or

$$N_1\xi^2 + N_2\eta^2 + N_3\zeta^2 = 0 \dots\dots\dots\dots (32)$$

Since any radius vector lying in this cone is of infinite length, the normal component of the stress across the plane perpendicular to it vanishes ; whence we see that all planes whose *normals* are generators of this cone suffer only tangential stress. It is therefore called the Normal Cone of Shearing Stress.

167.] **Second Stress Quadric ("Director Quadric").** Let us now construct the reciprocal quadric, whose equation referred to the principal axes is

$$\frac{\xi^2}{N_1} + \frac{\eta^2}{N_2} + \frac{\zeta^2}{N_3} = 1 \dots\dots\dots\dots (33)$$

If r be the radius vector drawn from the centre to the point (ξ, η, ζ) on the surface, and p the perpendicular from the centre on the tangent plane at (ξ, η, ζ), the direction-cosines of p referred to the principal axes are

$$p\xi/N_1,\ p\eta/N_2,\ p\zeta/N_3.$$

Now equations (5) give for the components parallel to $O\xi$, $O\eta$, $O\zeta$ of the stress across a plane the direction-cosines of whose normals referred to the same axes are $(\lambda_0, \mu_0, \nu_0)$

$$\left.\begin{aligned} F_0 &= N_1\lambda_0 \\ G_0 &= N_2\mu_0 \\ H_0 &= N_3\nu_0 \end{aligned}\right\} \dots\dots\dots\dots (34)$$

Hence the component stresses across the plane perpendicular to p (that is, the section conjugate to r) are given by

$$\left.\begin{aligned} F_0 &= p\xi \\ G_0 &= p\eta \\ H_0 &= p\zeta \end{aligned}\right\}.$$

Hence the resultant stress across the section perpendicular to p (or conjugate to r) acts in the direction of r; its amount is pr ; and the amount of its normal component is p^2.

168.] **Tangent Cone of Shearing Stress.** By considering the sign of the normal component, as in § 166, we see that if the stress at the origin be a *traction* in every direction (33) is an ellipsoid; if a pressure in every direction we have the alternative ellipsoid

$$\frac{\xi^2}{N_1} + \frac{\eta^2}{N_2} + \frac{\zeta^2}{N_3} = -1 \dots\dots\dots\dots (35)$$

while if it is a traction across some planes and a pressure across others, we have the pair of real conjugate hyperboloids (33) and (35).

These are separated by their asymptotic cone

$$\frac{\xi^2}{N_1}+\frac{\eta^2}{N_2}+\frac{\zeta^2}{N_3}=0 \quad\quad (36)$$

and it is easy to see that this cone envelopes all those planes through the origin which suffer only tangential stress. It is therefore called the Tangent Cone of Shearing Stress.

169.] **Third Stress Quadric ("Stress Ellipsoid").** It is obvious that equation (10) may be regarded as an equation to be satisfied by the components, parallel to the arbitrary axes, of the stress across any plane through the origin. Hence if we construct the quadric

$$(\mathfrak{p}x+\mathfrak{u}y+\mathfrak{t}z)^2+(\mathfrak{u}x+\mathfrak{q}y+\mathfrak{s}z)^2+(\mathfrak{t}x+\mathfrak{s}y+\mathfrak{r}z)^2=\mathfrak{R}^2 \quad\quad (37)$$

the radius vector r drawn to the point (x, y, z) on the surface will represent in magnitude and direction the stress across the central section whose normal is in the direction given by writing x, y, z for F, G, H in equations (9).

Transforming to the Principal Axes, we find that the radius vector r of the quadric

$$\frac{\xi^2}{N_1^2}+\frac{\eta^2}{N_2^2}+\frac{\zeta^2}{N_3^2}=1 \quad\quad (38)$$

represents in magnitude and direction the resultant stress across the central section whose direction-cosines referred to $O\xi$, $O\eta$, $O\zeta$ are given by

$$\left.\begin{aligned}\lambda_0&=\xi/N_1\\ \mu_0&=\eta/N_2\\ \nu_0&=\zeta/N_3\end{aligned}\right\} \quad\quad (39)$$

where (ξ, η, ζ) is the extremity of r. This quadric is of course always an ellipsoid.

If (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2), (ξ_3, η_3, ζ_3) be the coördinates of the extremities of three radii r_1, r_2, r_3, representing in magnitude and direction the resultant stresses across any three *orthogonal* central sections of this quadric, it follows from (39) that they must satisfy the relations

$$\left.\begin{aligned}\frac{\xi_2\xi_3}{N_1^2}+\frac{\eta_2\eta_3}{N_2^2}+\frac{\zeta_2\zeta_3}{N_3^2}&=0\\ \frac{\xi_3\xi_1}{N_1^2}+\frac{\eta_3\eta_1}{N_2^2}+\frac{\zeta_3\zeta_1}{N_3^2}&=0\\ \frac{\xi_1\xi_2}{N_1^2}+\frac{\eta_1\eta_2}{N_2^2}+\frac{\zeta_1\zeta_2}{N_3^2}&=0\end{aligned}\right\} \quad\quad (40)$$

which are the well-known conditions that r_1, r_2, r_3 may be *conjugate* semi-diameters of the quadric.

Hence we deduce that any three conjugate radii represent in magnitude and direction the resultant stresses across three orthogonal central sections. This also follows directly from equation (39), the geometrical interpretation of which is that r represents the stress across the section whose normal is the radius of the "auxiliary sphere" corresponding to r.

170.] **Relation between the Second and Third Quadrics.** If any radius vector from the common centre meet the Third Quadric in (ξ_1, η_1, ζ_1) and the Second in (ξ_2, η_2, ζ_2) and if r_1 be the length intercepted on it by the Third, we see by (39) that r_1 represents in magnitude and direction the stress across the plane

$$\frac{\xi\xi_1}{N_1} + \frac{\eta\eta_1}{N_2} + \frac{\zeta\zeta_1}{N_3} = 0 \qquad (41)$$

which is the same as

$$\frac{\xi\xi_2}{N_1} + \frac{\eta\eta_2}{N_2} + \frac{\zeta\zeta_2}{N_3} = 0,$$

that is, the central section of the Second Quadric conjugate to r_1.

Hence if r_1, r_2, r_3 be the lengths intercepted by the Third Quadric on any three conjugate radii of the Second, each represents in magnitude and direction the stress across the plane containing the other two. Thus the Third Quadric may be regarded as giving a graphical construction for the magnitudes of stresses, and the Second for the directions of the planes across which they act. We shall therefore distinguish them as the *Stress Ellipsoid* and the *Director Quadric*.

171.] In the cases where the Principal Stresses are of different signs, and there is consequently a real Tangent Cone of Shearing Stress (36), each generator of this cone represents *three coincident* conjugate radii, and the plane conjugate to any generator is the tangent plane to the cone along that generator. Thus if r be the length intercepted by the Third Quadric on any generator of the Tangent Cone of Shearing Stress, then r represents in magnitude and direction the shearing stress across the plane which touches the cone along that generator.

172.] **Fourth Stress Quadric.** Finally, let us describe that reciprocal of the Third Quadric whose equation is

$$(Px + Uy + Tz)^2 + (Ux + Qy + Sz)^2 + (Tx + Sy + Rz)^2 = 1 \qquad (42)$$

or

$$N_1^2\xi^2 + N_2^2\eta^2 + N_3^2\zeta^2 = 1 \qquad (43)$$

This is likewise always an ellipsoid.

It is obvious, by squaring and adding equations (5) or (34), that if r be the radius vector of this quadric perpendicular to any given central section, the amount of the resultant stress across that section is $1/r$, and its components parallel to the principal axes are

$$N_1\xi/r,\ N_2\eta/r,\ N_3\zeta/r \qquad (44)$$

ξ, η, ζ being the coördinates of the extremity of r.

Hence if (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) (ξ_3, η_3, ζ_3) be the extremities of three radii, perpendicular to three central sections the resultant stresses across which act in three orthogonal directions, we have from (44) the relations

$$\left.\begin{aligned} N_1^2\xi_2\xi_3 + N_2^2\eta_2\eta_3 + N_3^2\zeta_2\zeta_3 = 0 \\ N_1^2\xi_3\xi_1 + N_2^2\eta_3\eta_1 + N_3^2\zeta_3\zeta_1 = 0 \\ N_1^2\xi_1\xi_2 + N_2^2\eta_1\eta_2 + N_3^2\zeta_1\zeta_2 = 0 \end{aligned}\right\} \qquad (45)$$

which are the conditions that the three radii may be conjugate.

Hence the resultant stresses across any three central sections whose normals are conjugate radii act along three orthogonal radii.

173.] **Relation between the First and Fourth Quadrics.** We see from (44) that if (ξ_1, η_1, ζ_1) be the extremity of the radius vector r_1 of the Fourth Quadric, then the resultant stress across the central section perpendicular to r_1 acts along the normal to the plane

$$N_1\xi\xi_1 + N_2\eta\eta_1 + N_3\zeta\zeta_1 = 0 \qquad (46)$$

which is the section of the First Quadric conjugate to r_1.

Hence if r_1, r_2, r_3 be the intercepts by the Fourth Quadric on any three conjugate radii of the First, the resultant stress across the central section perpendicular to either acts in the direction perpendicular to the plane containing the other two; while the amounts of these resultant stresses are $1/r_1$, $1/r_2$, $1/r_3$ respectively.

Special Forms of Stress.

174.] **Hydrostatic Pressure.** All the preceding theorems apply to the most general form of the Stress, when N_1, N_2, N_3 are all unequal and of any sign, but none of them vanish.

The cases in which *two* of the principal stresses are equal are not worth working out in detail, as the results already obtained may be easily modified to suit them, if we remember that all the

Stress Quadrics become surfaces of revolution. The necessary and sufficient conditions are

$$\left.\begin{array}{l} P-\dfrac{TU}{S}=Q-\dfrac{US}{T}=R-\dfrac{ST}{U}, \\ \text{or} \quad \mathfrak{J}^2(\mathfrak{D}^2-4\mathfrak{J})-\mathfrak{D}\mathfrak{K}(4\mathfrak{D}^2-18\mathfrak{J})-27\mathfrak{K}^2=0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots(47)$$

[*Compare* § 120 (*iii.*).]

The case where *all three* of the principal stresses are equal is however remarkable.

If $N_1=N_2=N_3=-\Pi$, the discriminating cube (21) or (24) must reduce to

$$(\phi+\Pi)^3=0.$$

Hence we must have

$$\left.\begin{array}{l} \mathfrak{D}=-3\Pi \\ \mathfrak{J}=3\Pi^2 \\ \mathfrak{K}=-\Pi^3 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(48)$$

The stress-quadrics become spheres, the Third in particular becoming

$$\left.\begin{array}{ll} & \xi^2+\eta^2+\zeta^2=\Pi^2, \\ \text{and the Second} & \xi^2+\eta^2+\zeta^2=\Pi \end{array}\right\}.$$

Since any radius of a sphere coincides with the perpendicular from the centre on the tangent plane at its extremity, it follows that the stress across every central section is normal, and that it has the same value for each. Thus if Π is positive the stress at the origin consists of a normal pressure Π across every plane area which can be drawn through it. This is the nature of the stress which exists at every point of a fluid at rest under any forces, and it is therefore called Hydrostatic Pressure.

If Π is negative, we have simply to replace the pressure by a traction.

In order that the Stress Quadrics may be spheres we must obviously have

$$\left.\begin{array}{l} P=Q=R=-\Pi \\ S=T=U=0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(49)$$

Thus it is evident from equations (5) and § 157 that a homogeneous hydrostatic pressure can only be maintained by a uniform normal pressure of like amount applied over the whole bounding surface.

We may here notice another discrepancy in the numerical reckoning of Strain and Stress (*see* § 152). Three equal orthogonal contractions ϵ compound (§ 104) into a uniform compression of amount 3ϵ, while three equal orthogonal normal pressures Π compound into a hydrostatic pressure of amount Π.

Stress in Two Dimensions.

175.] **Plane of the Stress.** The remaining important types of stress are characterised by the vanishing of the third invariant $\mathfrak{K}$, and therefore also of one at least of the Principal Normal Stresses.

For the present we shall confine ourselves to the case in which only one of them, say N_3, vanishes. There is then no stress whatever across the small plane area drawn through O to coincide with the plane of $\xi\eta$.

The Stress Quadrics become cylinders with their generators parallel to $O\zeta$; and since in the third of equations (39) ν_0 cannot be greater than unity, it follows that at the extremity of every radius vector which represents the resultant stress across a *real* plane through the origin we must have $\zeta=0$.

Hence the directions of the resultant stresses across *all* plane areas drawn through O lie in the normal section by the plane of $\xi\eta$ which contains N_1 and N_2. The stress is therefore said to be entirely in two dimensions, and the plane of $\xi\eta$ is called the Plane of the Stress at O.

176.] **The Stress Conics.** It is obvious that all the graphic properties of the stress will depend upon curves in the plane of the stress, and especially on the normal sections of the Stress Cylinders by that plane. These curves we shall call the Stress Conics.

177.] **Case in which N_1 and N_2 have the same sign.** In this case $\mathfrak{J}$ is positive, while $\mathfrak{D}$ has the same sign as N_1 and N_2.

Assuming this sign to be positive, the Second and Third Stress Conics become the ellipses

$$\left.\begin{aligned} \frac{\xi^2}{N_1}+\frac{\eta^2}{N_2}&=1 \quad \text{.............................(50)} \\ \frac{\xi^2}{N_1^2}+\frac{\eta^2}{N_2^2}&=1 \quad \text{.............................(51)} \end{aligned}\right\}$$

The first of these is the *Director Conic*, replacing the Director Quadric of §§ 167, 170.

If (ξ, η) be the extremity of the radius vector representing in magnitude and direction the stress across the plane whose direction cosines referred to the principal axes are (λ, μ, ν) we have from equations (39)

$$\left.\begin{aligned} \lambda &= \frac{\xi}{N_1} \\ \mu &= \frac{\eta}{N_2} \end{aligned}\right\} \quad \text{.............................(52)}$$

and therefore

$$\frac{\xi^2}{N_1^2}+\frac{\eta^2}{N_2^2}=1-\nu^2 \quad \text{.............................(53)}$$

Thus the resultant stresses across all planes through O whose normals lie on a circular cone with axis $O\zeta$ and semi-vertical angle a, are represented in magnitude and direction by the radii of the ellipse

$$\frac{\xi^2}{N_1^2}+\frac{\eta^2}{N_2^2}=\sin^2 a \qquad (54)$$

To each such cone belongs one of these ellipses, the whole system being similar to and coaxial with the ellipse (51), which is the largest of all. In the limit when $a=0$, the ellipse (54) vanishes into the origin, so that, as we already know, the stress across the Principal Plane $\xi\eta$ is zero. As a increases, so does the size of the ellipse, and therefore the magnitude of the stress, until the limit is reached in which $a=\frac{1}{2}\pi$, when the ellipse (54) coincides with (51). The radii of (51) therefore represent the stresses across planes whose normals lie in the plane of the stress:—that is, across planes drawn through $O\zeta$. This system of *Stress Ellipses* replaces the Stress Ellipsoid of §§ 169, 170.

178.] Again the *trace* of a given plane on the plane of the stress (or the line in which the two planes intersect) is given by

$$\lambda\xi+\mu\eta=0,$$

and the projection of its normal on the plane of the stress by

$$\mu\xi-\lambda\eta=0.$$

Hence if (ξ_1, η_1) be the extremity of the radius vector representing the stress across this plane, we get from equations (52)—
for the trace of the plane

$$\frac{\xi\xi_1}{N_1}+\frac{\eta\eta_1}{N_2}=0 \qquad (55)$$

and for the projection of the normal

$$\frac{\xi\eta_1}{N_2}-\frac{\eta\xi_1}{N_1}=0 \qquad (56)$$

These equations afford us two geometrical constructions for determining the direction and magnitude of the stress across a given plane.

First Method. If (ξ_1, η_1) be the point on the stress-ellipse (54) whose radius vector represents the stress, and if (ξ_2, η_2) be the point on the auxiliary circle of that ellipse corresponding to (ξ_1, η_1), then (assuming that $N_1 > N_2$)

$$\xi_2=\xi_1,\quad \eta_2=\frac{N_1}{N_2}\cdot\eta_1;$$

thus (56) may be written

$$\xi\eta_2-\eta\xi_2=0.$$

Thus the projection of the normal to the plane is that radius of

the auxiliary circle which corresponds to the radius of the ellipse representing the stress.

Conversely, if the plane be given, we can construct the stress-ellipse (54) and its auxiliary circle

$$\xi^2 + \eta^2 = N_1^2 \sin^2 a \,; \qquad (57)$$

if we then project the normal to the given plane, and find the point on the ellipse corresponding to the extremity of that radius of the circle which coincides with the projection, the radius vector of this point represents in magnitude and direction the stress across the given plane.

We have seen in § 165 that when N_1 and N_2 are both positive this stress is always a **traction**.

Fig. 13.

It is well known that if OP_1, OP_2 be two conjugate radii of an ellipse, the corresponding radii OQ_1, OQ_2 of its auxiliary circle (Fig 13) are at right angles; and conversely.

Thus OP_1 represents the stress across a plane whose trace is OQ_2, and the projection of its normal OQ_1; while OP_2 represents the stress across a plane, making the same angle with $O\zeta$, whose trace is OQ_1, and the projection of its normal OQ_2.

Thus, conversely, if through any two perpendicular radii OQ_1, OQ_2 planes be drawn so that their normals may make the same angle a with $O\zeta$, the resultant stresses across them will be represented by conjugate radii of the ellipse (54); and, in particular, the stresses across any two orthogonal planes through $O\zeta$ are represented by conjugate radii of the ellipse (51).

Second Method. If (ξ_1, η_1) be as before the point on the ellipse (54) whose radius vector represents the stress across the given plane (λ, μ, ν), and if this radius (produced if necessary) meet the director-ellipse (50) in the point (ξ_2, η_2), we have

$$\xi_1 : \xi_2 :: \eta_1 : \eta_2.$$

Hence equation (55) giving the trace of the plane may be written

$$\frac{\xi\xi_2}{N_1} + \frac{\eta\eta_2}{N_2} = 0,$$

which represents the radius of (50) conjugate to the first.

Conversely, if the radius of the director-ellipse (50) conjugate to the trace of any given plane be drawn, the intercept on this by the stress-ellipse (54) represents in magnitude and direction the resultant stress across the given plane.

Since two conjugate radii of an ellipse never lie in the same quadrant, no plane through the origin is subject to simple Shearing Stress.

179.] If $\mathfrak{I}$ is positive and $\mathfrak{D}$ negative, N_1 and N_2 are both negative. All the theorems proved in the last two Articles will be equally true, the only change necessary being to substitute for (50) the equation

$$\frac{\xi^2}{N_1}+\frac{\eta^2}{N_2}=-1 \qquad (58)$$

The stress across every plane through the origin will be of the nature of a **pressure**.

180.] **Case in which $N_1 = N_2$.** If $\mathfrak{K}=0$, and $\mathfrak{I}=\frac{1}{4}\mathfrak{D}^2$, then $N_1=N_2=N$ (say); N having the same sign as $\mathfrak{D}$. The results of the previous Articles may be modified to suit this case, by writing everywhere "circle" for "ellipse," and "orthogonal" for "conjugate."

Thus the stress across any plane whose normal is inclined at an angle a to $O\zeta$ is represented in magnitude and direction by the radius of the circle

$$\xi^2+\eta^2=N^2\sin^2 a \qquad (59)$$

which is perpendicular to the trace of the plane.

In other words the stress across every such plane is $N\sin a$, and acts along the projection of its normal on the plane of the stress.

Every plane through the axis $O\zeta$ suffers a normal stress N.

The stress is obviously symmetrical about $O\zeta$, and the directions of $O\xi$ and $O\eta$ are indeterminate.

181.] **Case in which N_1 and N_2 have opposite signs.** If $\mathfrak{K}=0$, and $\mathfrak{I}$ is negative, one of the principal normal stresses will be a traction and the other a pressure, the sign of the greater of the two being the same as that of $\mathfrak{D}$; we shall suppose N_1 to be positive and N_2 negative.

Instead of the ellipse (50) we now have the pair of conjugate director hyperbolas

$$\frac{\xi^2}{N_1}+\frac{\eta^2}{N_2}=1 \qquad (60)$$

and

$$\frac{\xi^2}{N_1}+\frac{\eta^2}{N_2}=-1 \qquad (61)$$

separated by their asymptotes

$$\frac{\xi^2}{N_1} + \frac{\eta^2}{N_2} = 0 \qquad \text{.................................(62)}$$

The system of stress-ellipses (54) will of course remain unaltered.

To modify the results obtained by the first method of § 178, we must remember that, since N_2 is now negative, the coördinates of the point on the auxiliary circle corresponding to the point (ξ_1, η_1) on the ellipse (54) are now

$$\xi_2 = \xi_1, \quad \eta_2 = -\frac{N_1}{N_2} \cdot \eta_1$$

if N_1 be numerically greater than N_2;

or
$$\xi_2 = -\frac{N_2}{N_1} \cdot \xi_1, \quad \eta_2 = \eta_1$$

if N_1 be numerically less than N_2. The analogous construction for the present case is then as follows:—

To find the resultant stress across a plane whose normal makes an angle α with $O\xi$, project this normal on to the plane of the stress: let OQ be the radius of the auxiliary circle of the ellipse (54) with which this projection coincides. Find P, the point on the ellipse corresponding to Q, and draw the radius OP' of the ellipse, equally inclined to the major axis on the opposite side. Then OP' will represent in magnitude and direction the stress across the given plane—which may therefore be normal, tangential, or oblique.

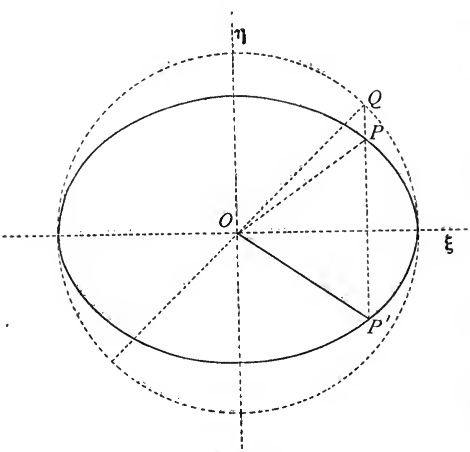


Fig. 14.

Again, modifying the results obtained by the second method of § 178, we see that the intercept made by the ellipse (54) on any radius which meets (60) represents in magnitude and direction the resultant **traction** across the plane drawn through the conjugate radius of (61) so that its normal makes an angle α with $O\xi$.

Similarly the intercept made by (54) on any radius which meets (61) represents the resultant **pressure** across a plane

drawn at the same inclination to $O\zeta$ through the conjugate radius of (60).

Either asymptote of the hyperbolas represents a pair of coincident conjugate radii; hence a plane drawn at any inclination a through either of the asymptotes (62) suffers a **shearing stress** represented in magnitude and direction by the intercept cut off on that asymptote by the ellipse (54).

182.] **Case in which $N_2 = -N_1$.** The last important case of stress in two dimensions occurs when $\mathfrak{K}=0$, $\mathfrak{D}=0$; $\mathfrak{J}$ being negative. We have then $N_2 = -N_1$. Assuming that N_1 is positive, and denoting it by N, the system of stress-ellipses (54) reduces to the system of circles

$$\xi^2 + \eta^2 = N^2 \sin^2 a \qquad (59)$$

as in § 180. Also the director hyperbolas (60) and (61) become the *rectangular* hyperbolas

$$\left.\begin{aligned} \xi^2 - \eta^2 &= N \\ \eta^2 - \xi^2 &= N \end{aligned}\right\} \qquad \begin{aligned} &(63) \\ &(64) \end{aligned}$$

If OP be the radius of the circle (59) which coincides with the projection of the normal to any one of the corresponding system of planes, and if OP' be the radius making the same angle as OP with $O\xi$, on the opposite side of it, OP' represents in magnitude and direction the resultant stress across the plane. The amount of this stress is therefore $\pm N \sin a$, and it may be normal, tangential, or oblique.

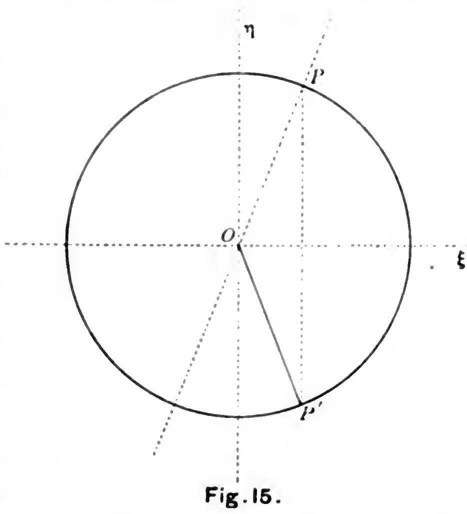


Fig. 15.

To determine its sign we must remember that every radius which meets (63) represents a **traction**, and every radius which meets (64) a **pressure.** Let OP, OQ be conjugate radii of (63) and (64), and let OY, OZ be perpendicular to them.

By the properties of the rectangular hyperbola, the asymptotes bisect the angles between the principal axes $O\xi$, $O\eta$; also OP and OQ are equally inclined to the asymptote which lies between them, and consequently OP and OZ are equally and oppositely inclined to $O\xi$, and OQ and OY to $O\eta$.

Now OP is the direction of the *traction* across any plane drawn through OQ, and therefore having the projection of its normal along OZ; and, similarly, OQ is the direction of the *pressure* across any plane drawn through OP, and therefore having the projection of its normal along OY.

In this case, therefore, the trace of a plane and the direction of the stress across it make equal angles with an asymptote; while the angle between the stress and the projection of the normal is bisected by $O\xi$ (the axis of *principal normal traction*), or by $O\eta$ (the axis of *principal normal pressure*), according as the stress is a traction or a pressure.

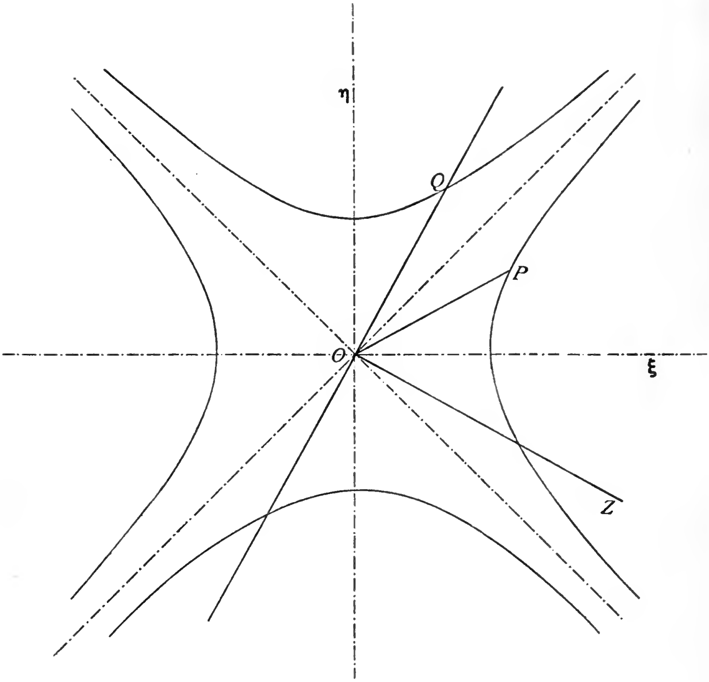


Fig. 16.

183.] Let us now follow the changes in the stress across a plane through O, as it moves round in such a manner that its normal describes a cone of semi-vertical angle α about $O\zeta$.

The *numerical magnitude* of the stress will of course always be the same—namely, $N \cdot \sin \alpha$. Let OQ (Fig. 16) represent the trace of the plane in any position, OZ the projection of the normal, and OP the direction of the stress. Then the angle $PO\xi$ is always equal to either of the angles $QO\eta$ and $ZO\xi$.

Let OQ coincide first of all with $O\eta$; then OZ and OP both coincide with $O\xi$. The normal component of the stress is a *traction* $N\sin^2\alpha$, and the tangential component $N\sin\alpha\cos\alpha$ acts along the line in which the plane is cut by that of $\zeta\xi$.

As OQ moves away from $O\eta$ towards the asymptote, the stress becomes more and more oblique, the angle POZ constantly increasing, until, when the plane actually passes through the asymptote, the normal traction has vanished altogether, and we have only a *shearing stress* of amount $N\sin\alpha$ acting along the asymptote.

As OQ passes the asymptote the normal component reappears as a *pressure*, OP having also passed the asymptote in the opposite direction. This normal component continually increases until, when OQ coincides with $O\xi$, and OP and OZ with $O\eta$, the resultant stress acts along $O\eta$, and consists of a normal *pressure* $N\sin^2\alpha$, together with a tangential component $N\sin\alpha\cos\alpha$ along the line of intersection of the given plane with that of $\eta\zeta$. This cycle of changes is then repeated in the reverse order until the trace of the plane once more coincides with $O\eta$.

184.] **Position of the Plane of the Stress.** Since this plane is perpendicular to the axis $O\zeta$ of zero stress, its direction-cosines referred to the arbitrary axes Ox, Oy, Oz are to be obtained by writing $N=0$ in equations (22).

We thus get the equations

$$\left.\begin{aligned} P\lambda + U\mu + T\nu &= 0 \\ U\lambda + Q\mu + S\nu &= 0 \\ T\lambda + S\mu + R\nu &= 0 \end{aligned}\right\} \qquad\qquad (65)$$

only two of which are independent, in virtue of the condition $\mathfrak{K} = 0$.

Taking this into account we find that equations (65) are equivalent to either of the pairs of equations

$$\mathfrak{s}\lambda = \mathfrak{t}\mu = \mathfrak{u}\nu \qquad\qquad (66)$$

or

$$\frac{\lambda}{\sqrt{\mathfrak{p}}} = \frac{\mu}{\sqrt{\mathfrak{q}}} = \frac{\nu}{\sqrt{\mathfrak{r}}} \qquad\qquad (67)$$

The equation of the Plane of the Stress in the case when $\mathfrak{K}=0$, referred to the arbitrary axes, may therefore be written in either of the forms

$$x\sqrt{\mathfrak{p}} + y\sqrt{\mathfrak{q}} + z\sqrt{\mathfrak{r}} = 0 \qquad\qquad (68)$$

or

$$\frac{x}{\mathfrak{s}} + \frac{y}{\mathfrak{t}} + \frac{z}{\mathfrak{u}} = 0 \qquad\qquad (69)$$

Stress in One Dimension.

185.] We now come finally to the case in which two roots of the discriminating cubic (21) vanish. If N denote the remaining root we must have

$$\left.\begin{array}{l}\mathfrak{J}=0\\ \mathfrak{K}=0\end{array}\right\}\ \dots\dots\dots\dots\dots\dots\dots\dots (70)$$

and $$\mathfrak{D}=N \dots\dots\dots\dots\dots\dots\dots\dots (71)$$

Supposing N_1 and N_2 to be the vanishing Principal Stresses, equations (39) show that at the extremity of every radius which represents the stress across a real plane we must have

$$\xi=0, \quad \eta=0.$$

Thus the resultant stress across *every* plane that can be drawn through O acts along $O\zeta$. The stress is therefore said to be in one dimension, and $O\zeta$ is called the Principal Axis of the Stress at O, the other two being indeterminate.

The third of equations (39) shows that the resultant stress across any plane (λ, μ, ν) is represented in magnitude and direction by the length

$$\zeta=\nu N \dots\dots\dots\dots\dots\dots\dots\dots (72)$$

measured along $O\zeta$.

If then we describe a circular cone about $O\zeta$ with semi-vertical angle a, the resultant stress across every plane whose normal lies in this cone is given by

$$\zeta=N\cos a \dots\dots\dots\dots\dots\dots\dots\dots (73)$$

Thus the stress is zero across every plane passing through $O\zeta$; and it follows that in this case no plane through O can suffer pure shearing stress.

If we describe about O a sphere of radius N, the projection on $O\zeta$ of the radius of the sphere coinciding with the normal to any plane represents in magnitude and direction the resultant stress across that plane.

N is obviously the maximum stress, and the stress across every plane through O is of the same sign as N or $\mathfrak{D}$.

186.] **Direction of the Axis.** The direction-cosines of $O\zeta$, referred to the arbitrary axes Ox, Oy, Oz, are given by equations (22).

Now it is well known that the conditions (70), when satisfied simultaneously, are equivalent to either of the sets of three

$$\mathfrak{p}=\mathfrak{q}=\mathfrak{r}=0 \dots\dots\dots\dots\dots\dots\dots\dots (74)$$

or $$\mathfrak{s}=\mathfrak{t}=\mathfrak{u}=0 \dots\dots\dots\dots\dots\dots\dots\dots (75)$$

that is, by

$$QR - S^2 = RP - T^2 = PQ - U^2 = 0,$$

or by the equivalent set

$$TU - PS = US - QT = ST - RU = 0.$$

Thus equations (22) may be written in either of the forms

$$\left.\begin{array}{l} S\lambda = T\mu = U\nu \quad \text{..............................(76)} \\ \text{or} \quad \dfrac{\lambda}{\sqrt{P}} = \dfrac{\mu}{\sqrt{Q}} = \dfrac{\nu}{\sqrt{R}} \quad \text{..............................(77)} \end{array}\right\}$$

and the equations of the Axis of the Stress are

$$\left.\begin{array}{l} Sx = Ty = Uz \quad \text{..............................(78)} \\ \text{or} \quad \dfrac{x}{\sqrt{P}} = \dfrac{y}{\sqrt{Q}} = \dfrac{z}{\sqrt{R}} \quad \text{..............................(79)} \end{array}\right\}$$

187.] **Heterogeneous Stress.** In general the standard components of the stress will vary from point to point of the body, all of them (§ 137) being continuous functions of the coordinates (x, y, z) of the point at which they act. The Principal Normal Stresses, and the direction-cosines of the Principal Axes at each point will therefore also be continuous functions of its coordinates.

All these theorems that we have proved for the Stress at the origin will be equally true (§ 158) for the Stress at any point P, if we refer the Quadrics, etc., to the principal axes at P, or to a system of axes through P parallel to the arbitrary axes Ox, Oy, Oz.

EXAMPLES.

1. Discuss the properties of the following stresses:—

(*i.*) $\{3a, \ -a, \ -a, \ 2a, \ 0, \ 0\}$;
(*ii.*) $\{0, \ 0, \ 0, \ a, \ a, \ a\}$;
(*iii.*) $\{a, \ a, \ 0, \ a, \ a, \ a\}$;
(*iv.*) $\{a, \ 0, \ 0, \ a, \ a, \ a\}$;
(*v.*) $\{13a, \ 10a, \ 5a, \ -6a, \ -3a, \ -2a\}$;
(*vi.*) $\{3a, \ -a, \ -2a, \ 3a, \ -a, \ -2a\}$.

Show that the principal normal stresses are respectively:—

(*i.*) $3a$, $3a$, $-a$;
(*ii.*) $2a$, $-a$, $-a$;
(*iii.*) $(\sqrt{3}+1)a$, $-(\sqrt{3}-1)a$, 0;
(*iv.*) $(\sqrt{2}+1)a$, $-(\sqrt{2}-1)a$, 0;
(*v.*) $14a$, $14a$, 0;
(*vi.*) $\sqrt{7}a$, $-\sqrt{7}a$, 0.

2. Prove that if through any point of a strained body a system of planes be drawn, such that the normal component of the stress across each has a given value **N**, the normals to these planes will generate a quadric cone.

3. If the stress be in two dimensions at the origin, and the plane of xy be made to coincide with the plane of the stress, show that

(*i.*) The principal normal stresses N_1, N_2 are the roots of the quadratic

$$(\phi - P)(\phi - Q) - U^2 = 0.$$

(*ii.*) The angles ψ_1, ψ_2 which $O\xi$ and $O\eta$ make with Ox are the roots of

$$\tan 2\psi = \frac{2U}{P-Q}.$$

(*iii.*) If $PQ > U^2$, four planes can be drawn through O at a given inclination a to the plane of the stress, so that the stress across each shall have a given normal component **N** (provided of course that **N** is taken within proper limits); and the projections on the plane of the shear of the normals to these four planes lie in the lines

$$(P - \mathbf{N}\operatorname{cosec}^2 a)x^2 + (Q - \mathbf{N}\operatorname{cosec}^2 a)y^2 + 2Uxy = 0.$$

Hence deduce the limits of **N** for a given value of a.

(*iv.*) What is the corresponding theorem when $PQ < U^2$?

(*v.*) Show that four planes can always be drawn at a given inclination a to the plane of the stress, such that the stress across each may have a given tangential component **T** (taken within proper limits).

Show that the projections on the plane of the stress of the normals to these planes are the lines

$$\begin{aligned}\sin^4 a(Px^2 + 2Uxy + Qy^2) + \mathrm{T}^2(x^2+y^2)^2 \\ = \sin^2 a(x^2+y^2)[(Px+Uy)^2 + (Ux+Qy)^2].\end{aligned}$$

Hence deduce the limits of T for a given value of a.

(*vi.*) Show that, for a given value of α, **N** is a maximum when the projection of the normal coincides with $O\xi$, and a minimum when it coincides with $O\eta$.

(*vii.*) Show that, for a given value of α, T is a minimum when the projection of the normal coincides with $O\xi$ or $O\eta$, and a maximum when it bisects either of the angles between these axes.

(*viii.*) Hence show that the two planes through O suffering greatest tangential stress are those which bisect the angles between the principal planes ξz and ηz.

4. Prove that, when two of the principal normal stresses are equal, the normals to those planes which suffer maximum tangential stress are all inclined at an angle of 45° to the direction of the third principal stress.

5. Show that, in general, the normals to planes through the origin, the stress across which has a given tangential component **T**, lie on the cone whose equation, referred to the principal axes, is

$$(\xi^2 + \eta^2 + \zeta^2)\{(N_1^2 - \mathrm{T}^2)\xi^2 + (N_2^2 - \mathrm{T}^2)\eta^2 + (N_3^2 - \mathrm{T}^2)\zeta^2\}$$
$$- (N_1\xi^2 + N_2\eta^2 + N_3\zeta^2)^2 = 0.$$

6. Show that if r be any radius vector of the surface

$$(N_2 - N_3)^2\eta^2\zeta^2 + (N_3 - N_1)^2\zeta^2\xi^2 + (N_1 - N_2)^2\xi^2\eta^2 = 1,$$

and **T** the tangential component of the stress at the origin across the section drawn through it perpendicular to r, then

$$\mathrm{T}r^2 = 1.$$

Show that the sections of this surface by the principal planes are conjugate rectangular hyperbolas, having the principal axes for their asymptotes. Hence, or otherwise, prove that the maximum tangential stress is suffered by the two planes $(1/\sqrt{2},\ 0,\ \pm 1/\sqrt{2})$, and that this maximum is $\frac{1}{2}(N_1 - N_3)$; N_1, N_2, N_3 being in descending order of magnitude.

CHAPTER IV.

POTENTIAL ENERGY OF STRAIN.

188.] **Introductory.** We saw in Chapter I. (§§ 21, 26, 27) that the Potential Energy of a perfectly elastic body, due to Strain produced at constant temperature, must always be equal to the work expended by external forces (including Applied Forces and Surface Tractions) in producing the strain; that this work (§ 31) is done against the Resistance (§ 135) offered by the body to stress, and is therefore equal to the work done by the Stresses (§ 135) during the Strain; and finally (§§ 27, 29, 34) that the Potential Energy and the Stress in any given state of the body are functions only of the actually existing Strain.

It is obvious that, since our new definition of Stress (§§ 131, 135) retains its essential characteristic (§ 29) of a purely mutual action between the component parts of the body, these theorems are as true for the perfectly elastic continuous mass with which we are now dealing as for the perfectly elastic molecular structure which we considered in *Section ii.* of Chapter I.

The course now to be taken by our investigation will therefore be as follows:—Regarding the six component stresses as functions only of the six analogous components of the strain which they suffice to maintain, under the given system of external forces, we shall first find an expression for the work done by them during an elementary increase in each of these components. This expression will involve the stresses and the increments of the strains, and we shall show that, in virtue of equations (3) and (5) of Chapter III., it is identically equal to the work that must be done by the Applied Forces and Surface Tractions, to produce the small increments in the displacements of their points of application which constitute the increment of Strain. We shall next employ the principle of superposition of small strains (§ 87) and stresses (§ 155) to express the six standard components of a small stress in terms of the six components of the corresponding small strain; and then, by eliminating the stresses from the

expression just found, we shall obtain the *differential* of the Potential Energy of strain, expressed as the *differential* of a function of the six component strains. Finally, integrating this from the natural state of the body $\{0, 0, 0, 0, 0, 0\}$ to the given state of strain $\{e, f, g, a, b, c\}$, we shall obtain the Potential Energy in the latter state as a function of e, f, g, a, b, c.

Work done by Stress during a small arbitrary variation of the Strain.

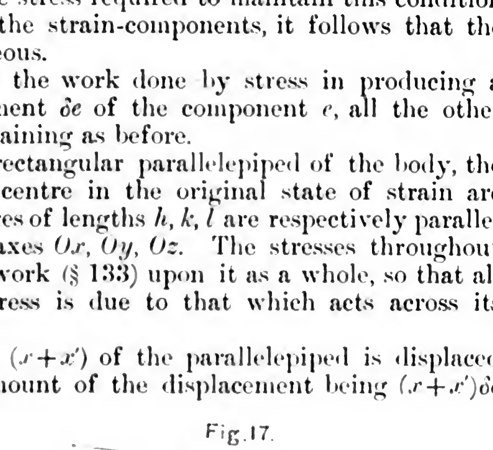
189.] **Work done in increasing a simple elongation.** Let us first suppose the body to be in equilibrium in the state of *homogeneous* strain $\{e, f, g, a, b, c\}$. Since the components $\{P, Q, R, S, T, U\}$ of the stress required to maintain this condition are functions only of the strain-components, it follows that the stress also is homogeneous.

Let us investigate the work done by stress in producing a small arbitrary increment δe of the component e, all the other strain-components remaining as before.

Consider a finite rectangular parallelepiped of the body, the coordinates of whose centre in the original state of strain are (x, y, z), and whose edges of lengths h, k, l are respectively parallel to the fixed arbitrary axes Ox, Oy, Oz. The stresses throughout its interior can do no work (§ 133) upon it as a whole, so that all the work done by stress is due to that which acts across its bounding surface.

Again, every point $(x+x')$ of the parallelepiped is displaced parallel to Ox, the amount of the displacement being $(x+x')\delta e$ [§ 88 and § 89, (i.)]. Hence only those components of the stress across its surface can do work which act parallel to Ox.

Fig. 17.



Now if we take a slice of elementary thickness, bounded by the parallel planes x' and $x' + dx$, every point in its perimeter suffers the same displacement; and, the stress-components at each point of this perimeter being as represented in Figure 17, the forces

acting parallel to Ox on the four edges of the slice are respectively

$$\left.\begin{array}{r} T\,.\,kdx' \\ U\,.\,ldx' \\ -T\,.\,kdx' \\ -U\,.\,ldx' \end{array}\right\}$$

It is therefore obvious that these forces together can do no work in such a displacement: and, this being true for every such slice, it follows that the only forces which can do any work on the parallelepiped, in increasing the elongation e, are the normal components of the *tensions* acting across the ends perpendicular to Ox.

Since P is a function of the strain-components, it will be altered to

$$P + \frac{\partial P}{\partial e}\,.\,\delta e$$

by the small increase of e.

Hence the work done *by* the tension acting across the *positive* end lies between

$$\left.\begin{array}{ll} & P\,.\,kl\left(x+\frac{h}{2}\right)\delta e \\ \text{and} & \left(P+\frac{\partial P}{\partial e}\,.\,\delta e\right)kl\left(x+\frac{h}{2}\right)\delta e \end{array}\right\}.$$

Similarly, the work done *against* the tension which acts across the *negative* end lies between

$$\left.\begin{array}{ll} & P\,.\,kl\left(x-\frac{h}{2}\right)\delta e \\ \text{and} & \left(P+\frac{\partial P}{\partial e}\,.\,\delta e\right)kl\left(x-\frac{h}{2}\right)\delta e \end{array}\right\}.$$

Hence, on the whole, the work done *by* stress lies between

$$\left.\begin{array}{ll} & P\delta e\,.\,hkl \\ \text{and} & \left(P+\frac{\partial P}{\partial e}\,.\,\delta e\right)\delta e\,.\,hkl \end{array}\right\}.$$

Thus, neglecting the square of δe, the whole work done by Stress in producing the small increment δe of the single component e is

$$P\delta e\,.\,hkl.$$

190.] **Strain and Stress Heterogeneous.** In precisely the same manner we may show that, if the strain be not homogeneous, the work done by stress on the *elementary* rectangular

parallelepiped $dxdydz$ having its centre at (x, y, z), in producing a small increment δe of the elongation of this element parallel to Ox, is simply

$$P\delta e \,.\, dxdydz \quad\text{.....................(1)}$$

where $P, e, \delta e$ are continuous functions of x, y, z.

The work done on the whole body by stress in producing any continuously distributed (but otherwise perfectly arbitrary) small variation δe in the elongation e throughout it is therefore

$$\iiint P\delta e \,.\, dxdydz \quad\text{.....................(2)}$$

and we see that, in varying a simple elongation, only the corresponding longitudinal traction (§ 148) can do any work.

Hence P, Q, R are the Simple Stresses (§ 33) corresponding to the Simple Strains e, f, g.

191.] **Work done in increasing a Simple Shear.** Let us next suppose the component shear a to suffer a small increment δa, the other components retaining their initial values.

If $O\xi$ and $O\zeta$ be the internal and external bisectors of the angle yOz we know (§§ 92, 100) that the shear a or $2s_1$ in the plane of yz is equivalent to an elongation s_1 in the direction of $O\xi$, together with a contraction s_1, or an elongation $(-s_1)$, in the direction of $O\zeta$. The small increment δa of the shear may therefore be resolved into the small increment δs_1 of the elongation parallel to $O\xi$, together with the small increment δs_1 of the contraction, or increment $(-\delta s_1)$ of the elongation parallel to $O\zeta$.

Again, a shearing stress of amount S in the plane of yz (or any parallel plane) may be resolved (§§ 150-152) into a longitudinal traction S parallel to $O\xi$, together with a longitudinal pressure S, or traction $(-S)$ parallel to $O\zeta$.

Hence, by superposition, we deduce from the last Article that the work done by stress on the element $dxdydz$, with its centre at the point (x, y, z) of the body, in producing the small increment δa of the shear of the element in the plane of yz, is

$$S\delta s_1 \,.\, dxdydz + (-S)(-\delta s_1)dxdydz\,;$$

that is
$$2S\delta s_1 \,.\, dxdydz,$$

or
$$S\delta a \,.\, dxdydz \quad\text{.....................(3)}$$

where $S, a, \delta a$ are continuous functions of x, y, z; δa being otherwise arbitrary.

The work done on the whole body by Stress in producing such a change throughout it is therefore

$$\iiint S\delta a \,.\, dxdydz \quad\text{.....................(4)}$$

and it follows that, in varying a simple shear, only the corresponding shearing stress (§ 149) can do any work.

Hence S, T, U are the Simple Stresses (§ 33) corresponding to the Simple Strains a, b, c.

192.] **Work done by Stress in any small arbitrary variation of the Strain.** Superposing (2) and (4), and the analogous formulæ for f, g, b, c, we see finally that the work done by Stress in producing small arbitrary and independent variations of all the strain-components throughout the body, such that the strain at any point (x, y, z) is altered from

$$\{e, f, g, a, b, c\}$$

to
$$\{e+\delta e, f+\delta f, g+\delta g, a+\delta a, b+\delta b, c+\delta c\}$$

is given by

$$\delta W = \iiint [P\delta e + Q\delta f + R\delta g + S\delta a + T\delta b + U\delta c]\,dx\,dy\,dz \ldots\ldots\ldots (5)$$

where $\{P, Q, R, S, T, U\}$ is the specification of the stress required to maintain the body in equilibrium in its original state of strain, under the given external forces.

Work done by the Applied Forces and Surface Tractions in producing a small variation of the Strain.

193.] **Expression for this Work.** As in the last Chapter, let X, Y, Z represent the components of the Applied Force per unit mass at the point (x, y, z) in the interior of the body, ρ the density at the same point, and F, G, H the components of the Surface Traction per unit area applied to the element dS of the bounding surface: these systems of forces and tractions constituting the system of "external forces" which, with the distribution of stress $\{P, Q, R, S, T, U\}$, holds the body in equilibrium in the original state of strain $\{e, f, g, a, b, c\}$.

Let the effect of the small arbitrary variation of the strain, considered in the last Article, be to change the component displacements u, v, w of any point (x, y, z), in the interior of the body or on its surface, to $u+\delta u$, $v+\delta v$, $w+\delta w$.

Then, by the principle of virtual velocities, the work that must be done by the external forces to produce this change is

$$\iiint \rho(X\delta u + Y\delta v + Z\delta w)\,dx\,dy\,dz + \iint (F\delta u + G\delta v + H\delta w)\,dS \ldots\ldots (6)$$

where the triple integral is taken throughout the volume of the body, and the double integral over the whole of its bounding surface.

By reasoning as in §§ 153, 154, we may show that it is indifferent, to the degree of approximation which we adopt for small strains, whether the integrals in expressions (5) and (6) be taken throughout the volume and over the surface of the body in

its natural or in its strained state; indeed the triple integral in (6) being integrated as to the element of *mass*, is absolutely identical in the two cases (§ 154).

We shall always suppose, for the sake of simplicity, that in these and similar cases *triple integrals are integrated throughout the volume and double integrals over the surface of the unstrained body.*

194.] **Identity of the two expressions for Work done in varying Strain.** Substituting for e, f, g, a, b, c in (5) from equations (59) of § 123, we get

$$\left.\begin{aligned}\delta e &= \delta\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\delta u, \text{ etc.}\\ \delta a &= \delta\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) = \frac{\partial}{\partial y}\delta w + \frac{\partial}{\partial z}\delta v, \text{ etc.}\end{aligned}\right\}\quad\text{.....................(7)}$$

and thus (5) becomes

$$\begin{aligned}\delta W = \iiint\Big\{ & P\,.\,\frac{\partial}{\partial x}\delta u + Q\,.\,\frac{\partial}{\partial y}\delta v + R\,.\,\frac{\partial}{\partial z}\delta w\\ & + S\left(\frac{\partial}{\partial y}\delta w + \frac{\partial}{\partial z}\delta v\right) + T\left(\frac{\partial}{\partial z}\delta u + \frac{\partial}{\partial x}\delta w\right)\\ & + U\left(\frac{\partial}{\partial x}\delta v + \frac{\partial}{\partial y}\delta u\right)\Big\}\,dxdydz\\ = \iiint\Big\{ & P\,.\,\frac{\partial}{\partial x}\delta u + U\,.\,\frac{\partial}{\partial x}\delta v + T\,.\,\frac{\partial}{\partial x}\delta w\\ & + U\,.\,\frac{\partial}{\partial y}\delta u + Q\,.\,\frac{\partial}{\partial y}\delta v + S\,.\,\frac{\partial}{\partial y}\delta w\\ & + T\,.\,\frac{\partial}{\partial z}\delta u + S\,.\,\frac{\partial}{\partial z}\delta v + R\,.\,\frac{\partial}{\partial z}\delta w\Big\}\,dxdydz.\end{aligned}$$

Integrating by parts, as in § 146,

$$\begin{aligned}\delta W = & \iint(P\delta u + U\delta v + T\delta w)\lambda dS\\ & - \iiint\left(\frac{\partial P}{\partial x}\delta u + \frac{\partial U}{\partial x}\delta v + \frac{\partial T}{\partial x}\delta w\right)dxdydz\\ & + \iint(U\delta u + Q\delta v + S\delta w)\mu dS\\ & - \iiint\left(\frac{\partial U}{\partial y}\delta u + \frac{\partial Q}{\partial y}\delta v + \frac{\partial S}{\partial y}\delta w\right)dxdydz\\ & + \iint(T\delta u + S\delta v + R\delta w)\nu dS\\ & - \iiint\left(\frac{\partial T}{\partial z}\delta u + \frac{\partial S}{\partial z}\delta v + \frac{\partial R}{\partial z}\delta w\right)dxdydz\end{aligned}$$

Rearranging the order of the terms,

$$\delta W = \iint\{(P\lambda + U\mu + T\nu)\delta u + (U\lambda + Q\mu + S\nu)\delta v + (T\lambda + S\mu + R\nu)\delta w\}dS$$
$$-\iiint\left\{\left(\frac{\partial P}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial T}{\partial z}\right)\delta u + \left(\frac{\partial U}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial S}{\partial z}\right)\delta v + \left(\frac{\partial T}{\partial x}+\frac{\partial S}{\partial y}+\frac{\partial R}{\partial z}\right)\delta w\right\}dxdydz \quad \text{......(8)}$$

Hence, in virtue of equations (3) and (5) of Chapter III., we have finally

$$\delta W = \iint(F\delta u + G\delta v + H\delta w)dS + \iiint\rho(X\delta u + Y\delta v + Z\delta w)dxdydz \quad \text{.... (9)}$$

Thus the work done by Stress during an infinitely small change of Strain is always equal to the work done on the body by external forces in producing the change; and either is of course equal to the corresponding infinitely small increase of the Potential Energy of the Strain.

195.] **Case in which motion is taking place.** If relative motion of parts of the body is taking place, so that the initial and final states of strain are only states through which the body passes; as, for instance, when the body is vibrating about a stable state of strained equilibrium, maintained by suitable forces; we may show by employing equations (4) of Chapter III., that the expression (6) for the work done on the body by the external forces is equal to the increase δW of the potential energy, together with the accompanying increase of the kinetic energy $\mathfrak{T}$.

This latter is of course

$$\delta\mathfrak{T} = \delta\iiint\tfrac{1}{2}\rho(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)dxdydz,$$

or, since u, v, w are the *variable* portions of the coördinates of any point,

$$= \delta\iiint\tfrac{1}{2}\rho(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)dxdydz$$
$$= \iiint\rho(\dot{u}\ddot{u} + \dot{v}\ddot{v} + \dot{w}\ddot{w})\delta t\,.\,dxdydz,$$

where δt is the small interval of time occupied by the change. This again is equal to

$$\iiint\rho(\ddot{u}\delta u + \ddot{v}\delta v + \ddot{w}\delta w)dxdydz \quad \text{..................(10)}$$

Thus the expression (6) for the work done by the external forces—which must now of course be equal to the total change of energy, both potential and kinetic—diminished by the expres-

sion (10) for the corresponding increase of kinetic energy alone, becomes

$$\iiint \rho\{(X-\ddot{u})\delta u + (Y-\ddot{v})\delta v + (Z-\ddot{w})\delta w\}dxdydz$$
$$+\iint (F\delta u + G\delta v + H\delta w)dS \ldots\ldots(11)$$

By equations (4) and (5) of Chapter III., (11) is identical with (8), and therefore with (5).

Potential Energy of Strain.

196.] **Energy per Unit Volume.** Let W be the total potential energy of the body when held in equilibrium in the state of strain $\{e, f, g, a, b, c\}$ by the distribution of stress $\{P, Q, R, S, T, U\}$.

Also let V denote the measure of this potential energy per unit volume of the unstrained body, so that

$$W = \iiint V dxdydz \ldots\ldots(12)$$

We have shewn that the infinitesimal increment of V, due to arbitrary and independent infinitesimal increments of the strain-components, is given by

$$\delta V = P\delta e + Q\delta f + R\delta g + S\delta a + T\delta b + U\delta c \ldots\ldots(13)$$

Now the potential energy and the components of the stress in any given state of strain are functions only of the components of that strain. Hence when the increments of the strain-components in (13) are sufficiently reduced, each side must become the perfect differential of some function V of the six independent variables e, f, g, a, b, c.

Thus we may write

$$dV = Pde + Qdf + Rdg + Sda + Tdb + Udc \ldots\ldots(14)$$

and also

$$dV = \frac{\partial V}{\partial e}de + \frac{\partial V}{\partial f}df + \frac{\partial V}{\partial g}dg + \frac{\partial V}{\partial a}da + \frac{\partial V}{\partial b}db + \frac{\partial V}{\partial c}dc \ldots\ldots(15)$$

whence we deduce that

$$\left.\begin{aligned} P = \frac{\partial V}{\partial e},\quad Q = \frac{\partial V}{\partial f},\quad R = \frac{\partial V}{\partial g} \\ S = \frac{\partial V}{\partial a},\quad T = \frac{\partial V}{\partial b},\quad U = \frac{\partial V}{\partial c} \end{aligned}\right\} \ldots\ldots(16)$$

(*Compare* §§ 32, 33. *See Errata for those Articles.*)

197.] **Stress in terms of Strain. "Hooke's Law."** Since stress is a function only of the strain ultimately produced by it, it follows that if a single small stress $\{P, Q, R, S, T, U\}$ produce the small strain $\{e, f, g, a, b, c\}$, then two small stresses, each equal to $\{P, Q, \text{etc.}\}$, applied successively to the body, will produce two successive small strains, each equal to $\{e, f, \text{etc.}\}$. But, by the principle of superposition, the two successive small stresses are equivalent to a single small stress $\{2P, 2Q, 2R, 2S, 2T, 2U\}$, and the two successive small strains to a single small strain $\{2e, 2f, 2g, 2a, 2b, 2c\}$.

Thus the single stress $\{2P, 2Q, \text{etc.}\}$ will produce and maintain the strain $\{2e, 2f, \text{etc.}\}$.

This result may obviously be extended so long as the strain and stress remain small, so that ultimately we see that, if n be any finite multiplier, the stress,

$$\{nP, nQ, nR, nS, nT, nU\}$$

will suffice to maintain the strain

$$\{ne, nf, ng, na, nb, nc\}.$$

Hence we deduce, solely from the principle of superposition of small strains and stresses, that if a perfectly elastic solid be in equilibrium in a given state of small strain, under a given small stress, and if the strain be increased in any finite ratio, the stress required to maintain it will be increased in the same ratio.

In other words, *the six components of stress are linear functions of the six components of the corresponding strain.*

This law was discovered *experimentally* by Robert Hooke, and first made public by him in 1678. (For the various ways in which it has been arrived at *theoretically, see* Appendix III., *below.*)

198.] **Coefficients of Elasticity.** From equations (16) we see that the partial derivatives of V as to each of the strain-components must in general be linear functions of all the six components.

And, finally, it appears that the potential energy per unit volume of a perfectly elastic solid under *small* strain is a *homogeneous quadratic function* of the six component strains.

We may then assume

$$\left.\begin{aligned}2V = \kappa_{11}e^2 + \kappa_{22}f^2 + \kappa_{33}g^2 + \kappa_{44}a^2 + \kappa_{55}b^2 + \kappa_{66}c^2 \\ + 2\kappa_{23}fg + 2\kappa_{31}ge + 2\kappa_{12}ef \\ + 2\kappa_{56}bc + 2\kappa_{64}ca + 2\kappa_{45}ab \\ + 2\kappa_{14}ea + 2\kappa_{15}eb + 2\kappa_{16}ec \\ + 2\kappa_{24}fa + 2\kappa_{25}fb + 2\kappa_{26}fc \\ + 2\kappa_{34}ga + 2\kappa_{35}gb + 2\kappa_{36}gc\end{aligned}\right\}\dots\dots\dots(17)$$

where the 21 "Elastic Coefficients" are, for a homogeneous body, absolute constants, depending only on the elastic properties of the body, the constant temperature at which it is maintained, and the directions of the arbitrarily chosen axes of reference.

If the body be not homogeneous, the coefficients will be functions also of the position of the point in the neighbourhood of which V is given by (17). We shall, however, always suppose that we are dealing with naturally homogeneous bodies (§ 43, *but see* § 220, *below*).

In general, the coefficients must be supposed all independent of one another; and in fact we cannot with certainty attribute to them any property whatever, except that they are finite, and that for every possible form of small strain they must make V positive (§ 21).

Differentiating (17), and substituting in (16), we get

$$\left.\begin{aligned} P &= \kappa_{11}e + \kappa_{12}f + \kappa_{13}g + \kappa_{14}a + \kappa_{15}b + \kappa_{16}c \\ Q &= \kappa_{21}e + \kappa_{22}f + \kappa_{23}g + \kappa_{24}a + \kappa_{25}b + \kappa_{26}c \\ R &= \kappa_{31}e + \kappa_{32}f + \kappa_{33}g + \kappa_{34}a + \kappa_{35}b + \kappa_{36}c \\ S &= \kappa_{41}e + \kappa_{42}f + \kappa_{43}g + \kappa_{44}a + \kappa_{45}b + \kappa_{46}c \\ T &= \kappa_{51}e + \kappa_{52}f + \kappa_{53}g + \kappa_{54}a + \kappa_{55}b + \kappa_{56}c \\ U &= \kappa_{61}e + \kappa_{62}f + \kappa_{63}g + \kappa_{64}a + \kappa_{65}b + \kappa_{66}c \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(18)$$

where $\kappa_{12} = \kappa_{21}$, etc., the double notation being employed solely for the sake of symmetry.

199.] Average Stress during change of Strain. Hence we find, by comparing (18) with (17), as we might have deduced directly from (16) by Euler's theorem on homogeneous functions,

$$V = \tfrac{1}{2}(Pe + Qf + Rg + Sa + Tb + Uc) \ldots\ldots\ldots\ldots\ldots\ldots(19)$$

whence by (12)

$$W = \tfrac{1}{2}\iiint(Pe + Qf + Rg + Sa + Tb + Uc)\,dx\,dy\,dz \ldots\ldots\ldots(20)$$

From (14) we have, by integration,

$$V = \int_{\{0,\,0,\,0,\,0,\,0,\,0\}}^{\{e,\,f,\,g,\,a,\,b,\,c\}} (P\,de + Q\,df + R\,dg + S\,da + T\,db + U\,dc).$$

Hence the interpretation of (19) is that the *average value* of the stress, while the body is being brought from its natural state to the state of strain $\{e, f, g, a, b, c\}$ is $\{\tfrac{1}{2}P, \tfrac{1}{2}Q, \tfrac{1}{2}R, \tfrac{1}{2}S, \tfrac{1}{2}T, \tfrac{1}{2}U\}$; that is, one-half of the stress required to maintain it in the specified state of strain.

This might also have been deduced directly from the principle of superposition. For in each of the intermediate states of strain the stress (being always a function only of the actually existing

strain) must be such as would keep the body in equilibrium in that state; but, by the principle of superposition, if we have any number of states of strain, and the corresponding stresses given, the average of all these stresses will suffice to maintain equilibrium in the state of strain which is the average of all the given states. Now, the path by which a perfectly elastic solid is brought to a given state of strain being without effect on the stress required to maintain it in its final state, the average value of the strain may be taken to be simply $\{\frac{1}{2}e, \frac{1}{2}f, \frac{1}{2}g, \frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c\}$, and the stress corresponding to this is, by § 197,

$$\{\tfrac{1}{2}P, \tfrac{1}{2}Q, \tfrac{1}{2}R, \tfrac{1}{2}S, \tfrac{1}{2}T, \tfrac{1}{2}U\}.$$

This latter expression therefore represents the average value of the stress during the change.

200.] **Strain in terms of Stress.** By elimination between equations (18) we can obtain the six component strains as linear functions of the six component stresses

$$\left.\begin{array}{c} e = K_{11}P + K_{12}Q + K_{13}R + K_{14}S + K_{15}T + K_{16}U \\ \text{etc., etc.} \\ a = K_{41}P + K_{42}Q + K_{43}R + K_{44}S + K_{45}T + K_{46}U \\ \text{etc., etc.} \end{array}\right\} \qquad (21)$$

where $K_{12} = K_{21}$, etc.

Substituting in (19) we obtain V as a homogeneous quadratic function of the stress-components.

$$\begin{aligned} 2V = K_{11}P^2 + K_{22}Q^2 + K_{33}R^2 + K_{44}S^2 + K_{55}T^2 + K_{66}U^2 \\ + 2K_{23}QR + 2K_{31}RP + 2K_{12}PQ \\ + 2K_{56}TU + 2K_{64}US + 2K_{45}ST \\ + 2K_{14}PS + 2K_{15}PT + 2K_{16}PU \\ + 2K_{24}QS + 2K_{25}QT + 2K_{26}QU \\ + 2K_{34}RS + 2K_{35}RT + 2K_{36}RU \end{aligned} \qquad (22)$$

whence, by differentiation and comparison with (21),

$$\left.\begin{array}{c} e = \dfrac{\partial V}{\partial P},\ f = \dfrac{\partial V}{\partial Q},\ g = \dfrac{\partial V}{\partial R} \\ a = \dfrac{\partial V}{\partial S},\ b = \dfrac{\partial V}{\partial T},\ c = \dfrac{\partial V}{\partial U} \end{array}\right\} \qquad (23)$$

201.] **Asymmetrical Elasticity.** We have defined a homogeneous body (§ 43), in the most general terms, as being such that any two equal and similar portions, *similarly situated* in the body, possess identical elastic properties. In the most general case of homogeneity we may therefore suppose the elastic properties of the body to vary in different directions; that is to say, the specification of the stress required to maintain a given strain

will depend not only on the specification of the strain but also on the directions of the axes of reference. The equations of the last three Articles are applicable to this most general case of *Asymmetrical Homogeneity;* the 21 elastic coefficients, and also the 21 reciprocal coefficients $K_{11} \ldots K_{66}$ (which are functions of the former), being taken to be all independent of one another and of the *position* of the origin, but varying with the *directions* of the axes of reference.

Crystalline Symmetry.

202.] **Planes and Axes of Rectangular Symmetry.** Many natural solids are found to possess different degrees of symmetry in their elastic properties. Such solids are in general called *crystalline*, and their elastic symmetry is found to be in invariable relation to certain lines and planes connected with their constant external form of crystallisation. We now proceed to investigate the analytical conditions for various degrees of elastic symmetry, confining ourselves to the cases in which the lines and planes of symmetry are *rectangular.*

203.] **One Plane of Symmetry.** Let us suppose the elastic properties of the body symmetrical about the plane of xy, or any parallel plane (*see* § 201): so that, for example, the specification referred to Ox, Oy, Oz of the stress required to maintain a given uniform elongation in the direction (λ, μ, ν) will be the same as the specification referred to Ox, Oy, and Oz *reversed*, of the stress required to maintain an equal elongation in the direction $(\lambda, \mu, -\nu)$. This latter becomes (λ, μ, ν) when Oz is reversed; and since we know by § 113 that the specification of the elongation depends only on (λ, μ, ν), it follows that the condition that xy may be a plane of elastic symmetry is that the reversal of Oz leaves unaltered the specification of stress in terms of the specification of strain.

Consequently the expression (17) for the potential energy in terms of the strain-components must also remain unchanged when Oz is reversed.

Now the effect of reversing Oz is to change the signs of z and w hence by § 123 the signs of a and b are reversed, the other components remaining as before. Thus if xy is a plane of symmetry all those terms in the expression for V which contain odd powers of a and b, except their product ab, must vanish.

$$\left.\begin{array}{l} \kappa_{56} = 0, \;\; \kappa_{64} = 0 \\ \kappa_{14} = 0, \;\; \kappa_{15} = 0 \\ \kappa_{24} = 0, \;\; \kappa_{25} = 0 \\ \kappa_{34} = 0, \;\; \kappa_{35} = 0 \end{array}\right\}.$$

I

And, finally,

$$
\begin{aligned}
2V = \kappa_{11}e^2 + \kappa_{22}f^2 + \kappa_{33}g^2 + \kappa_{44}a^2 + \kappa_{55}b^2 + \kappa_{66}c^2 \\
+ 2(\kappa_{23}fg + \kappa_{31}ge + \kappa_{12}ef) + 2\kappa_{45}ab \\
+ 2(\kappa_{16}ec + \kappa_{26}fc + \kappa_{36}gc) \quad \text{......(24)}
\end{aligned}
$$

Thus the number of elastic coefficients is reduced to 13.

204.] **Three Planes of Symmetry.** By three successive applications of the results of the last Article, we may show that, if *all three* of the coördinate planes are planes of elastic symmetry, all the terms in V involving odd powers of a, b, or c must disappear. In addition therefore to the above conditions we must now have

$$
\left.\begin{aligned}
\kappa_{45} = 0 \\
\kappa_{16} = \kappa_{26} = \kappa_{36} = 0
\end{aligned}\right\}.
$$

Thus we may write

$$
\begin{aligned}
2V = \kappa_{11}e^2 + \kappa_{22}f^2 + \kappa_{33}g^2 + \kappa_{44}a^2 + \kappa_{55}b^2 + \kappa_{66}c^2 \\
+ 2(\kappa_{23}fg + \kappa_{31}ge + \kappa_{12}ef) \quad \text{......(25)}
\end{aligned}
$$

and the number of the elastic coefficients is reduced to 9.

This may be called *complete rectangular symmetry;* it belongs to the "tessaral" class of crystals whose form of crystallisation is a rectangular parallelepiped, the planes of elastic symmetry being parallel to the pairs of opposite faces.

Equations (18) become

$$
\left.\begin{aligned}
P &= \kappa_{11}e + \kappa_{12}f + \kappa_{13}g \\
Q &= \kappa_{21}e + \kappa_{22}f + \kappa_{23}g \\
R &= \kappa_{31}e + \kappa_{32}f + \kappa_{33}g \\
S &= \kappa_{44}a \\
T &= \kappa_{55}b \\
U &= \kappa_{66}c
\end{aligned}\right\} \quad \text{......(26)}
$$

by which we see that the relations between the elongations and normal tractions perpendicular to the "principal planes" (or planes of symmetry) of the crystal, and between the shear in each of these planes and the corresponding shearing stress form four independent systems.

205.] **One Axis of Symmetry.** Let us next suppose that there is one direction in the crystal about which its elastic properties have a certain degree of symmetry. Any line Oz drawn in this direction may be called the Axis of the crystal, and its elastic properties will be arranged with more or less symmetry in the plane of xy, or any other plane perpendicular to the axis. There are two principal degrees of such symmetry, which we will consider separately.

(*i.*) *Uniaxial Crystalline Symmetry.* In this case, which is common to Iceland Spar, and other crystals, called in Optics "uniaxial," there are two orthogonal planes through the crystalline axis, such that the elastic properties of the body are not only symmetrical about each (or about any planes parallel to either), but they also bear exactly the same relations to one as to the other. Thus these two planes (which we shall take for the planes of yz and zx) may be interchanged without affecting the form of the Potential Energy, or the relations of Stress and Strain.

It is thus obvious that V must involve e and f symmetrically, and also a and b. Thus we may write

$$2V = \kappa_{11}(e^2 + f^2) + \kappa_{33}g^2 + \kappa_{44}(a^2 + b^2) + \kappa_{66}c^2 + 2\kappa_{23}(fg + ge) + 2\kappa_{12}ef \ldots (27)$$

Thus the number of elastic coefficients is reduced to 6, and equations (18) become

$$\left.\begin{aligned} P &= \kappa_{11}e + \kappa_{12}f + \kappa_{23}g \\ Q &= \kappa_{12}e + \kappa_{11}f + \kappa_{23}g \\ R &= \kappa_{23}(e + f) + \kappa_{33}g \\ S &= \kappa_{44}a \\ T &= \kappa_{44}b \\ U &= \kappa_{66}c \end{aligned}\right\}.$$

Such crystals may be said to have *square symmetry* about their axis.

(*ii.*) *Complete Circular Symmetry about an Axis.* In this case, which does not occur in any natural crystal, but which is artificially brought about in wires drawn from masses of metal naturally possessing the highest degree of symmetry (*see* § 207, *below*), the elastic properties of the body are absolutely symmetrical *in all directions* perpendicular to the axis; so that, if this be Oz as before, it is absolutely indifferent in what directions we take Ox and Oy.

It is obvious that in this case the expression (27) for V must remain the same form when Ox and Oy are turned through any angle ω in their own plane. Let us take ω so small that its square and higher powers may be neglected: the effect of rotating the axes will then be to change x, y, u, v into $x + \omega y$, $y - \omega x$, $u + \omega v$, $v - \omega u$; z and w remaining unaltered. The effect on the strain-components will be to change e, f, g, a, b, c into $e + \omega c$, $f - \omega c$, g, $a - \omega b$, $b + \omega a$, $c - 2\omega e + 2\omega f$, respectively.

Hence, neglecting the square of ω, the expression (27) for $2V$ is transformed into

$$2V + 2\omega(\kappa_{11} - 2\kappa_{66} - \kappa_{12})(ec - fc).$$

The term involving ω must vanish for all values of ω, and there-

fore, since the strain-components must be assumed independent, we must have

$$\kappa_{12} = \kappa_{11} - 2\kappa_{66}.$$

Thus

$$2V = \kappa_{11}(e^2 + f^2) + \kappa_{33}g^2 + \kappa_{44}(a^2 + b^2) + \kappa_{66}c^2 + 2\kappa_{23}(fg + ge) + 2(\kappa_{11} - 2\kappa_{66})ef \quad \text{.......(28)}$$

and the number of the elastic coefficients is reduced to 5.

206.] **Three interchangeable Planes of Symmetry.** Let us now start afresh from the case of § 204, and suppose that the elastic properties of the body are not only symmetrical about the three coördinate planes, but that they also bear precisely the same relations to each of these planes. It is then evident that the coördinate axes may be interchanged in any manner without affecting the form of V—that is to say, the expression (25) must be so modified that it may involve e, f, g symmetrically, and also a, b, c. Thus we must have

$$\left.\begin{array}{l} \kappa_{11} = \kappa_{22} = \kappa_{33} \\ \kappa_{44} = \kappa_{55} = \kappa_{66} \\ \kappa_{23} = \kappa_{31} = \kappa_{12} \end{array}\right\}.$$

And, finally,

$$2V = \kappa_{11}(e^2 + f^2 + g^2) + \kappa_{44}(a^2 + b^2 + c^2) + 2\kappa_{23}(fg + ge + ef) \quad \text{......(29)}$$

Thus the number of elastic coefficients is reduced to 3.

This may be called *complete cubical symmetry.* It occurs in Rock Salt. It is obvious that if any cubical portion of the body could be removed and replaced with any pair of its faces occupying the positions originally belonging to any other pair, and then made once more continuous with the rest of the body, the elastic properties of the whole would be absolutely unaffected.

Equations (18) become in this case

$$\left.\begin{array}{l} P = \kappa_{11}e + \kappa_{23}(f + g) \\ Q = \kappa_{11}f + \kappa_{23}(g + e) \\ R = \kappa_{11}g + \kappa_{23}(e + f) \\ S = \kappa_{44}a \\ T = \kappa_{44}b \\ U = \kappa_{44}c \end{array}\right\}.$$

Isotropy.

207.] **Definition.** Let us finally suppose that the body not only satisfies the conditions of homogeneity (§ 43), but is such that any two equal and similar portions, *however they be situated in* the body, possess identical elastic properties.

These properties are then quite independent of direction, and the body may be said to possess complete *spherical symmetry;* so that, if any spherical portion were rotated through any angle about any axis, and again made continuous with the rest, the body would remain elastically unchanged.

All such bodies are said to be *elastically isotropic.*

All other bodies, whether crystalline or asymmetrical, are called in contradistinction *æolotropic.*

Jellies, indiarubber, glass slowly cooled, and metals in their ordinary state may be considered as homogeneous isotropic bodies. The great traction which is applied in manufacturing metal wires, by pulling them through small holes in a perforated plate produces a permanent *set* (§ 12) which, when the wires are cooled and freed from tension, results in the crystalline state described in § 205 (*ii.*).

A somewhat similar effect is produced, in a less marked degree, by the various processes of rolling and hammering to which bars and plates of wrought iron and steel are subjected, in the course of manufacture. (*See* Appendix IV., Section B, "Ductile Metals.")

208.] **Energy and Stress.** It is obvious that, for an isotropic solid, all axes are axes of complete circular symmetry, and all planes are interchangeable planes of symmetry. Thus the conditions of § 205 (*ii.*) and of § 206 must be satisfied simultaneously, and by comparing (28) and (29) we see that

$$\left.\begin{aligned} \kappa_{11} &= \kappa_{33} \\ \kappa_{44} &= \kappa_{66} \\ \kappa_{23} &= \kappa_{11} - 2\kappa_{66} \end{aligned}\right\};$$

and, finally, for an isotropic solid,

$$2V = \kappa_{11}(e^2 + f^2 + g^2) + \kappa_{44}(a^2 + b^2 + c^2) + 2(\kappa_{11} - 2\kappa_{44})(fg + ge + ef)\ldots(30)$$

Thus the number of the elastic coefficients is reduced to 2.

Equations (18) now become

$$\left.\begin{aligned} P &= \kappa_{11}e + (\kappa_{11} - 2\kappa_{44})(f + g) \\ Q &= \kappa_{11}f + (\kappa_{11} - 2\kappa_{44})(g + e) \\ R &= \kappa_{11}g + (\kappa_{11} - 2\kappa_{44})(e + f) \\ S &= \kappa_{44}a \\ T &= \kappa_{44}b \\ U &= \kappa_{44}c \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

As we have now arrived at the most perfect conceivable degree of symmetry, it is evident that we are not justified in assuming any general relation between the two remaining coefficients, and they must therefore be supposed independent.

The insurmountable objection to all the old molecular theories, founded on Boscovitch's assumption (§ 37), is that they give an invariable ratio $\kappa_{11} : \kappa_{44} = 3$ between these coefficients for *all* isotropic solids—thus leaving in effect only one independent constant.

It was first pointed out by Stokes, in 1845, that natural solids afford a series of values for this ratio, varying within wide extremes, and not even showing a tendency to approximate to an ideal limit. (*See* Chapter XII.)

209.] **The Potential Energy as an Invariant of the Strain.** Since in an isotropic body the directions of the axes of reference cannot affect the form of the Potential Energy, which now depends solely on the specification of the Strain, it follows that V must be an Invariant of the Strain, and therefore a function of the invariants D, J, K (§ 111) of the various Strain Quadrics.

Now, by § 198, V must be a homogeneous quadratic function of the strain-components, and the forms of the invariants, which are homogeneous functions of the first, second, and third degrees respectively, show us at once that the only relation that can be assumed between them and V is

$$2V = \alpha D^2 + \beta J$$

where α and β are absolute constants. Substituting for D and J the values given by equations (39) of § 111, we get

$$\begin{aligned} 2V &= \alpha(e+f+g)^2 + \beta(fg - s_1^2 + ge - s_2^2 + ef - s_3^2) \\ &= \alpha(e^2+f^2+g^2) - \frac{\beta}{4}(a^2+b^2+c^2) \\ &\quad + 2\left(\alpha + \frac{\beta}{2}\right)(fg+ge+ef), \end{aligned}$$

which becomes identical with (30) on assuming

$$\alpha = \kappa_{11},\ \beta = -4\kappa_{44}.$$

The Elastic Moduli of an Isotropic Solid.

210.] **Modulus of Rigidity.** We see at once from equations (31) that a shear in either of the coördinate planes requires only the corresponding shearing stress (§ 149) to produce and maintain it, and that this stress bears to the shear the constant ratio $\kappa_{44} : 1$.

Since the directions of the axes of reference are perfectly indifferent, it follows that κ_{44} represents the shearing stress that

must be applied in any plane to produce and maintain the unit of shear in that plane; analogically speaking, that is to say. A shearing stress which would produce such an enormous distortion of the body as the unit shear (*see* Appendix II.) would certainly not obey the proportional law, except perhaps in one or two substances of exceptionally perfect elasticity. The units of strain (and all finite strains) really lie altogether outside our *theory*, and it is only by direct *experiment* that we can determine the degree of approximation to which it represents their laws. Such statements as the above must always be understood to be made under this reserve. (*See* Appendix IV., *below*.)

This quantity is usually called the *Modulus of Rigidity*, or simply the *Rigidity* of the body; it is also known as its *Elasticity of Figure*.

We shall in future denote it by the symbol n.

211.] **Modulus of Compression.** Let us now suppose that the strain throughout the body is a homogeneous *cubical compression*, uniform in all directions of amount Δ. Then (§ 112) we shall have everywhere

$$\left.\begin{array}{l} e=f=g=-\tfrac{1}{3}\Delta \\ a=b=c=0 \end{array}\right\},$$

and by equations (31)

$$\left.\begin{array}{l} P=Q=R=-(\kappa_{11}-\tfrac{4}{3}\kappa_{44})\Delta \\ S=T=U=0 \end{array}\right\}.$$

By § 174 we see that the stress at every point of the body will be a homogeneous *hydrostatic pressure*, of amount

$$\Pi=(\kappa_{11}-\tfrac{4}{3}\kappa_{44})\Delta\,;$$

and to maintain this strain and stress we must apply a uniform normal pressure Π over the whole bounding surface of the body.

Thus the quantity $(\kappa_{11}-\frac{4}{3}\kappa_{44})$ represents the uniform normal pressure which must be applied over the surface to produce the unit of cubical compression throughout the body.

This quantity is usually called the *Modulus of Compression*, the *Bulk-modulus of Elasticity*, or the *Elasticity of Volume*.

We shall in future denote it by k.

Of course a uniform normal *traction* over the bounding surface, of amount $k\Delta$, will in like manner produce a uniform cubical *dilatation* Δ throughout the body.

The reciprocal modulus $1/k$ is often called the *Compressibility* of the body, denoting as it does the cubical compression produced by a uniform surface pressure of unit magnitude.

212.] **The New Notation.** Writing then

$$\kappa_{44}=n,\ \kappa_{11}=k+\tfrac{4}{3}n,$$

in equations (30) and (31), they become

$$\left.\begin{aligned}P&=(k+\tfrac{4}{3}n)e+(k-\tfrac{2}{3}n)(f+g)\\Q&=(k+\tfrac{4}{3}n)f+(k-\tfrac{2}{3}n)(g+e)\\R&=(k+\tfrac{4}{3}n)g+(k-\tfrac{2}{3}n)(e+f)\\S&=na\\T&=nb\\U&=nc\end{aligned}\right\}\quad\text{.....................(32)}$$

and

$$2V=(k+\tfrac{4}{3}n)(e^2+f^2+g^2)+2(k-\tfrac{2}{3}n)(fg+ge+ef)$$
$$+n(a^2+b^2+c^2)\text{..................................(33)}$$

The latter may also be written

$$2V=(k-\tfrac{2}{3}n)\Delta^2+2n(e^2+f^2+g^2)+n(a^2+b^2+c^2)\text{...........(34)}$$

where Δ denotes, *as it invariably will in future*, the cubical dilatation (not necessarily uniform in all directions) at the point (x, y, z); so that (§§ 102, 103)

$$\Delta=e+f+g\text{..................................(35)}$$

Many formulæ are simplified by the use of the symbol m, where

$$m=k+\tfrac{1}{3}n\text{..................................(36)}$$

the fraction $\frac{1}{3}n$ being thus eliminated. For instance, the first three of equations (32) become

$$\left.\begin{aligned}P&=(m+n)e+(m-n)(f+g)\\Q&=(m+n)f+(m-n)(g+e)\\R&=(m+n)g+(m-n)(e+f)\end{aligned}\right\}\text{.....................(37)}$$

213.] **Young's Modulus.** This is the theoretical value (§ 210) of the longitudinal traction in any direction which will by itself produce unit elongation in the same direction.

Its value in terms of k and n can be deduced from equations (32) by putting

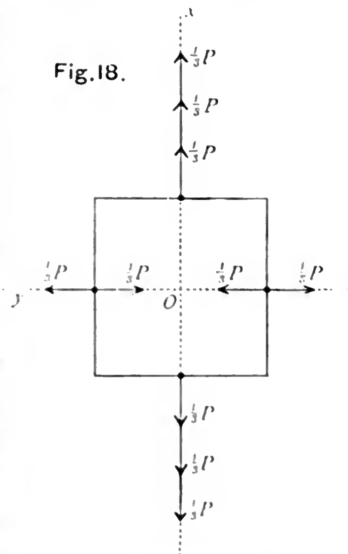
$$e=1,\ Q=R=S=T=U=0;$$

the value of P obtained by eliminating f and g from the remaining equations being the required modulus.

It will however be more instructive to determine it by the following analytical method:—

Consider a unit cube of the body, with its edges parallel to the axes, subjected to a homogeneous longitudinal traction P parallel to Ox. Each of the faces perpendicular to Ox will suffer a normal traction P per unit area, while the other faces will suffer no stress at all. Divide the traction P on each of the x-faces into three equal *tractions*, each $\frac{1}{3}P$, and apply to each of the four remaining faces a normal *traction* $\frac{1}{3}P$ per unit area, and an equal normal *pressure*. By the principle of superposition this system of stresses will be equivalent to the first, and we are at liberty to recompound them in any way we like. (Figure 18 represents all the stresses but those on the z-faces.)

Fig. 18.



First collect the normal *tractions* $\frac{1}{3}P$ over all the six faces: by § 211 these will produce a *cubical dilatation*, uniform in all directions, of amount $P/3k$, and this by § 105 may be resolved into a uniform *elongation* of amount $P/9k$ parallel to each of the axes.

Next combine the second of the three normal *tractions* on the x-faces with the equal normal *pressures* on the y-faces. By § 150 these are equivalent to a *shearing stress* of amount $\frac{1}{3}P$ in the plane of xy, which by § 210 must produce a *shear* of amount $P/3n$, and of the same type—that is, having its axes of elongation and contraction parallel to Ox and Oy. By § 100 (*see also* § 152) this may be resolved into an *elongation* $P/6n$ parallel to Ox and a *contraction* $P/6n$ parallel to Oy.

Similarly the last of the three normal *tractions* $\frac{1}{3}P$ on the x-faces, combined with the normal *pressures* $\frac{1}{3}P$ on the z-faces, will produce an *elongation* $P/6n$ parallel to Ox and a *contraction* $P/6n$ parallel to Oz.

On the whole, then, a longitudinal traction of amount P parallel to Ox produces the elongations—

$$\left.\begin{array}{ll} P\left(\dfrac{1}{9k}+\dfrac{1}{6n}+\dfrac{1}{6n}\right) & \text{parallel to } Ox \\ P\left(\dfrac{1}{9k}-\dfrac{1}{6n}\right) & \text{parallel to } Oy \\ P\left(\dfrac{1}{9k}-\dfrac{1}{6n}\right) & \text{parallel to } Oz \end{array}\right\}.$$

Denoting these by e, f, g, we have

$$\left.\begin{aligned} P &= \frac{9kn}{3k+n} \cdot e \\ f = g &= -\frac{3k-2n}{2(3k+n)} \cdot e \end{aligned}\right\}.$$

Thus if q denotes Young's Modulus

$$q = \frac{9kn}{3k+n} = \frac{3kn}{m} \quad \text{.........................(38)}$$

In all known solids $k > \frac{2}{3}n$, so that there is always a lateral *contraction* in the directions perpendicular to that of the applied tension.

If we employ the symbol σ to denote the ratio $(-f/e)$ of lateral contraction to longitudinal elongation in this case, we shall have

$$\sigma = \frac{3k-2n}{2(3k+n)} = \frac{m-n}{2m} \quad \text{.........................(39)}$$

214.] **Strain in terms of Stress.** If we solve equations (32) for the strain-components, we find

$$e = \frac{3k+n}{9kn} \cdot P - \frac{3k-2n}{18kn}(Q+R)$$

etc., etc.,

or, substituting from (38) and (39),

$$\left.\begin{aligned} e &= \frac{1}{q} \cdot P - \frac{\sigma}{q}(Q+R) \\ f &= \frac{1}{q} \cdot Q - \frac{\sigma}{q}(R+P) \\ g &= \frac{1}{q} \cdot R - \frac{\sigma}{q}(P+Q) \\ a &= \frac{1}{n} \cdot S \\ b &= \frac{1}{n} \cdot T \\ c &= \frac{1}{n} \cdot U \end{aligned}\right\} \quad \text{.....................(40)}$$

Thus, by (19),

$$2V = -\frac{\sigma}{q}(P+Q+R)^2 + \frac{1}{2n}(P^2+Q^2+R^2) + \frac{1}{n}(S^2+T^2+U^2) \text{......(41)}$$

215.] **Principal Axes of Strain and Stress.** If the principal axes of the *Strain* at any point of the body are parallel to the axes of reference, we have *at that point*, by equations (32) of § 83,

$$e = \epsilon_1, f = \epsilon_2, \ g = \epsilon_3; \ a = b = c = 0.$$

Thus, at the same point,

$$\left.\begin{aligned} P &= (m+n)\epsilon_1 + (m-n)(\epsilon_2+\epsilon_3) \\ Q &= (m+n)\epsilon_2 + (m-n)(\epsilon_3+\epsilon_1) \\ R &= (m+n)\epsilon_3 + (m-n)(\epsilon_1+\epsilon_2) \\ S &= 0 \\ T &= 0 \\ U &= 0 \end{aligned}\right\}.$$

Thus the stresses across small plane areas drawn through the point, perpendicular to the axes of reference, are wholly normal, and by § 163 the axes of reference are also parallel to the principal axes of the *Stress* at the point.

Conversely, it may be shown that, if the axes of reference are taken parallel to the principal axes of the stress at any point of the body, they must necessarily be parallel to the principal axes of the strain at the same point.

Hence we deduce that, *at every point of an isotropic body, the Principal Axes of the Strain and of the Stress are coincident*, and that the principal elongations ϵ_1, ϵ_2, ϵ_3 and the principal normal stresses N_1, N_2, N_3 are connected by the equations

$$\left.\begin{aligned} N_1 &= (m+n)\epsilon_1 + (m-n)(\epsilon_2+\epsilon_3) \\ N_2 &= (m+n)\epsilon_2 + (m-n)(\epsilon_3+\epsilon_1) \\ N_3 &= (m+n)\epsilon_3 + (m-n)(\epsilon_1+\epsilon_2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

or by

$$\left.\begin{aligned} \epsilon_1 &= \frac{1}{q}N_1 - \frac{\sigma}{q}(N_2+N_3) \\ \epsilon_2 &= \frac{1}{q}N_2 - \frac{\sigma}{q}(N_3+N_1) \\ \epsilon_3 &= \frac{1}{q}N_3 - \frac{\sigma}{q}(N_1+N_2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (42a)$$

The corresponding formulæ for V are

$$\left.\begin{aligned} 2V &= (m+n)(\epsilon_1^2+\epsilon_2^2+\epsilon_3^2) + 2(m-n)(\epsilon_2\epsilon_3+\epsilon_3\epsilon_1+\epsilon_1\epsilon_2) \\ &= (m-n)\Delta^2 + 2n(\epsilon_1^2+\epsilon_2^2+\epsilon_3^2) \end{aligned}\right\} \quad \ldots\ldots (43)$$

or $$2V = -\frac{\sigma}{q}(N_1+N_2+N_3)^2 + \frac{1}{2n}(N_1^2+N_2^2+N_3^2) \ldots\ldots\ldots\ldots (43a)$$

These results might of course have been deduced directly from § 209, coupled with the corresponding theorem that V must also be an invariant of the stress.

They evidently apply also to the crystalline forms of §§ 204–206 (since in them also the shearing stress and the shear vanish together independently of the elongations and normal stresses), but not to any lower degree of symmetry.

216.] **Lines and Tubes of Stress. Tie Lines and Strut Lines. Principal Surfaces of the Strain.** The components of strain and stress being supposed continuous functions of the coördinates throughout the body, so also will be the direction-cosines of the Principal Axes at each point, given by equations (29) of § 79, or by equations (22) of § 163. Hence if we draw the principal axis $P\xi$ at any point P, corresponding to the continuous elongation ϵ_1 and the continuous normal stress N_1, and if an elementary length PP' be taken along $P\xi$, and the corresponding principal axis $P'\xi'$ be drawn at P', the change in direction from $P\xi$ to $P'\xi'$ will be a small quantity of the same order of dimensions as the elementary length PP'. If this process be continued we get a broken line $PP'P''P'''$........., composed of elements PP', $P'P''$........., each of which coincides with the principal axis for ϵ_1 and N_1 at one of its extremities.

Proceeding to the limit, in which the lengths of these elements are indefinitely reduced, we have a curve such that the tangent to it at any point P is the principal axis $P\xi$ for ϵ_1 and N_1 at that point. It is thus possible to draw a *system of continuous curves in the body enveloping the principal axis $P\xi$ at every point through which they pass.*

The differential equations of this system are

$$\left.\begin{aligned} &\frac{e\,dx + s_3\,dy + s_2\,dz}{dx} = \frac{s_3\,dx + f\,dy + s_1\,dz}{dy} = \frac{s_2\,dx + s_1\,dy + g\,dz}{dz} = \epsilon_1 \\ &\text{or} \\ &\frac{P\,dx + U\,dy + T\,dz}{dx} = \frac{U\,dx + Q\,dy + S\,dz}{dy} = \frac{T\,dx + S\,dy + R\,dz}{dz} = N_1 \end{aligned}\right\} \quad \text{......(44)}$$

where of course for ϵ_1 and N_1 are to be substituted the proper functions of x, y, z.

Since ϵ_1 is a root of equation (28) § 79, and N_1 of equation (21) § 163, only two equations of each set are independent.

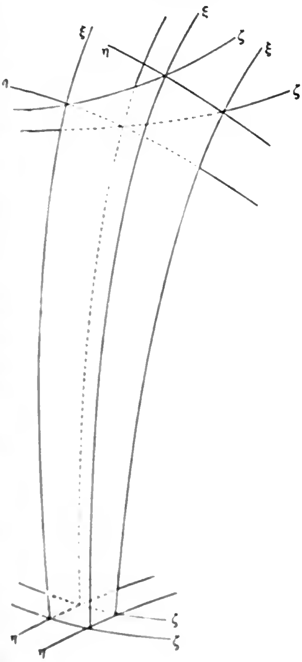
We get a second system of curves enveloping all the principal axes $P\eta$, corresponding to ϵ_2 and N_2, at the points through which they pass, and a third system everywhere enveloping $P\zeta$.

It is obvious that these three systems of curves cut everywhere orthogonally; and that the strain at each point consists of an elongation of each of the three curves which pass through it (with or without rotation), while the stress consists of a normal traction across each of the three elementary plane areas which can be drawn through the point to touch two of the curves.

These curves are called **Lines of Stress.**

Let us take two consecutive ζ-lines, and also two consecutive η-lines which intersect the former; these four curves will enclose an elementary figure which is ultimately a plane rectangle. If now we draw the ξ-curves through every point of the perimeter of this area, we shall form a *tube* of elementary section, called a **Tube of Stress** (Figure 19.)

Fig. 19.



The normal section of the tube at any point is an approximately plane rectangle bounded by consecutive Stress lines of the η and ζ systems, while each of its sides may be looked upon as composed of approximately plane rectangles bounded by the edges of the tube and by two consecutive curves of the η system or of the ζ system.

The stress across every section of the elementary *fibre* of the body bounded by the tube is wholly in the direction of its length; and the stress across any element of its surface (the tube of stress) is wholly normal to the element.

It is thus obvious that the body may be supposed divided in three different ways into systems of curvilinear fibres, which transmit stress through the body in the direction of their length, while the action between adjacent fibres is, at every point, wholly normal to their common surface.

We shall adopt the terms used to denote the functions of beams in engineering structures, and call these fibres *Ties* when they transmit a tension, and *Struts* when they transmit a thrust in the direction of their length.

The Stress lines which form the walls of the tubes will accordingly be called *Tie-lines* or *Strut-lines*.

Thus equations (44) are the differential equations of a system of tie-lines or strut-lines according as N_1 is positive or negative.

If $N_1 = 0$ we have a system of lines of zero stress.

If we draw several adjacent Tubes of Stress (of the ξ-system, let us say) as in Figure 20, it is obvious that any set of conterminous normal sections of these tubes will form adjacent elements of a continuous surface. Each such surface will contain a complete system of the η-curves, and also a complete system of

the ζ-curves, and will everywhere be cut normally by the ξ-curves.

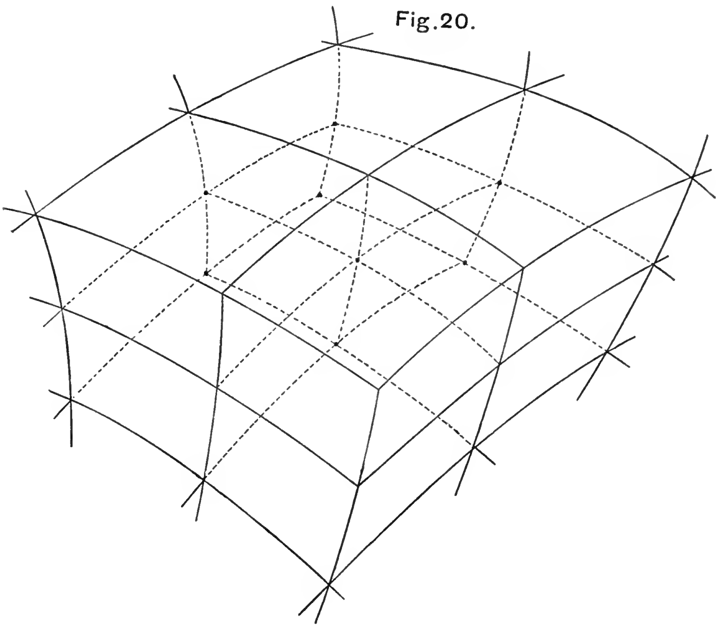
Thus we can construct three orthogonal systems of surfaces throughout the body, such that

(*i.*) The curves of intersection of the three surfaces which pass through any point P are the Lines of Stress at P, and therefore have for their tangents the principal axes of the strain $P\xi$, $P\eta$, $P\zeta$.

(*ii.*) The tangent planes to the three surfaces at P are the principal planes of the strain.

(*iii.*) Each of the elements of volume (ultimately rectangular parallelepipeds) into which the body is divided by consecutive

Fig. 20.



surfaces of the three systems, is subjected only to elongations in the directions of its edges (with or without rotation) and suffers *no shear* whatever (consequently remaining rectangular).

These surfaces may be called the **Principal Surfaces** of the Strain or of the Stress. We shall return to them in the next Chapter.

If *one* of the principal stresses vanishes, each of the system of principal surfaces which is cut orthogonally by the lines of zero stress envelopes the *Plane of the Stress* (§ 175) at every point through which it passes. The differential

equation of this system is therefore, by equations (66) and (67) of § 184,

$$\left.\begin{aligned} \sqrt{\mathfrak{p}}\,.\,dx + \sqrt{\mathfrak{q}}\,.\,dy + \sqrt{\mathfrak{r}}\,.\,dz = 0 \\ \text{or} \qquad \frac{dx}{\mathfrak{s}} + \frac{dy}{\mathfrak{t}} + \frac{dz}{\mathfrak{u}} = 0 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(45)$$

and those of the lines of zero stress are

$$\left.\begin{aligned} \frac{dx}{\sqrt{\mathfrak{p}}} = \frac{dy}{\sqrt{\mathfrak{q}}} = \frac{dz}{\sqrt{\mathfrak{r}}} \\ \text{or} \qquad \mathfrak{s}dx = \mathfrak{t}dy = \mathfrak{u}dz \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(45a)$$

When *two* of the principal stresses vanish (§§ 185, 186) only one principal axis at each point is determinate. Thus we have only one determinate system of lines of stress, given by equations (76) and (77) of § 186, namely,

$$\left.\begin{aligned} \frac{dx}{\sqrt{P}} = \frac{dy}{\sqrt{Q}} = \frac{dz}{\sqrt{R}} \\ \text{or} \qquad Sdx = Tdy = Udz \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(46)$$

and only one determinate system of principal surfaces, given by

$$\left.\begin{aligned} \sqrt{P}dx + \sqrt{Q}dy + \sqrt{R}dz = 0 \\ \text{or} \qquad \frac{dx}{S} + \frac{dy}{T} + \frac{dz}{U} = 0 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(46a)$$

In this case any two systems whatever of surfaces which cut those and each other orthogonally may be taken as the other two systems of principal surfaces, and their curves of intersection with the determinate system will give two systems of lines of zero stress.

In *homogeneous* stress and strain, the Lines of Stress are *straight lines*, and the Principal surfaces are orthogonal systems of *parallel planes*.

Equations of Equilibrium and Motion.

217.] **In terms of the Component Strains.** Substituting for the component stresses in equations (3) of § 142 their values (32) in terms of the component strains, we get for the equations of equilibrium

$$\left.\begin{aligned} (m+n)\frac{\partial e}{\partial x} + (m-n)\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) + n\left(\frac{\partial c}{\partial y} + \frac{\partial b}{\partial z}\right) + \rho X = 0 \\ (m+n)\frac{\partial f}{\partial y} + (m-n)\left(\frac{\partial g}{\partial y} + \frac{\partial e}{\partial y}\right) + n\left(\frac{\partial a}{\partial z} + \frac{\partial c}{\partial x}\right) + \rho Y = 0 \\ (m+n)\frac{\partial g}{\partial z} + (m-n)\left(\frac{\partial e}{\partial z} + \frac{\partial f}{\partial z}\right) + n\left(\frac{\partial b}{\partial x} + \frac{\partial a}{\partial y}\right) + \rho Z = 0 \end{aligned}\right\} \quad\ldots\ldots(47)$$

The equations of motion (4) of § 143 similarly become

$$\left.\begin{aligned}(m+n)\frac{\partial e}{\partial x}+(m-n)\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial x}\right)+n\left(\frac{\partial c}{\partial y}+\frac{\partial b}{\partial z}\right)+\rho(X-\ddot{u})=0\\(m+n)\frac{\partial f}{\partial y}+(m-n)\left(\frac{\partial g}{\partial y}+\frac{\partial e}{\partial y}\right)+n\left(\frac{\partial a}{\partial z}+\frac{\partial c}{\partial x}\right)+\rho(Y-\ddot{v})=0\\(m+n)\frac{\partial g}{\partial z}+(m-n)\left(\frac{\partial e}{\partial z}+\frac{\partial f}{\partial z}\right)+n\left(\frac{\partial b}{\partial x}+\frac{\partial a}{\partial y}\right)+\rho(Z-\ddot{w})=0\end{aligned}\right\}\dots(48)$$

Lastly, the boundary conditions (5) of § 144 take the form

$$\left.\begin{aligned}\lambda[(m+n)e+(m-n)(f+g)]+\mu nc+\nu nb=F\\\lambda nc+\mu[(m+n)f+(m-n)(g+e)]+\nu na=G\\\lambda nb+\mu na+\nu[(m+n)g+(m-n)(e+f)]=H\end{aligned}\right\}\dots\dots\dots\dots(49)$$

where F, G, H are the components of surface traction, and (λ, μ, ν) the direction-cosines of the outward normal.

218.] **In terms of the Displacements.** Substituting for the component strains in these equations their values from equations (59) of § 123, we get for the equations of equilibrium

$$(m+n)\frac{\partial^2 u}{\partial x^2}+(m-n)\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial z\partial x}\right)+n\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$
$$+n\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\rho X=0,$$

etc., etc.

Rearranging the order of the terms, these equations become

$$\left.\begin{aligned}m\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+n\nabla^2 u+\rho X=0\\m\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+n\nabla^2 v+\rho Y=0\\m\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+n\nabla^2 w+\rho Z=0\end{aligned}\right\}\dots\dots\dots\dots(50)$$

or, since

$$\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z},\dots\dots\dots\dots(35)$$

$$\left.\begin{aligned}m\frac{\partial\Delta}{\partial x}+n\nabla^2 u+\rho X=0\\m\frac{\partial\Delta}{\partial y}+n\nabla^2 v+\rho Y=0\\m\frac{\partial\Delta}{\partial z}+n\nabla^2 w+\rho Z=0\end{aligned}\right\}\dots\dots\dots\dots(51)$$

From these we may at once deduce the equations of motion

$$\left.\begin{aligned} m\frac{\partial\Delta}{\partial x}+n\nabla^2 u+\rho(X-\ddot{u})&=0\\ m\frac{\partial\Delta}{\partial y}+n\nabla^2 v+\rho(Y-\ddot{v})&=0\\ m\frac{\partial\Delta}{\partial z}+n\nabla^2 w+\rho(Z-\ddot{w})&=0 \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(52)$$

It is equally easy, by a slightly different transformation, to throw these equations into Lamé's form

$$\left.\begin{aligned} (m+n)\frac{\partial\Delta}{\partial x}-2n\left(\frac{\partial\theta_3}{\partial y}-\frac{\partial\theta_2}{\partial z}\right)+\rho(X-\ddot{u})&=0\\ (m+n)\frac{\partial\Delta}{\partial y}-2n\left(\frac{\partial\theta_1}{\partial z}-\frac{\partial\theta_3}{\partial x}\right)+\rho(Y-\ddot{v})&=0\\ (m+n)\frac{\partial\Delta}{\partial z}-2n\left(\frac{\partial\theta_2}{\partial x}-\frac{\partial\theta_1}{\partial y}\right)+\rho(Z-\ddot{w})&=0 \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(52a)$$

where θ_1, θ_2, θ_3 are the component rotations. If we substitute for them their values from equations (59) of § 123, this form is at once seen to be identical with (52).

The boundary conditions are

$$\left.\begin{aligned} &\lambda\left[(m+n)\frac{\partial u}{\partial x}+(m-n)\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]+\mu n\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\\ &\qquad+\nu n\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=F\\ &\lambda n\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\mu\left[(m+n)\frac{\partial v}{\partial y}+(m-n)\left(\frac{\partial w}{\partial z}+\frac{\partial u}{\partial x}\right)\right]\\ &\qquad+\nu n\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)=G\\ &\lambda n\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\mu n\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\\ &\qquad+\nu\left[(m+n)\frac{\partial w}{\partial z}+(m-n)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]=H \end{aligned}\right\}\quad\ldots\ldots(53)$$

Equations of Motion and Equilibrium obtained from the Potential Energy.

219.] We obtained equations (47-53) by substitution in the equations of Stress given in Chapter III. These relations however have not been elsewhere assumed in the present Chapter, except in §§ 194, 195, to prove the equality of the small total increase of energy and the corresponding small amount of work expended on the body by the external forces.

We now propose to show, by an application of the principle of Virtual Velocities which is strictly the converse of that of §§ 194, 195, that, assuming the expression (34) for the Potential Energy per unit volume, the equations of motion and equilibrium (47, 48, 49) can be immediately deduced.

Introducing the symbol m into (34) we have

$$2V = (m-n)\Delta^2 + 2n(e^2+f^2+g^2) + n(a^2+b^2+c^2).$$

Thus if W be the total Potential Energy of the Strain

$$2W = \iiint\{(m-n)\Delta^2 + 2n(e^2+f^2+g^2) + n(a^2+b^2+c^2)\}dxdydz \ldots (54)$$

by equation (12), § 196.

We shall consider the most general case, in which motion is taking place, for the case of statical equilibrium can always be deduced from it by making all the velocities and accelerations zero.

The kinetic energy of the motion is then

$$\mathfrak{T} = \iiint \tfrac{1}{2}\rho(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)dxdydz.$$

Let us now suppose that the Applied Forces and Surface Tractions are allowed to do work on the body by producing a very small variation of the strain.

Let δu, δv, δw be the consequent small increments of the displacements of any point (x, y, z), in the body or on its surface, from its natural position. These may be supposed quite arbitrary and independent (but each must be a continuous function of the coördinates).

Let δe, δf, δg, δa, δb, δc, δW and $\delta\mathfrak{T}$ be the corresponding small increments in the strain-components, and in the potential and kinetic energies. From the principle of Conservation of Energy we know that the work done on the body must be equal to the increment $\delta W + \delta\mathfrak{T}$ of its total energy.

Now $$\delta\mathfrak{T} = \iiint\rho(\ddot{u}\delta u + \ddot{v}\delta v + \ddot{w}\delta w)dxdydz \ldots\ldots (55)$$

as in § 195.

And the work done by external forces is, as before,

$$\iiint\rho(X\delta u + Y\delta v + Z\delta w)dxdydz + \iint(F\delta u + G\delta v + H\delta w)dS \ldots (56)$$

Since this is equal to $\delta\mathfrak{T} + \delta W$, we get from (55) and (56)

$$\delta W = \iiint\rho\{(X-\ddot{u})\delta u + (Y-\ddot{v})\delta v + (Z-\ddot{w})\delta w\}dxdydz$$
$$+\iint(F\delta u + G\delta v + H\delta w)dS \ldots\ldots (57)$$

But, from (54),

$$\delta W = \iiint\{(m-n)\Delta\delta\Delta + 2n(e\delta e + f\delta f + g\delta g)$$
$$+ n(a\delta a + b\delta b + c\delta c)\}dxdydz,$$

where, as in equations (7) of § 194,

$$\left.\begin{aligned}
\delta e &= \delta\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\delta u, \text{ etc.}\\
\delta\Delta &= \frac{\partial}{\partial x}\delta u + \frac{\partial}{\partial y}\delta v + \frac{\partial}{\partial z}\delta w,\\
\delta a &= \frac{\partial}{\partial y}\delta w + \frac{\partial}{\partial z}\delta v, \text{ etc.}
\end{aligned}\right\}\cdot\cdot$$

Thus

$$\begin{aligned}
\delta W = \iiint\Big\{&(m-n)\Delta\left(\frac{\partial}{\partial x}\delta u + \frac{\partial}{\partial y}\delta v + \frac{\partial}{\partial z}\delta w\right)\\
&+ 2n\left(e\frac{\partial}{\partial x}\delta u + f\frac{\partial}{\partial y}\delta v + g\frac{\partial}{\partial z}\delta w\right) + na\left(\frac{\partial}{\partial y}\delta w + \frac{\partial}{\partial z}\delta v\right)\\
&+ nb\left(\frac{\partial}{\partial z}\delta u + \frac{\partial}{\partial x}\delta w\right) + nc\left(\frac{\partial}{\partial x}\delta v + \frac{\partial}{\partial y}\delta u\right)\Big\}\,dxdydz\\
= \iiint\Big\{&[(m-n)\Delta + 2ne]\frac{\partial}{\partial x}\delta u + nc\frac{\partial}{\partial x}\delta v + nb\frac{\partial}{\partial x}\delta w\\
&+ nc\frac{\partial}{\partial y}\delta u + [(m-n)\Delta + 2nf]\frac{\partial}{\partial y}\delta v + na\frac{\partial}{\partial y}\delta w\\
&+ nb\frac{\partial}{\partial z}\delta u + na\frac{\partial}{\partial z}\delta v + [(m-n)\Delta + 2ng]\frac{\partial}{\partial z}\delta w\Big\}\,dxdydz.
\end{aligned}$$

Integrating by parts as in §§ 146 and 194,

$$\begin{aligned}
\delta W = &\iint\{[(m-n)\Delta + 2ne]\delta u + nc\,.\,\delta v + nb\,.\,\delta w\}\lambda dS\\
&- \iiint\left\{\frac{\partial}{\partial x}[(m-n)\Delta + 2ne]\,.\,\delta u + \frac{\partial}{\partial x}(nc)\,.\,\delta v + \frac{\partial}{\partial x}(nb)\,.\,\delta w\right\}dxdydz\\
&+ \iint\{nc\,.\,\delta u + [(m-n)\Delta + 2nf]\delta v + na\,.\,\delta w\}\mu dS\\
&- \iiint\left\{\frac{\partial}{\partial y}(nc)\,.\,\delta u + \frac{\partial}{\partial y}[(m-n)\Delta + 2nf]\delta v + \frac{\partial}{\partial y}(na)\,.\,\delta w\right\}dxdydz\\
&+ \iint\{nb\,.\,\delta u + na\,.\,\delta v + [(m-n)\Delta + 2ng]\delta w\}\nu dS\\
&- \iiint\left\{\frac{\partial}{\partial z}(nb)\,.\,\delta u + \frac{\partial}{\partial z}(na)\,.\,\delta v + \frac{\partial}{\partial z}[(m-n)\Delta + 2ng]\delta w\right\}dxdydz.
\end{aligned}$$

Rearranging the order of the terms,

$$\begin{aligned}
\delta W = &\iint\{\lambda[(m-n)\Delta + 2ne] + \mu nc + \nu nb\}\delta u\,.\,dS\\
&+ \iint\{\lambda nc + \mu[(m-n)\Delta + 2nf] + \nu na\}\delta v\,.\,dS\\
&+ \iint\{\lambda nb + \mu na + \nu[(m-n)\Delta + 2ng]\}\delta w\,.\,dS\\
&- \iiint\left\{\frac{\partial}{\partial x}[(m-n)\Delta + 2ne] + \frac{\partial}{\partial y}(nc) + \frac{\partial}{\partial z}(nb)\right\}\delta u\,.\,dxdydz\\
&- \iiint\left\{\frac{\partial}{\partial x}(nc) + \frac{\partial}{\partial y}[(m-n)\Delta + 2nf] + \frac{\partial}{\partial z}(na)\right\}\delta v\,.\,dxdydz\\
&- \iiint\left\{\frac{\partial}{\partial x}(nb) + \frac{\partial}{\partial y}(na) + \frac{\partial}{\partial z}[(m-n)\Delta + 2ng]\right\}\delta w\,.\,dxdydz \ldots(58)
\end{aligned}$$

Equating this to (57) we get

$$\begin{aligned}
&\iint\{\lambda[(m-n)\Delta+2ne]+\mu nc+\nu nb-F\}\delta u\,.\,dS\\
&+\iint\{\lambda nc+\mu[(m-n)\Delta+2nf]+\nu na-G\}\delta v\,.\,dS\\
&+\iint\{\lambda nb+\mu na+\nu[(m-n)\Delta+2ng]-H\}\delta w\,.\,dS\\
&-\iiint\left\{\frac{\partial}{\partial x}[(m-n)\Delta+2ne]+\frac{\partial}{\partial y}(nc)+\frac{\partial}{\partial z}(nb)+\rho(X-\ddot{u})\right\}\delta u\,.\,dxdydz\\
&-\iiint\left\{\frac{\partial}{\partial x}(nc)+\frac{\partial}{\partial y}[(m-n)\Delta+2nf]+\frac{\partial}{\partial z}(na)+\rho(Y-\ddot{v})\right\}\delta v\,.\,dxdydz\\
&-\iiint\left\{\frac{\partial}{\partial x}(nb)+\frac{\partial}{\partial y}(na)+\frac{\partial}{\partial z}[(m-n)\Delta+2ng]+\rho(Z-\ddot{w})\right\}\delta w\,.\,dxdydz\\
&=0.
\end{aligned}$$

This then is the condition to be satisfied in any small arbitrary variation of the strain. Since δu, δv, δw are independent as well as arbitrary, each of the double and triple integrals must vanish separately for all values that they may assume. We must therefore have identically

$$\left.\begin{aligned}
\lambda[(m-n)\Delta+2ne]+\mu nc+\nu nb-F=0\\
\lambda nc+\mu[(m-n)\Delta+2nf]+\nu na-G=0\\
\lambda nb+\mu na+\nu[(m-n)\Delta+2ng]-H=0
\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(59)$$

at all points of the surface, and

$$\left.\begin{aligned}
\frac{\partial}{\partial x}[(m-n)\Delta+2ne]+\frac{\partial}{\partial y}(nc)+\frac{\partial}{\partial z}(nb)+\rho(X-\ddot{u})=0\\
\frac{\partial}{\partial x}(nc)+\frac{\partial}{\partial y}[(m-n)\Delta+2nf]+\frac{\partial}{\partial z}(na)+\rho(Y-\ddot{v})=0\\
\frac{\partial}{\partial x}(nb)+\frac{\partial}{\partial y}(na)+\frac{\partial}{\partial z}[(m-n)\Delta+2ng]+\rho(Z-\ddot{w})=0
\end{aligned}\right\}\ldots\ldots(60)$$

throughout the interior of the body.

For the case of equilibrium we have only to put $\ddot{u}=\ddot{v}=\ddot{w}=0$, whence

$$\left.\begin{aligned}
\frac{\partial}{\partial x}[(m-n)\Delta+2ne]+\frac{\partial}{\partial y}(nc)+\frac{\partial}{\partial z}(nb)+\rho X=0\\
\frac{\partial}{\partial x}(nc)+\frac{\partial}{\partial y}[(m-n)\Delta+2nf]+\frac{\partial}{\partial z}(na)+\rho Y=0\\
\frac{\partial}{\partial x}(nb)+\frac{\partial}{\partial y}(na)+\frac{\partial}{\partial z}[(m-n)\Delta+2ng]+\rho Z=0
\end{aligned}\right\}\ldots\ldots\ldots\ldots(61)$$

If m and n be treated as constants, equations (59) (60) (61) are obviously identical with (49) (48) (47) above; and by substitution from (32) or (40) they are easily reduced to (5) (4) (3) of the last Chapter.

220.] **Heterogeneous Isotropy.** The extremely general method by which the equations of the last Article were obtained, by the assumption only that the Potential Energy per unit volume of an isotropic solid was of the form (34) with only two independent coefficients, enables us, however, to interpret them in a more general light. It is easy to imagine a heterogeneous solid, such that every elementary portion of it is strictly isotropic—and consequently possesses only two independent elastic moduli—while the values of these moduli—(say the Rigidity and the Bulk-modulus) vary continuously from one element to another (*see* § 198).

The quantities n and k, with their derivatives m, q, σ, will then be functions of the position in the body of the element considered, though not of the strain to which it is subjected.

Equations (59) (60) (61) will then represent the conditions of motions or equilibrium, not only for a homogeneously isotropic body (m, n, ρ absolute constants), but also for any heterogeneously isotropic body, in which m, n, ρ are any continuous functions of (x, y, z).

221.] **Absolute Moduli, Weight Moduli, Length Moduli.** The moduli k, n, q, being *stresses*, are of course, like all other stresses, measured by the force applied per unit area. The numerical measure of a modulus and its physical dimensions therefore depend both on the unit of length and the unit of force which we adopt.

(*i.*) The *gravitation* measure of force is the one most commonly adopted. On this system a modulus represents the *weight* that must be applied per unit area to produce unit strain of the corresponding type. Thus the moduli may be given in pounds or tons per square inch, or preferably in grammes per square centimetre.

(*ii.*) Since what we call the *weight* of a gramme is simply the force exerted by the earth on a given *mass* known as a gramme, the gravitation measure of force, and therefore also of the moduli, varies from point to point of the earth's surface, which the resistance of the body to stress of course does not. In order therefore to make these measurements available in all countries, the forces ought to be reduced to *absolute measure*, which (since the absolute unit of force, called in the British system the *poundal*, and in the metric system the *dyne*, is that force which produces the unit acceleration in the unit mass) is done by multiplying the gravitation measure by the numerical value of the acceleration produced by gravity at the spot where the measurements are made.

Each modulus will then represent the number of absolute units of force to be applied to the unit of area:—say the number

of poundals per square inch, or of dynes per square centimetre. It should be mentioned, however, that the discrepancies in our experimental data—due to variation of material, etc.—far more than cover the small variations of gravity. Rules for reducing stresses and moduli from one system to another will be given with the tables at the end of Appendix IV., *below*.

(*iii.*) A very convenient method of measuring the moduli is to express them in terms of the *length* of a bar of the material, of unit section, whose weight is equal to the force per unit area that is to be applied. This, like (*i.*), is a local measure.

When the moduli are expressed in this system they are called *Length-moduli*, and their numerical measures are called the lengths of the moduli.

Thus if k, $\mathfrak{k}$ denote the weight-modulus and length-modulus of a given material, expressed in the C.G.S. system of units, a force equal to the weight of k grammes, or to the weight of a bar of the material one square centimetre in section and $\mathfrak{k}$ centimetres long, distributed uniformly over each square centimetre of the surface of any body of the same substance, will produce in it the unit of cubical compression. And similarly for the other moduli.

Thus $k = \rho\mathfrak{k}$, etc., where ρ is the density of the body in grammes per cubic centimetre.

222.] **Resilience. Strength. Tenacity. Modulus of Rupture.** When a given elastic body is brought to a given state of strain, and then set free, the work which it is able to do in virtue of its elasticity, in returning to its natural state, is called the *Resilience of the given body for the given strain.*

This we know (§ 31) to be equal to the work done on the body in straining it, or to its potential energy in the given state of strain.

When we speak simply of the *Resilience of a material* for a given *type* of strain, we mean its potential energy *per unit volume* when strained to its *elastic limit* (§§ 12-14) *for that particular type.*

For a brittle substance (§ 13) with comparatively narrow limits of elasticity, within which the proportional law of § 197 may be taken as very approximately true (*see* Appendix IV., *below*), the resilience will be given at once by substituting in any of the expressions obtained for V in this Chapter the limiting values of the strain-components, any increase of which would produce rupture or marked permanent set.

For example if E, A represents the limits of elongation and shear for a brittle isotropic solid, its resilience for elongation is

$$\tfrac{1}{2}(m+n)E^2,$$

and its resilience for shear is

$$\tfrac{1}{2}nA^2.$$

The limits of safety for linear elongation and contraction, as well as for cubical dilatation and compression are generally different.

Thus (the sign of a shear being a purely *geometrical* convention, devoid of *physical* meaning) a natural isotropic solid has *five* principal resiliences.

The resilience thus defined is measured as energy per unit volume. This might be expressed in terms of an absolute unit (*e.g.*, ergs per cubic centimetre, or foot-poundals per cubic foot), but in practice the (local) gravitation measure of energy is adopted, the unit of which is equal to the work done in raising the unit of mass through the unit height against gravity.

Thus the resilience is usually expressed in gramme-centimetres per cubic centimetre, or in foot-pounds or foot-tons per cubic foot.

There is also a length measure of resilience, defined as the height through which unit mass of the body must be raised to do work against (local) gravity equal to the resilience per unit volume.

Thus if V be the resilience of a given material for a given type of strain in gramme-centimetres per cubic-centimetre, and $\mathfrak{V}$ in centimetres per gramme; and if n, k, q, $\mathfrak{n}$, $\mathfrak{k}$, $\mathfrak{q}$ be the weight moduli and length moduli on the C.G.S. system:

$$V/\mathfrak{V} = k/\mathfrak{k} = n/\mathfrak{n} = q/\mathfrak{q} = \rho \qquad \text{(62)}$$

where ρ is the density of the body in grammes per cubic centimetre (or its specific gravity); whence we should have, if the proportional law held up to the point of rupture,

$$\mathfrak{V} = \tfrac{1}{2}\mathfrak{n}.A^2$$

for torsion, and so on.

The **Strength** of a material for a given type of strain is measured by the stress which will produce the limiting or extreme strain of that type. Thus we have an Elastic Strength, corresponding to the limit of perfect elasticity, and an Ultimate Strength corresponding to the point of rupture, for each type of strain.

The **Tenacity** is the ultimate strength for elongation—that is the value of the longitudinal stress which produces the extreme elongation E. If the proportional law held up to this point, the tenacity would be qE, in grammes-weight per square centimetre.

The **Length-modulus of rupture** is the tenacity expressed as a length. On the same supposition this would be $\mathfrak{q}E$ centimetres. (*See the table in* Appendix IV.)

Possible Discontinuity of Strain and Stress.

223.] **Limitations.** We have hitherto confined ourselves to the consideration of those cases of strain in which not only the displacements but also the strain-components themselves (§ 52) are perfectly continuous functions of position throughout the whole body. And in accordance with this limitation the Applied Forces and Surface Tractions (§ 136) and *consequently* also the Stress (§ 137) have also invariably been taken as continuous, and therefore (§ 197) suitable to maintain such a strain.

Discontinuity in the Applied Forces never occurs in actual structures to any important extent, but the consideration of discontinuous Surface Tractions and Pressures is of the utmost practical importance, since, for obvious reasons, the component parts of a complicated structure must necessarily bear upon one another by definite and circumscribed portions of their bounding surfaces.

Let us now therefore consider how far our theory, as at present developed, can take account of such discontinuity. We must first investigate the nature and extent of the discontinuity (if any) permissible in the displacements and the components of strain and stress, and hence deduce the characteristics of the discontinuous systems of external forces with which we are able to deal.

224.] **The component displacements.** To begin with, we may observe that, even though in passing from one region of the body to another the displacements may become discontinuous *in form*, they cannot in any case present discontinuity of *magnitude*. For if it were otherwise, two points immediately in contact with the separating surface (and practically coincident with one another in the natural state) would suffer displacements differing by quantities of a higher order of magnitude than their initial distance, and rupture of the body would take place over portions or the whole of the surface of discontinuity.

If then with the notation of § 51 we suppose the component displacements in any one region of the body given by

$$\left.\begin{aligned} u_1 &= \phi_1(x,\ y,\ z) \\ v_1 &= \chi_1(x,\ y,\ z) \\ w_1 &= \psi_1(x,\ y,\ z) \end{aligned}\right\},$$

and in any contiguous region by

$$\left.\begin{aligned} u_2 &= \phi_2(x,\ y,\ z) \\ v_2 &= \chi_2(x,\ y,\ z) \\ w_2 &= \psi_2(x,\ y,\ z) \end{aligned}\right\},$$

we must have, at every point of the surface of discontinuity,

$$\left.\begin{array}{l}\phi_1=\phi_2\\ \chi_1=\chi_2\\ \psi_1=\psi_2\end{array}\right\}.$$

Example. The distribution of displacement

$$\left.\begin{array}{l}u_1=0\\ v_1=cx\\ w_1=0\end{array}\right\}\qquad \left.\begin{array}{l}u_2=0\\ v_2=\frac{1}{2}c\beta\left(1+\dfrac{x^2}{\beta^2}\right)\\ w_2=0\end{array}\right\}$$

is not permissible, unless the boundary between the two regions is the plane $x=\beta$.

225.] **The component strains.** Let P be any point (x, y, z) in the surface of discontinuity, and let P_1 and P_2 be points in the normal through P, on either side of the surface, and let τ be the indefinitely small distance of either from P.

Then if (λ, μ, ν) be the direction-cosines of the normal, reckoned positive in the direction PP_2, the projections upon the axes of reference of the elementary distance P_1P_2 in the natural state are

$$\left.\begin{array}{l}h=x_2-x_1=2\lambda\tau\\ k=y_2-y_1=2\mu\tau\\ l=z_2-z_1\;=2\nu\tau\end{array}\right\}.$$

Now the component displacements of P_1 are

$$u_1=\phi_1-\tau\left(\lambda\frac{\partial\phi_1}{\partial x}+\mu\frac{\partial\phi_1}{\partial y}+\nu\frac{\partial\phi_1}{\partial z}\right)+\frac{\tau^2}{2}\left(\lambda^2\frac{\partial^2\phi_1}{\partial x^2}+\ldots\ldots\right)+\ldots\ldots$$

etc., etc.

and those of P_2 are

$$u_2=\phi_2+\tau\left(\lambda\frac{\partial\phi_2}{\partial x}+\mu\frac{\partial\phi_2}{\partial y}+\nu\frac{\partial\phi_2}{\partial z}\right)+\frac{\tau^2}{2}\left(\lambda^2\frac{\partial^2\phi_2}{\partial x^2}+\ldots\ldots\right)+\ldots\ldots$$

etc., etc.

In these expressions the values of ϕ_1, ϕ_2, and their derivatives may be taken as those which they have at P, so that $\phi_1=\phi_2$, etc.

If, as in § 52, strained lengths of these projections be denoted by $h+\delta h$, etc., we shall have

$$\delta h=u_2-u_1=\tau\left[\lambda\left(\frac{\partial\phi_2}{\partial x}+\frac{\partial\phi_1}{\partial x}\right)+\mu\left(\frac{\partial\phi_2}{\partial y}+\frac{\partial\phi_1}{\partial y}\right)+\nu\left(\frac{\partial\phi_2}{\partial z}+\frac{\partial\phi_1}{\partial z}\right)\right]$$
$$+\frac{\tau^2}{2}\left[\lambda^2\left(\frac{\partial^2\phi_2}{\partial x^2}-\frac{\partial^2\phi_1}{\partial x^2}\right)+\ldots\ldots\right]+\ldots\ldots$$

with similar expressions for δk and δl.

Comparing these with §§ 52-54, we see that the elongation of any elementary straight line which crosses the surface of discontinuity is simply to be taken as the sum of the elongations of the two portions into which it is divided by the surface; and if the component displacements satisfy the conditions of § 52, each in its own region and up to its bounding surface,—that is to say: if ϕ_1, χ_1, ψ_1, and all their derivatives be finite, infinitely small, or zero for all points whose displacements they represent, and also ϕ_2, χ_2, ψ_2—the strain-components at every point of the body, including those which lie on the surfaces of discontinuity, will, as before, depend entirely on the first derivatives of these functions, and the form of our theory of small strains will not be altered.

It must be particularly observed that the conditions of § 52 are only imposed on each strain-component-function within and up to the boundaries of its own region, and (so far as the conditions of *strain* are concerned) no relations need be assumed between the values which any two distinct forms take at the dividing surface.

In other words: while the displacements may be discontinuous in form from one region of the body to another, but must be continuous in value throughout the whole body; the strain-components admit of discontinuity both of form and value from one region to another, provided always that the discontinuities of value only occur coincidently with the discontinuities of form.

226.] **The Stress-Components.** Since the stress across any element of a surface of *strain-discontinuity* must, like that across any other surface in the body, be a purely mutual action between the two portions of matter immediately in contact with it on either side, it is obvious that even if the stress becomes discontinuous as to its form in crossing the surface, it must be to a certain extent continuous in value. For if we take an element dS of the surface (practically coinciding with an element of the tangent plane at its centre) and form two discs of elementary thickness, bounded by two elementary plane areas parallel to dS on either side, the theorem of § 137 must apply rigorously to each of these discs, so that the components of the stresses across their further faces can only differ by quantities of the same order of magnitude as the thickness of the two discs combined.

That is, if we draw a small plane area very close to a surface of discontinuity, and parallel to the tangent plane at its nearest point, the components of the stress across this area will preserve continuity of value while the area moves parallel to itself across the surface of discontinuity.

The analytical conditions can easily be deduced from § 144. Let ABC in Figure 9 represent an element of the surface of dis-

continuity separating region (I.) from region (II.). Let O be any point (x_1, y_1, z_1) in the first region. Complete the rectangular parallelepiped of which ABC is a diagonal plane, and let O' (x_2, y_2, z_2) be its opposite angle in region (II.)

The stress at O will have for its components $P_1, Q_1, R_1, S_1, T_1, U_1$, continuous functions of (x_1, y_1, z_1) throughout region (I.); and similarly the components $P_2, Q_2, R_2, S_2, T_2, U_2$ of the stress at O' will be functions of (x_2, y_2, z_2) continuous throughout region (II.)

If the signs of $(\lambda_{12}, \mu_{12}, \nu_{12}$ be taken as in § 144 they will represent the direction-cosines of the normal drawn towards region (II.) We can then show, as in that Article, by diminishing indefinitely the distance OO', that the components, in the *positive* directions of the axes, of the stress exerted across ABC by the tetrahedron $O'ABC$ on the tetrahedron $OABC$ must be

$$\left.\begin{array}{l} P_1\lambda_{12} + U_1\mu_{12} + T_1\nu_{12} \\ U_1\lambda_{12} + Q_1\mu_{12} + S_1\nu_{12} \\ T_1\lambda_{12} + S_1\mu_{12} + R_1\nu_{12} \end{array}\right\}.$$

And similarly, the components, in the *negative* directions of the axes, of the stress exerted across ABC by the tetrahedron $OABC$ on the tetrahedron $O'ABC$ must be

$$\left.\begin{array}{l} P_2\lambda_{12} + U_2\mu_{12} + T_2\nu_{12} \\ U_2\lambda_{12} + Q_2\mu_{12} + S_2\nu_{12} \\ T_2\lambda_{12} + S_2\mu_{12} + R_2\nu_{12} \end{array}\right\}.$$

Since these stresses must be identically equal and opposite, the conditions to be satisfied at a surface of *stress-discontinuity* are

$$\left.\begin{array}{l} \lambda_{12}(P_1 - P_2) + \mu_{12}(U_1 - U_2) + \nu_{12}(T_1 - T_2) = 0 \\ \lambda_{12}(U_1 - U_2) + \mu_{12}(Q_1 - Q_2) + \nu_{12}(S_1 - S_2) = 0 \\ \lambda_{12}(T_1 - T_2) + \mu_{12}(S_1 - S_2) + \nu_{12}(R_1 - R_2) = 0 \end{array}\right\} \quad \text{.........(63)}$$

If therefore the law of § 197 is to hold throughout, and if the body be everywhere homogeneous and isotropic, the relations which must exist between the strain-components at a surface of discontinuity are

$$\left.\begin{array}{l} \lambda_{12}[(m-n)(\Delta_1 - \Delta_2) + 2n(e_1 - e_2)] + \mu_{12}n(c_1 - c_2) + \nu_{12}n(b_1 - b_2) = 0 \\ \lambda_{12}n(c_1 - c_2) + \mu_{12}[(m-n)(\Delta_1 - \Delta_2) + 2n(f_1 - f_2)] + \nu_{12}n(a_1 - a_2) = 0 \\ \lambda_{12}n(b_1 - b_2) + \mu_{12}n(a_1 - a_2) + \nu_{12}[(m-n)(\Delta_1 - \Delta_2) + 2n(g_1 - g_2)] = 0 \end{array}\right\} \text{...(64)}$$

These, with the relations

$$u_1 = u_2,\ v_1 = v_2,\ w_1 = w_2 \text{.......................(65)}$$

between the component displacements, are the conditions to be satisfied at every point of a surface in the body at which the strain and stress become discontinuous in form.

227.] **The External Forces.** Our next object is to discover the systems of discontinuous Applied Forces and Surface Tractions which are capable of producing strains which satisfy these conditions. For this purpose we shall employ the method of § 219.

We shall suppose for simplicity that there is only one surface of discontinuity in the body, and to make this assumption as general as possible we shall suppose that it cuts the bounding surface of the body.

Let us call the two portions into which the volume of the body may thus be supposed divided regions (I.) and (II.), and let the suffixes 1 and 2 be used to distinguish all quantities belonging to them.

Let Σ_1, Σ_2 be the portions of the external surface of the body belonging to the two regions, and Σ_{12} the surface of discontinuity. We shall write the direction-cosines of the normal to the latter at any point $(\lambda_{12}, \mu_{12}, \nu_{12})$ when drawn towards (II.) and $(\lambda_{21}, \mu_{21}, \nu_{21})$ when drawn towards (I.), so that we have identically

$$\lambda_{12}+\lambda_{21}=\mu_{12}+\mu_{21}=\nu_{12}+\nu_{21}=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(66)$$

If the body be isotropic and homogeneous, its total potential energy is W, where

$$2W=\iiint\{(m-n)\Delta_1{}^2+2n(e_1{}^2+f_1{}^2+g_1{}^2)+n(a_1{}^2+b_1{}^2+c_1{}^2)\}dx_1dy_1dz_1$$
$$+\iiint\{(m-n)\Delta_2{}^2+2n(e_2{}^2+f_2{}^2+g_2{}^2)+n(a_2{}^2+b_2{}^2+c_2{}^2)\}dx_2dy_2dz_2$$

and its kinetic energy (if in motion) is $\mathfrak{T}$ where

$$2\mathfrak{T}=\iiint\rho(\dot{u}_1{}^2+\dot{v}_1{}^2+\dot{w}_1{}^2)dx_1dy_1dz_1$$
$$+\iiint\rho(\dot{u}_2{}^2+\dot{v}_2{}^2+\dot{w}_2{}^2)dx_2dy_2dz_2.$$

Let us now suppose, as in § 219, that small arbitrary variations δu, δv, δw of the displacements are produced throughout the body. Then the work done by the external forces will be

$$\iiint\rho(X_1\delta u_1+Y_1\delta v_1+Z_1\delta w_1)dx_1dy_1dz_1$$
$$+\iiint\rho(X_2\delta u_2+Y_2\delta v_2+Z_2\delta w_2)dx_2dy_2dz_2$$
$$+\iint(F_1\delta u_1+G_1\delta v_1+H_1\delta w_1)d\Sigma_1$$
$$+\iint(F_2\delta u_2+G_2\delta v_2+H_2\delta w_2)d\Sigma_2.$$

[It is hardly necessary to remark that no work can be done on the body as a whole by stress across Σ_{12}.]

The increment in the kinetic energy is

$$\delta \mathfrak{T} = \iiint \rho(\ddot{u}_1 \delta u_1 + \ddot{v}_1 \delta v_1 + \ddot{w}_1 \delta w_1) dx_1 dy_1 dz_1$$
$$+ \iiint \rho(\ddot{u}_2 \delta u_2 + \ddot{v}_2 \delta v_2 + \ddot{w}_2 \delta w_2) dx_2 dy_2 dz_2.$$

Thus the increment of the potential energy must be

$$\begin{aligned}\delta W = &\iiint \rho \{(X_1 - \ddot{u}_1)\delta u_1 + (Y_1 - \ddot{v}_1)\delta v_1 + (Z_1 - \ddot{w}_1)\delta w_1\} dx_1 dy_1 dz_1 \\ &+ \iiint \rho \{(X_2 - \ddot{u}_2)\delta u_2 + (Y_2 - \ddot{v}_2)\delta v_2 + (Z_2 - \ddot{w}_2)\delta w_2\} dx_2 dy_2 dz_2 \\ &+ \iint (F_1 \delta u_1 + G_1 \delta v_1 + H_1 \delta w_1) d\Sigma_1 \\ &+ \iint (F_2 \delta u_2 + G_2 \delta v_2 + H_2 \delta w_2) d\Sigma_2 \quad \text{......(67)}\end{aligned}$$

But we also have, on substituting the values of the strain-components,

$$\begin{aligned}\delta W = \iiint &\Big\{ [(m-n)\Delta_1 + 2ne_1] \frac{\partial \delta u_1}{\partial x_1} + nc_1 \frac{\partial \delta v_1}{\partial x_1} + nb_1 \frac{\partial \delta w_1}{\partial x_1} \\ &+ nc_1 \frac{\partial \delta u_1}{\partial y_1} + [(m-n)\Delta_1 + 2nf_1] \frac{\partial \delta v_1}{\partial y_1} + na_1 \frac{\partial \delta w_1}{\partial y_1} \\ &+ nb_1 \frac{\partial \delta u_1}{\partial z_1} + na_1 \frac{\partial \delta v_1}{\partial z_1} + [(m-n)\Delta_1 + 2ng_1] \frac{\partial \delta w_1}{\partial z_1} \Big\} dx_1 dy_1 dz_1 \\ + \iiint &\Big\{ [(m-n)\Delta_2 + 2ne_2] \frac{\partial \delta u_2}{\partial x_2} + nc_2 \frac{\partial \delta v_2}{\partial x_2} + nb_2 \frac{\partial \delta w_2}{\partial x_2} \\ &+ nc_2 \frac{\partial \delta u_2}{\partial y_2} + [(m-n)\Delta_2 + 2nf_2] \frac{\partial \delta v_2}{\partial y_2} + na_2 \frac{\partial \delta w_2}{\partial y_2} \\ &+ nb_2 \frac{\partial \delta u_2}{\partial z_2} + na_2 \frac{\partial \delta v_2}{\partial z_2} + [(m-n)\Delta_2 + 2ng_2] \frac{\partial \delta w_2}{\partial z_2} \Big\} dx_2 dy_2 dz_2.\end{aligned}$$

Let us first integrate by parts the first line of each of these integrals.

We thus get

$$\begin{aligned}&\iint \{[(m-n)\Delta_1 + 2ne_1]\delta u_1 + nc_1 \delta v_1 + nb_1 \delta w_1\} \lambda_1 d\Sigma_1 \\ &+ \iint \{[(m-n)\Delta_1 + 2ne_1]\delta u_1 + nc_1 \delta v_1 + nb_1 \delta w_1\} \lambda_{12} d\Sigma_{12} \\ &- \iiint \Big\{ \frac{\partial}{\partial x_1}[(m-n)\Delta_1 + 2ne_1] \,.\, \delta u_1 + \frac{\partial}{\partial x_1}(nc_1) \,.\, \delta v_1 \\ &\qquad + \frac{\partial}{\partial x_1}(nb_1) \,.\, \delta w_1 \Big\} dx_1 dy_1 dz_1 \\ &+ \iint \{[(m-n)\Delta_2 + 2ne_2]\delta u_2 + nc_2 \delta v_2 + nb_2 \delta w_2\} \lambda_2 d\Sigma_2 \\ &+ \iint \{[(m-n)\Delta_2 + 2ne_2]\delta u_2 + nc_2 \delta v_2 + nb_2 \delta w_2\} \lambda_{21} d\Sigma_{12} \\ &- \iiint \Big\{ \frac{\partial}{\partial x_2}[(m-n)\Delta_2 + 2ne_2] \,.\, \delta u_2 + \frac{\partial}{\partial x_2}(nc_2) \,.\, \delta v_2 \\ &\qquad + \frac{\partial}{\partial x_2}(nb_2) \,.\, \delta w_2 \Big\} dx_2 dy_2 dz_2.\end{aligned}$$

Treating each line in the same way, the integral to be taken over Σ_{12} is found to be

$$\begin{aligned}\iint\Big[&\{\lambda_{12}[(m-n)\Delta_1+2ne_1]+\mu_{12}nc_1+\nu_{12}nb_1\}\delta u_1\\&+\{\lambda_{21}[(m-n)\Delta_2+2ne_2]+\mu_{21}nc_2+\nu_{21}nc_2\}\delta u_2\\&+\{\lambda_{12}nc_1+\mu_{12}[(m-n)\Delta_1+2nf_1]+\nu_{12}na_1\}\delta v_1\\&+\{\lambda_{21}nc_2+\mu_{21}[(m-n)\Delta_2+2nf_2]+\nu_{21}na_2\}\delta v_2\\&+\{\lambda_{12}nb_1+\mu_{12}na_1+\nu_{12}[(m-n)\Delta_1+2ng_1]\}\delta w_1\\&+\{\lambda_{21}nb_2+\mu_{21}na_2+\nu_{21}[(m-n)\Delta_2+2ng_2]\}\delta w_2\Big]d\Sigma_{12};\end{aligned}$$

and in virtue of equations (64), (65), (66) this vanishes identically.

We have then, finally,

$$\begin{aligned}\delta W=\iint\Big[&\{\lambda_1[(m-n)\Delta_1+2ne_1]+\mu_1nc_1+\nu_1nb_1\}\delta u_1\\&+\{\lambda_1nc_1+\mu_1[(m-n)\Delta_1+2nf_1]+\nu_1na_1\}\delta v_1\\&+\{\lambda_1nb_1+\mu_1na_1+\nu_1[(m-n)\Delta_1+2ng_1]\}\delta w_1\Big]d\Sigma_1\end{aligned}$$

$$\begin{aligned}+\iint\Big[&\{\lambda_2[(m-n)\Delta_2+2ne_2]+\mu_2nc_2+\nu_2nb_2\}\delta u_2\\&+\{\lambda_2nc_2+\mu_2[(m-n)\Delta_2+2nf_2]+\nu_2na_2\}\delta v_2\\&+\{\lambda_2nb_2+\mu_2na_2+\nu_2[(m-n)\Delta_2+2ng_2]\}\delta w_2\Big]d\Sigma_2\end{aligned}$$

$$\begin{aligned}-\iiint\Big[&\Big\{\frac{\partial}{\partial x_1}[(m-n)\Delta_1+2ne_1]+\frac{\partial}{\partial y_1}(nc_1)+\frac{\partial}{\partial z_1}(nb_1)\Big\}\delta u_1\\&+\Big\{\frac{\partial}{\partial x_1}(nc_1)+\frac{\partial}{\partial y_1}[(m-n)\Delta_1+2nf_1]+\frac{\partial}{\partial z_1}(na_1)\Big\}\delta v_1\\&+\Big\{\frac{\partial}{\partial x_1}(nb_1)+\frac{\partial}{\partial y_1}(na_1)+\frac{\partial}{\partial z_1}[(m-n)\Delta_1+2ng_1]\Big\}\delta w_1\Big]dx_1dy_1dz_1\end{aligned}$$

$$\begin{aligned}-\iiint\Big[&\Big\{\frac{\partial}{\partial x_2}[(m-n)\Delta_2+2ne_2]+\frac{\partial}{\partial y_2}(nc_2)+\frac{\partial}{\partial z_2}(nb_2)\Big\}\delta u_2\\&+\Big\{\frac{\partial}{\partial x_2}(nc_2)+\frac{\partial}{\partial y_2}[(m-n)\Delta_2+2nf_2]+\frac{\partial}{\partial z_2}(na_2)\Big\}\delta v_2\\&+\Big\{\frac{\partial}{\partial x_2}(nb_2)+\frac{\partial}{\partial y_2}(na_2)+\frac{\partial}{\partial z_2}[(m-n)\Delta_2+2ng_2]\Big\}\delta w_2\Big]dx_2dy_2dz_2.\end{aligned}$$

Equating this to (67), we get, since δu, δv, δw are arbitrary and independent, throughout region (I.),

$$\left.\begin{aligned}&\frac{\partial}{\partial x_1}[(m-n)\Delta_1+2ne_1]+\frac{\partial}{\partial y_1}(nc_1)+\frac{\partial}{\partial z_1}(nb_1)+\rho(X_1-\ddot{u}_1)=0\\&\frac{\partial}{\partial x_1}(nc_1)+\frac{\partial}{\partial y_1}[(m-n)\Delta_1+2nf_1]+\frac{\partial}{\partial z_1}(na_1)+\rho(Y_1-\ddot{v}_1)=0\\&\frac{\partial}{\partial x_1}(nb_1)+\frac{\partial}{\partial y_1}(na_1)+\frac{\partial}{\partial z_1}[(m-n)\Delta_1+2ng_1]+\rho(Z_1-\ddot{w}_1)=0\end{aligned}\right\}\ldots\ldots(68)$$

throughout region (II.) a similar set of equations; over the portion of the external surface belonging to region (I.)

$$\left.\begin{aligned}\lambda_1[(m-n)\Delta_1+2ne_1]+\mu_1 nc_1+\nu_1 nb_1&=F_1\\ \lambda_1 nc_1+\mu_1[(m-n)\Delta_1+2nf_1]+\nu_1 na_1&=G_1\\ \lambda_1 nb_1+\mu_1 na_1+\nu_1[(m-n)\Delta_1+2ng_1]&=H_1\end{aligned}\right\}\text{..............}(69)$$

and a similar set of equations over that portion of the external surface which belongs to region (II.)

It appears from the two groups of equations (68) that the Applied Forces must be continuous in form and value throughout the same regions as the strain and stress, but that they are not necessarily continuous in value at the surfaces where they become discontinuous in form.

Similarly, from the two groups of equations (69) it appears that the Surface Tractions are continuous all over such portion of the external surface as forms part of the boundary of a region of strain-continuity, but they are at liberty to become discontinuous in value as well as in form on crossing a curve in which the external surface of the body is cut by any internal surface of discontinuity.

228.] **Summary.** A homogeneous isotropic solid may be supposed subjected to a distribution of Applied Force, the *form* of which as a function of the coördinates is continuous throughout definite regions of the body, but becomes discontinuous at the surfaces separating these regions, the *magnitude* of the force being also continuous within each region and either continuous or finitely discontinuous at the separating surfaces; provided that the forms of the surfaces of discontinuity bear such relations to the force-functions in the regions on either side that solutions of equations (51) or (52) can be obtained giving u, v, w as continuous functions of the coördinates throughout each region, and satisfying equations (63) and (64) at each point of a surface of form-discontinuity.

The distribution of Surface Tractions must be so related to the distribution of Applied Force, that these values of u, v, w may also satisfy equations (53) at all points of the bounding surface, whence we deduce that the Surface Traction cannot be anywhere discontinuous in form except on crossing the curves (if any) in which the bounding surface is cut by the surfaces of force-discontinuity, and that it may be either continuous or finitely discontinuous in value at these curves.

The body may also be subjected to a system of Surface Traction *only* which is continuous in form over definite areas of the bounding surface, and either continuous or finitely discontinuous in value at the separating curves; provided that solutions of the

equations deduced by making $X=Y=Z=0$ in equations (51) or (52) can be obtained which will give u, v, w as continuous functions of the coördinates, within regions separated by surfaces which meet the bounding surface of the body in the *curves of traction-discontinuity*, and will satisfy equations (65) and (64) at every point of each such surface, and equations (53) at all points of the bounding surface of the body.

Theoretically the latter problem always admits of solution.

229.] *Example of given discontinuous strain.* These results will be made much more intelligible by a simple illustration of the converse problem:—given a distribution of strain, discontinuous in form but satisfying the conditions (65) and (64) at every point of the surfaces of discontinuity, to find the distribution of discontinuous Force and Surface Traction which will maintain the strain. This problem is always determinate (§ 146).

Let us then consider the equilibrium of a beam of isotropic material, subjected to the strain proposed in the example at the end of § 224.

We will suppose the beam of rectangular form, and of length 2β, the origin being taken at the centre of one of its ends, and the axis of x in the direction of its length.

Region (I.) will extend from the plane of yz (the nearer end) to the plane $x=\beta$ (in the middle of the beam); region (II.) comprising the further half.

The component displacements in the two regions are

$$\left.\begin{aligned} u_1 &= 0 \\ v_1 &= cx \\ w_1 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} u_2 &= 0 \\ v_2 &= \tfrac{1}{2}c\beta\left(1+\frac{x^2}{\beta^2}\right) \\ w_2 &= 0 \end{aligned}\right\}$$

which satisfy equations (65) at all points of the surface of discontinuity of form, $x=\beta$.

The component strains are

$$\left.\begin{aligned} e_1 = f_1 = g_1 = a_1 = b_1 &= 0 \\ c_1 &= c \\ e_2 = f_2 = g_2 = a_2 = b_2 &= 0 \\ c_2 &= c\frac{x}{\beta} \end{aligned}\right\}$$

which satisfy equations (64).

The body being in equilibrium, the distribution of Applied Force in the two regions is, from equations (47).

$$\left.\begin{aligned} &X_1 = Y_1 = Z_1 = 0 \\ &X_2 = Z_2 = 0, \quad Y_2 = -\frac{nc}{\rho\beta} \end{aligned}\right\}$$

The force is therefore discontinuous *in value* as well as *in form*, being zero everywhere in region (I.) and constant in magnitude and direction everywhere in region (II.).

If the axes of y and z be taken perpendicular to the sides of the beam, we have from equations (49):—

On the sides perpendicular to Oy $(0, \pm 1, 0)$

$$\left.\begin{array}{l} F_1 = \pm nc,\ G_1 = 0,\ H_1 = 0 \\ F_2 = \pm nc\frac{x}{\beta},\ G_2 = 0,\ H_2 = 0 \end{array}\right\};$$

on the sides perpendicular to Oz $(0, 0, \pm 1)$

$$\left.\begin{array}{l} F_1 = G_1 = H_1 = 0 \\ F_2 = G_2 = H_2 = 0 \end{array}\right\};$$

and on the ends $(\pm 1, 0, 0)$

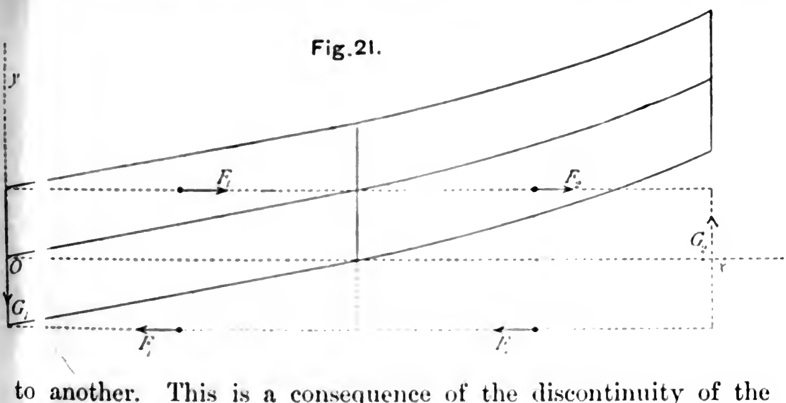
$$\left.\begin{array}{l} F_1 = 0,\ G_1 = -nc,\ H_1 = 0 \\ F_2 = 0,\ G_2 = 2nc,\ H_2 = 0 \end{array}\right\}:$$

It is obvious that on those sides (perpendicular to Oy) on which the Traction becomes discontinuous *in form*, this discontinuity occurs only at the lines in which the sides are cut by the plane of strain-discontinuity $x = \beta$.

In this example the Traction is continuous *in value* over the whole of each side.

It may be well to point out that the Surface Traction *necessarily* becomes discontinuous in passing from one side of the beam

Fig. 21.



to another. This is a consequence of the discontinuity of the direction-cosines in equations (49), and must occur wherever there is discontinuity in the *curvature of the surface*, even when—or rather, especially when—the strain is perfectly continuous throughout.

Figure 21 represents on a greatly exaggerated scale the

straining of the beam, which is everywhere in two dimensions, in planes parallel to xy.

The system of planes in the body initially parallel to zx are strained into cylindrical surfaces with generators parallel to Oz, each consisting of two portions—

(*i.*) A plane in region (I.) meeting the nearer end of the beam (plane of yz) in the same straight line as before the strain.

(*ii.*) A parabolic cylinder in region (II.), touching (*i.*) along the generator in which it is cut by the plane of discontinuity $x=\beta$, and having the plane of yz for its axial plane.

The dotted lines represent the state of things before the strain.

APPENDIX III.

Hooke's Law.

In 1676 Robert Hooke* published his *Description of Helioscopes, &c.*, on page 31 of which appeared the following paragraph:—

"3. *The true Theory of* Elasticity *or* Springiness, *and a particular Explication thereof in several Subjects in which it is to be found: And the way of computing the velocity of Bodies moved by them.* ceiiinosssttuu."

In a second treatise, published in 1678, (*Lectures de Potentia Restitutiva, or of Spring*, p. [1]), the anagram was explained as follows:—

"About two years since I printed this Theory in an Anagram at the end of my Book of the Descriptions of Helioscopes, viz., *ceiiinosssttuu, id est, Ut tensio sic vis;* That is, The Power of any Spring is in the same proportion with the tension thereof: That is, if one power stretch or bend it one space, two will bend it two, and three will bend it three, and so forward. Now as the Theory is very short, so the way of trying it is very easie."

His proof of his theory is purely experimental, and is based upon the following examples:—a spiral spring drawn out, a watch spring made to coil or uncoil, a long wire suspended vertically and stretched, and a wooden beam fixed (at one end) in a horizontal position, and loaded.

It is obvious that by *Tensio* Hooke meant extension or distortion (*i.e.* Strain) and by *Vis* the external force or couple producing the strain.

* For these quotations from Hooke I am indebted to Professor Tait's *Properties of Matter*, Ch. XI., Art. 221.

It was not till nearly a century and a half after Hooke's time that Stress (*pression intérieure*) came to be clearly defined in the sense in which we now employ it, although in 1807 Young (*Lectures on Natural Philosophy*) foreshadowed the modern and more general interpretation of Hooke's results by a series of investigations on the value of the modulus known by his name (§ 213), which he defined, for a given material, as the ratio of the force employed to elongate or compress a rod *of unit section* to the small elongation or contraction produced by it.

In all the examples adduced by Hooke, however, (as we shall see in Chapter VII.), the Stress is simply proportional* (for small strains) to the external force or couple, and consequently his anagram may justly be interpreted as the first enunciation of the grandly simple law of § 197,

Stress is proportional to Strain,

which has always been associated with his name.

A long series of observers, including Young, Coulomb, Wertheim, Kirchhoff, Hodgkinson, Kupffer, Tresca, etc., have repeated and varied Hooke's experiments during two centuries, with the result that the law has been firmly established *as an experimental fact*, and the various moduli ascertained for a great number of materials (*see* Appendix IV., *below*).

It was not till 1821 that Navier constructed, on the basis of Boscovitch's hypothesis (§ 37), the first mathematical theory of elasticity, confining himself to what are now called (at the suggestion of Cauchy) isotropic solids. Navier, followed by Poisson, and at first by Cauchy, obtained expressions for the intermolecular stresses (§ 28) which, as a necessary consequence of Boscovitch's hypothesis, reduced for small strains to linear functions of the first derivatives of the relative displacements of the molecules.

During the next year Cauchy gave a definition of Stress (*pression ou tension intérieure*) suitable to the hypothesis of continuous matter (§ 131), and obtained the equations of equilibrium, and of resolution and composition of stress (§§ 141-144) on that hypothesis.

In 1827 Cauchy finally rejected the inconsistent analysis of Navier and Poisson, and in this and the following years he developed Boscovitch's molecular theory to its furthest limit.

* This may be at once demonstrated for the case of a suspended wire stretched by a weight. If W be the weight, P the longitudinal stress, e the elongation, and A, A' the natural and strained sectional areas of the wire; then, with the notation of § 213 (neglecting as insignificant the weight of the wire itself),

$$W = A'P\,;\quad P = qe\,;$$
$$A' = A(1+f+g) = A(1-2\sigma e):$$
$$\therefore\quad W = qAe(1-2\sigma e) = qA\,.\,e, \text{ ultimately.}$$

In the next year he showed how the stress per unit area (§ 131) across any surface in the body might be deduced from the intermolecular stresses (§ 28) between the two systems of molecules separated by it. Applying this method to the results of his molecular theory he obtained for P, Q, R, S, T, U—in the case of a homogeneous crystal with three orthogonal planes of symmetry—formulæ identical in form with those of § 204, but without any necessary relation between the pairs of coefficients κ_{12} and κ_{21}, etc.

Substituting these values in the equations of equilibrium of (§ 142), he showed that they became identical in form with those deduced directly from the molecular theory.

Hooke's law was thus established, for crystalline and isotropic solids, as a necessary consequence of the molecular hypothesis, and it only remained to discover an independent proof that would be applicable to a purely *mathematical* method of treating the subject, independently of all hypotheses as to the intermolecular reactions.

Cauchy himself, followed by Lamé and (for many years) Barré de St. Venant, being convinced that every mathematical theory of elasticity must ultimately rest upon his development of Boscovitch's hypothesis, made no attempt at such independent proof, but felt himself justified (with the support of the experimental law) in assuming, for the most general case possible, equations of the form of (18) § 198 (leaving however the pairs of coefficients κ_{12}, κ_{21}, etc., independent).

In 1837 George Green applied his grand conception of the potential to the theory of elasticity, which he treated from a purely mathematical point of view. He showed that the potential energy of the strain, as well as the stress components, must be functions simply of the components of the actually existing strain.

He then *assumed* that V (§ 196) could be developed in a *complete* series of homogeneous functions of the strain components, viz.,

$$V = V_0 + V_1 + V_2 + V_3 + \dots\dots\dots\dots$$

It was then easy to show that V_0 and V_1 must be zero (energy and stress being measured from the natural state), and that in consequence, when strain was indefinitely reduced,

$$V = V_2 ;$$

whence the linearity of the stress components follows at once.

It must be observed that Green advanced no *theoretical* grounds for assuming the presence of the term V_2 in the expansion of V.

In 1845 however Stokes wrote as follows:—

"The capability which solids possess of being put into a state

of isochronous vibration" [*e.g.*, the fact that a tuning fork maintains its pitch unaltered during the whole time that its note is audible] "shews that the pressures" [stresses] "called into action by small displacements" [*i.e.*, relative displacements of the parts of bodies, or strains] "depend on homogeneous functions of those displacements of one dimension.

"I shall suppose moreover, according to the general principle of the superposition of small quantities, that the pressures due to different displacements are superimposed, and consequently that the pressures are linear functions of the displacements. Since squares of α, β and γ" [our u, v, w] "are neglected, these pressures may be referred to a unit of surface in the natural state or after displacement indifferently, and a pressure which is normal to any surface after displacement may be regarded as normal to the original position of that surface."

The first paragraph is an extension of the experimental proof of Hooke's law from static stresses, to which it had been hitherto confined, to those which exist at each instant in a body which is passing through continuously varying states of small strain.

The second paragraph sums up the principle of superposition in connection with the theory of elasticity, as developed in §§ 87, 88 and 153-155, and applied in § 197. This is the first indication of a purely mathematical proof of Hooke's law, capable of universal application independently of any hypothesis as to the intimate structure of matter.

It was not however till twenty-two years later that the importance of this demonstration was properly appreciated. Most British and German mathematicians—Kirchhoff, Clerk Maxwell and Rankine (1850), Sir W. Thomson (1855 and 1856), Kirchhoff (1858), Neumann (1859), Clebsch (1862), etc.—continued to rely on the experimental law of Hooke and Stokes, and on Green's tacit assumption that the second powers of the strain components were necessarily present in the expansion of the potential energy, and their first powers in those of the stress components. Rankine however inclined strongly towards a molecular theory of his own, which he regarded as an extension of Navier and Poisson's hypothesis. Meanwhile in France Lamé in 1852 still followed Navier, but in 1859 adopted Cauchy's view. Barré de St. Venant, a staunch follower of Cauchy, contents himself in his celebrated memoir on the torsion of prisms (1853), with the following:—

" . . . l'expérience prouve que l'*effort est proportionnel aux effets*, tant que ceux-ci restent très-petits, et non à des puissances de ces effets, autres que la première ; ce à quoi il n'y aurait pourtant aucune impossibilité *mathématique*. C'est même en cela que consiste le fameux principe *ut tensio sic vis*, avancé par Hooke et employé par Mariotte, il y a bientôt deux siècles. Admettons donc avec tout le monde que les pressions [*stresses*]

sont fonctions linéaires des dilatations [*elongations*] et des glissements [*shears*] tant qu'ils sont très-petits, . . . ": supported by a summary of Cauchy's proof.

In 1867, however, appeared the first edition of Thomson and Tait's *Natural Philosophy*, in which Green's energy method and Stokes' application of the principle of superposition were employed in combination exactly as we have adopted them in the present treatise (§§ 196-198).

This method of demonstration is now universally recognised by British and German mathematicians.

Barré de St. Venant however, among whose great services to the theory of elasticity his unceasing protest* against the assumption of Green and his followers will certainly not be reckoned the least, has never recognised it, and still declines to admit that Hooke's law can be based upon any *purely mathematical* proof, independently of a theory of intermolecular reactions.

It appears to me that the fact—on which St. Venant so strongly insists, in showing that any theory based upon the assumption of intermolecular forces which are functions only of the distance must necessarily lead to Hooke's law—of the essentially *differential* nature of strains and stresses affords us a proof quite independent of any such theory.

We have chosen, with the object of simplifying our analysis (§ 109), as our system of Strain Coördinates (§ 32) the six orthogonal components of small strain, which by definition vanish in one particular state of the body. This system, admirably adapted as it is for expressing small deviations from the natural state of the body, should be looked upon as exhibiting the *process* by which the change of state is produced, rather than as defining absolutely any particular state.

For our present purpose it will be advantageous to refer the configuration of the body to a more general system of Strain Coördinates. Let us take the following:

$$\left.\begin{array}{l}\mathfrak{e} = 1 + e,\ \ \mathfrak{f} = 1 + f,\ \ \mathfrak{g} = 1 + g,\\ \mathfrak{a} = \dfrac{\pi}{2} - a,\ \mathfrak{b} = \dfrac{\pi}{2} - b,\ \mathfrak{c} = \dfrac{\pi}{2} - c.\end{array}\right\}$$

These do not vanish for any strain within the limits of our theory, but are always positive; and in the natural state

$$\left.\begin{array}{l}\mathfrak{e}_0 = \mathfrak{f}_0 = \mathfrak{g}_0 = 1\\ \mathfrak{a}_0 = \mathfrak{b}_0 = \mathfrak{c}_0 = \dfrac{\pi}{2}\end{array}\right\},$$

* See his annotated edition of Navier's *Leçons sur Mécanique Appliquée*, App. V. (1864); his history of Hooke's law in Moigno's *Leçons de Mécanique Analytique*, Leçon XXII. (1868); and his annotated edition of Clebsch's *Théorie de l'Elasticité des Corps Solides*, note on § 11 (1883).

while the components of Strain are given by the differences

$$\left.\begin{array}{l} e = \mathfrak{e} - \mathfrak{e}_0,\ f = \mathfrak{f} - \mathfrak{f}_0,\ g = \mathfrak{g} - \mathfrak{g}_0, \\ a = \mathfrak{a}_0 - \mathfrak{a},\ b = \mathfrak{b}_0 - \mathfrak{b},\ c = \mathfrak{c}_0 - \mathfrak{c}. \end{array}\right\}$$

Similarly with the components of Stress. P, Q, R, S, T, U, being the tractions due to strain, must be regarded as the *differences* between the tractions existing in the strained state of the body and those present in the natural state. According to some authorities, the latter are identically zero throughout the body, but it is safest to make the most general assumption, and our argument will not be affected.

Let then $\mathfrak{P}, \mathfrak{Q}, \mathfrak{R}, \mathfrak{S}, \mathfrak{T}, \mathfrak{U}$ represent the components of the *total* tractions at any point of the body in the strained state $\mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, and let $\mathfrak{P}_0, \mathfrak{Q}_0, \mathfrak{R}_0, \mathfrak{S}_0, \mathfrak{T}_0, \mathfrak{U}_0$ be their values in the natural state; so that

$$\left.\begin{array}{l} P = \mathfrak{P} - \mathfrak{P}_0,\ Q = \mathfrak{Q} - \mathfrak{Q}_0,\ R = \mathfrak{R} - \mathfrak{R}_0, \\ S = \mathfrak{S} - \mathfrak{S}_0,\ T = \mathfrak{T} - \mathfrak{T}_0,\ U = \mathfrak{U} - \mathfrak{U}_0 \end{array}\right\}.$$

Whatever hypothesis we adopt as to the nature and origin of the mutual reactions between contiguous portions of the body, it is obvious that (the temperature being constant and uniform) we must assume that they depend solely on the configuration of the body, and that they vary in a definite and perfectly continuous manner throughout all continuous changes of state, within the limits of perfect elasticity. Since, for all we know to the contrary, the total traction components may be capable of assuming either sign, it is possible that they may pass through the value zero for particular values of the strain-coördinates. But it is not possible, under the assumed conditions of continuity, that the *rate of variation* of any traction component with any strain coördinate which it involves can change sign, or vanish, for any value of that coördinate. For example, if any traction component $\mathfrak{P}$ be continuously increased, within the limits of perfect elasticity, and if at any stage of the process any strain coördinate $\mathfrak{a}$ be found to increase with $\mathfrak{P}$, we cannot suppose that in any other stage—however limited—the value of $\mathfrak{a}$ can decrease or even remain stationary.

Hence, taking any one component $\mathfrak{P}$, we may assume a relation of the form

$$\mathfrak{P} = \phi(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}),$$

where ϕ is some continuous function of the independent strain coördinates; such that, if the first derivative of $\mathfrak{P}$ as to any one of these coördinates vanishes for any value of the coördinate, it must vanish for all values:—that is to say, $\mathfrak{P}$ must be altogether independent of that coördinate.

In the natural state

$$\mathfrak{P}_0 = \phi(\mathfrak{e}_0, \mathfrak{f}_0, \mathfrak{g}_0, \mathfrak{a}_0, \mathfrak{b}_0, \mathfrak{c}_0),$$

and by Taylor's theorem,

$$\begin{aligned}\mathfrak{P} = \mathfrak{P}_0 &+ (\mathfrak{e} - \mathfrak{e}_0)\left[\frac{\partial\mathfrak{P}}{\partial\mathfrak{e}}\right]_0 + (\mathfrak{f} - \mathfrak{f}_0)\left[\frac{\partial\mathfrak{P}}{\partial\mathfrak{f}}\right]_0 \\ &+ (\mathfrak{g} - \mathfrak{g}_0)\left[\frac{\partial\mathfrak{P}}{\partial\mathfrak{g}}\right]_0 + (\mathfrak{a} - \mathfrak{a}_0)\left[\frac{\partial\mathfrak{P}}{\partial\mathfrak{a}}\right]_0 \\ &+ (\mathfrak{b} - \mathfrak{b}_0)\left[\frac{\partial\mathfrak{P}}{\partial\mathfrak{b}}\right]_0 + (\mathfrak{c} - \mathfrak{c}_0)\left[\frac{\partial\mathfrak{P}}{\partial\mathfrak{c}}\right]_0 \\ &+ \tfrac{1}{2}\{\dots\dots\dots\} + \dots\dots\dots\dots\dots\dots\end{aligned}$$

Thus, substituting for the differences,

$$\begin{aligned}P = e\left[\frac{\partial\mathfrak{P}}{\partial\mathfrak{e}}\right]_0 + f\left[\frac{\partial\mathfrak{P}}{\partial\mathfrak{f}}\right]_0 + g\left[\frac{\partial\mathfrak{P}}{\partial\mathfrak{g}}\right]_0 - a\left[\frac{\partial\mathfrak{P}}{\partial\mathfrak{a}}\right]_0 - b\left[\frac{\partial\mathfrak{P}}{\partial\mathfrak{b}}\right]_0 - c\left[\frac{\partial\mathfrak{P}}{\partial\mathfrak{c}}\right]_0 \\ + \tfrac{1}{2}\left\{ e^2\left[\frac{\partial^2\mathfrak{P}}{\partial\mathfrak{e}^2}\right]_0 + \dots\dots\dots \right\} + \dots\dots\dots\dots\dots\dots\end{aligned}$$

By what has just been said, if the coefficient of the first power of any difference vanishes, that difference does not occur at all in the expansion of $\mathfrak{P}$. Hence it follows that if the coefficient of the first power of any strain component vanishes, that strain component does not appear at all in the expansion of P.

In other words, the expansion of any stress component contains the first powers of all those strain components of which it is a function.

Ultimately therefore, when the strain is very small, each stress component must be a linear function of all those strain components upon which it depends.

Thus Hooke's law is demonstrated, independently of any hypothesis as to the origin of stress.

APPENDIX IV.

Elastic Properties of Natural Materials.

We have already indicated (§§ 12 and 13) a rough subdivision of solid materials into the *brittle*, whose range of elasticity is practically coextensive with their power of resisting rupture, and the *malleable, plastic, or ductile*, capable of enduring stress which very greatly exceeds their elastic limits, and distinguished by their ability to acquire a permanent *set* under such stresses.

We now proceed to a more detailed account of the behaviour of natural materials under stresses varying from zero to the point

of rupture, and we shall find it convenient to subdivide the *malleable* bodies into two classes, namely—

A. The *Plastic*, which acquires a set whenever subjected to stress exceeding a certain definite limit, characteristic of the material; but whose mechanical qualities are in no way modified thereby. This class, which includes the so-called "soft solids" (such as clay and wax) as well as lead amongst the metals, has for us an almost purely theoretical interest. We shall find that it merges insensibly into the class of *Fluids*, and we shall naturally be led to give under this head a fuller account of that property of *viscosity* which, although it is manifested to a small extent (§ 16) by all solids under small elastic strains, is in its fuller development confined to fluids and to malleable solids strained beyond the limits of their elasticity.

B. The *Ductile*, the limits of whose elasticity are extended by every stress which produces a set, and whose *hardness* or resistance to further set depends in consequence upon the greatest stress to which they may have already been subjected, as well as upon the intrinsic qualities of the material. This class is by far the most interesting and important from a practical point of view, including as it does nearly all those metals that are most frequently employed in structures.

My general authority for the experimental facts on which the following account is based is Prof. Cotterill's *Applied Mechanics*, Chapter XVIII., where many references to the original memoirs, etc., will be found. Figures 23, 25, 27, 28 are taken from the same source. The account of the behaviour of a bar of ductile metal, when very cautiously elongated to the point of rupture, is in substance reproduced, together with Figures 24, 24 A, 25 A, and 28, from two letters by Prof. Alex. B. W. Kennedy, published in *Nature*, vol. xxxi., p. 504, and vol. xxxii., p. 269. I have also drawn freely from the very interesting discussion on Mr. Hackney's paper "On the Adoption of Standard Forms of Test-pieces for Bars and Plates," reported in the *Proceedings of the Institution of Civil Engineers*, vol. lxxvi., pp. 70-158. Figures 24 and 26 are reproduced, on a more convenient scale, from that report, and are due to Prof. Kennedy.

A.—*Plastic Solids and Viscous Fluids.*

A "perfectly plastic" solid—which is as much an abstraction as a "perfectly elastic" or "perfectly rigid" solid—is defined by the following properties:—

(*i.*) It possesses perfect elasticity of bulk (§ 14) under purely Hydrostatic Pressures (§ 174) whether positive or negative; that is, for uniform cubical dilatations or compressions unaccompanied by distortion (§ 211). This bulk-elasticity is limited in the direction of dilatation only by the tenacity (§ 222) of the material, and on the side of compression is theoretically without limit.

(*ii.*) Its elasticity of form (§ 14) is perfect for all distortions

within a certain perfectly defined and usually very narrow limit, which is characteristic of the material.

Plasticity. The utmost resistance S which a perfectly plastic material can offer to distorting stress coincides with the limit of its elasticity of form, and it follows from § 135 that it is impossible to *maintain* in the interior of such a body a shearing stress exceeding S by however little. The excess is, in fact, entirely unbalanced by any resistance on the part of the body, which in consequence relieves itself by continuous change of shape without change of volume, until—if the circumstances permit—the maximum shearing stress is reduced to the limit S. A perfectly plastic solid may therefore be distorted to any extent, *however great*, by the continuous application of a shearing stress exceeding S by any amount *however small.*

Moreover, since S is the limit of elastic resistance to distortion, the resilience (§ 222) of the body is precisely the same as if it were only strained to its elastic limits, and consequently the large distortion produced by the excess of shearing stress is not recoverable, but remains as a *permanent set* (§ 12) when the stress is removed.

Flow. This continuous and permanent change of shape, without change in the volume or density of any part, is called Flow; and the tendency to flow, *without modification of any mechanical property,* under continuously applied and constant distorting stress, however little in excess of *a definite elastic limit* S, is called Plasticity.

Fluidity. Those substances for which S is absolutely zero, but which nevertheless possess perfect elasticity of volume, are called Perfect Fluids. A perfect fluid is therefore totally devoid of rigidity (§ 210), and offers no resistance whatever to shearing stress: and a purely hydrostatic pressure is the only form of stress that can be maintained within it, even for an instant.

The characteristic property of perfect fluids is therefore their tendency to flow freely under any distorting stress *however small:* and this property is called Fluidity.

Solidity. Since the quantity S—which we may call the *measure of solidity*—may be indefinitely small, it is obvious that no strict line of demarcation can be drawn between fluids and plastic solids, but that a series of the latter arranged in descending order of solidity may be supposed to pass insensibly into the former group. Even if the fluids and plastic solids with which we have to deal in nature were free from *viscosity* (*see below*), as we have hitherto supposed, the universal and unavoidable presence of the shearing stress due to gravity would render it difficult practically to distinguish a perfectly plastic solid of quite conceivably small *solidity* from a perfect fluid of the same density and compressibility.

It should be observed that the solidity $\mathbf{S}$ is a *limit*, and not a *modulus*. Thus it is quite possible for a plastic body of very small solidity to possess very large moduli of compression and rigidity. In such a case the limiting shear $\mathbf{S}/n$ which the body can suffer without flow, is of course very small.

The exact nature of Flow will be better understood by working out a simple example.

Let a right circular cylinder of perfectly plastic material, the solidity of which is $\mathbf{S}$, be placed with its base upon a perfectly smooth and rigid horizontal plate, and let another smooth and rigid plate be laid on the top of the cylinder and loaded until the total weight applied is $\mathbf{W}$. Let h_0 be the initial height of the cylinder, and A_0 the initial area of its base.

We will assume for simplicity that its ends can slip freely over the surfaces of the plates, and that the action of gravity upon it may be neglected.

The load $\mathbf{W}$ will then be uniformly distributed over the upper surface, and the stress throughout the cylinder will be a homogeneous longitudinal pressure $\mathbf{W}/A_0$. There will be no longitudinal stress in any horizontal direction, and therefore the principal normal stresses will be at every point

$$N_1 = -\mathbf{W}/A_0,\ N_2 = N_3 = 0.$$

Hence it follows from Example 6 on Chapter III. that at every point there exists a shearing stress of amount $\mathbf{W}/2A_0$, tending to diminish the height and increase the diameter of the body.

If $\mathbf{W}/2A_0$ be less than $\mathbf{S}$, the whole stress will be within the elastic limit of the material, and the strain (wholly elastic) will be, with the notation of § 213,

$$\epsilon_1 = -\mathbf{W}/A_0 q,\ \epsilon_2 = \epsilon_3 = +\sigma\mathbf{W}/A_0 q.$$

If now $\mathbf{W}$ be increased to the value $2A_0\mathbf{S}$, the cylinder will be strained precisely to the limit of its elasticity of form, and we shall have

$$\epsilon_1 = -2\mathbf{S}/q,\ \epsilon_2 = \epsilon_3 = +2\sigma\mathbf{S}/q.$$

The resilience per unit volume is then by equation (43) of § 215,

$$V = \tfrac{1}{2}\{(m-n)(4\sigma-2)^2 + 2n(8\sigma^2+4)\}\mathbf{S}^2/q^2,$$

whence on reduction

$$V = 2\mathbf{S}^2/q.$$

This condition of the cylinder is represented by the axial section $ABCD$ in Figure 22. Its dimensions are

$$\left.\begin{aligned} h &= h_0(1+\epsilon_1) = h_0(1-2\mathbf{S}/q) \\ A &= A_0(1+\epsilon_2+\epsilon_3) = A_0(1+4\sigma\mathbf{S}/q) \end{aligned}\right\}$$

Now let **W** be increased. Since the limit of solidity has already been reached there will be at each point a shearing stress of amount

$$\mathbf{W}/2A - \mathbf{S}$$

in excess of the resistance offered, and to relieve itself from this stress the body will begin to flow.

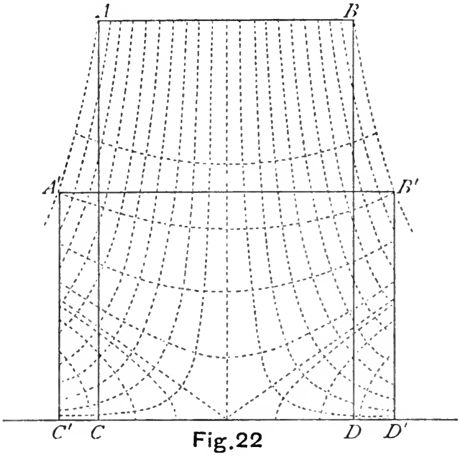


Fig.22

From the symmetry of the conditions the centre O of the base will remain at rest, and if $\epsilon_1', \epsilon_2', \epsilon_3'$ be at any moment the additional elongations due to flow, we must have

$$\epsilon_1' + \epsilon_2' + \epsilon_3' = 0, \quad \epsilon_2' = \epsilon_3'$$

Thus $$\epsilon_2' = \epsilon_3' = -\tfrac{1}{2}\epsilon_1',$$

and by § 126 the equipotential surfaces will be the hyperboloids of revolution

$$2\xi^2 = \eta^2 + \zeta^2 \pm C^2,$$

$O\xi$ being vertical.

The lines of displacement (§ 127)—which in this case preserve their form during the whole time that flow is taking place, and are called **Lines of Flow**—are therefore the system of curves satisfying the differential equations

$$-\frac{d\xi}{2\xi} = \frac{d\eta}{\eta} = \frac{d\zeta}{\zeta}.$$

The solution is to be symmetrical as to η and ζ, and therefore the equations of the Lines of Flow are

$$\left.\begin{aligned} \xi(\eta^2 + \zeta^2) &= \text{constant} \\ \eta/\zeta &= \text{constant} \end{aligned}\right\}.$$

Every point of the body will describe that line of flow which passes through its initial position (*see* Figure 22), and this process will continue until the maximum shearing stress at each point has been again reduced to the limiting value **S** by the expansion of the surface over which the constant load **W** is applied.

If h', A' be the final height and sectional area, we have

$$\left.\begin{aligned} \mathbf{W} &= 2\mathbf{S}A' \\ A'h' &= Ah \end{aligned}\right\}$$

and therefore

$$\left.\begin{aligned} A' &= \frac{\mathbf{W}}{2\mathbf{S}} \\ h' &= \frac{2\mathbf{S}A_0h_0}{\mathbf{W}}\left[1 + \frac{(4\sigma - 2)\mathbf{S}}{q}\right] \\ &= \frac{2\mathbf{S}A_0h_0}{\mathbf{W}}\left(1 - \frac{2\mathbf{S}}{3k}\right) \end{aligned}\right\}.$$

The cylinder is now in the condition represented by $A'B'C'D'$.

Since the principal stresses are once more

$$X_1 = -2\mathbf{S}, \quad X_2 = X_3 = 0,$$

the resilience is by equation (43a) of § 215

$$V = \tfrac{1}{2}\left(\frac{1}{2n} - \frac{\sigma}{q}\right)4\mathbf{S}^2$$

$$= \frac{2\mathbf{S}^2}{q},$$

which is precisely the same as before flow began.

If therefore the load be removed the elastic recovery of the cylinder will be exactly that which corresponds to this elastic stress. That is, there will be a vertical elongation $2\mathbf{S}/q$ and a contraction $2\sigma\mathbf{S}/q$ in every horizontal direction. Thus if (h'', A'') be the form in which the body is left on removal of the load

$$h'' = h'(1 + 2\mathbf{S}/q)$$

$$= \frac{2\mathbf{S}A_0h_0}{\mathbf{W}}(1 + 4\sigma\mathbf{S}/q)$$

and

$$A'' = A'(1 - 4\sigma\mathbf{S}/q)$$

$$= \frac{\mathbf{W}}{2\mathbf{S}}(1 - 4\sigma\mathbf{S}/q).$$

The final *volume* is

$$A''h'' = A_0h_0.$$

so that the cubical compression is entirely recovered, as of course it ought to be. The body is simply permanently deformed, without alteration of its volume, density, or solidity.

For an example of the continued flow of a plastic body under shearing stress exceeding **S** we have only to suppose the figure reversed, and the load applied so as to transmit a *tension* along the cylinder. If $\mathbf{W}=2A_0\mathbf{S}$ the elastic yielding of the material under the longitudinal traction $2\mathbf{S}$ will diminish the area over which the load is applied to $A_0(1-4\sigma\mathbf{S}/q)$, and simultaneously increase the maximum shearing stress to $\mathbf{S}(1+4\sigma\mathbf{S}/q)$.

Flow will therefore begin in the reverse direction to that followed under the former circumstances, and every point in the body will retrace approximately its former course. The effect of this flow will of course be to diminish continually the area over which the load is applied, and therefore to increase continually the shearing stress. If **W** be continually diminished the

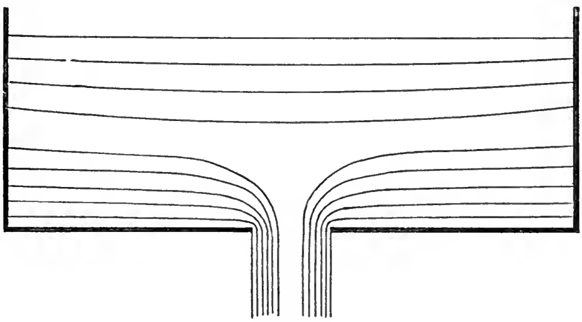


Fig. 23

shearing stress may be kept as little as we please in excess of **S**, and the cylinder may be indefinitely attenuated and elongated.

This of course assumes that the tenacity (§ 222) of the material is more than twice as great as its solidity. If this is not the case the cylinder would be ruptured instead of flowing under the assumed circumstances.

But it follows from Example 6 on Chapter III. that flow may be produced without risk of rupture by introducing a lateral *pressure* in addition to the longitudinal traction. If Π be this pressure, the conditions to be satisfied are

$$\left.\begin{array}{r} N_1 < \text{tenacity} \\ \tfrac{1}{2}(N_1+\Pi) > \text{solidity} \end{array}\right\}.$$

Figure 23 represents an experiment of Tresca's on the flow of lead. A series of flat circular plates of lead were placed in a rigid cylinder, having a small orifice in the centre of its base, and

forcibly compressed. The lead issues as a jet from the orifice, and the lines of flow indicated by the distorted boundaries of the plates bear a striking resemblance to the corresponding lines in water issuing from an orifice in a horizontal plate.

Tresca found lead to be very fairly plastic, and ascertained its solidity to be about

$$\mathbf{S} = 200{,}000 \text{ grammes per square centimetre}$$
$$= 2850 \text{ pounds per square inch.}$$

The quality of manufactured lead is however more variable than that of any other metal.

Viscosity. The properties which we have ascribed to "perfectly plastic" solids and "perfect fluids" are modified in actual materials by the universal presence of more or less viscosity (§ 16).

(*i.*) According to the theory of fluid viscosity advanced by Stokes in 1845, and subsequently extended and verified by himself, Clerk Maxwell, Poiseuille, O. E. Meyer, Helmholtz and Piotrowski, and others, this property consists in a kind of sliding friction between layers of molecules, only called into play when relative motion of the layers is taking place in a direction tangential to their common surface.

Viscosity and Shearing Motion (§ 95) must therefore be regarded as inseparable: and, since flow is merely continuous shear, it follows that flow is always opposed by viscosity.

On the other hand, a uniform cubical dilatation or compression (§§ 104, 105, 112)—whether homogeneous or not—is specially characterised by the absence of shear, and this form of strain consequently possesses the unique property of being absolutely unaffected by the existence of viscosity.

(*ii.*) The amount of viscous resistance of a given solid or fluid, at a given uniform temperature, depends only upon, and increases continuously with the *rate at which shear takes place*, and invariably *vanishes with this rate*:—or, in other words, infinitely small resistance is offered by viscosity to infinitely slow flowing.

The existence of viscosity in a material does not therefore affect the conditions of *equilibrium* under stress, but only resists and modifies the process (other than simple dilatation or compression) by which a body passes from one state of strain to another; a relation being introduced between the *magnitude of the stress* producing the change of state, and the *time occupied* by the change.

Although a fluid may, in virtue of its viscosity, offer immense resistance to *sudden* or *rapid* distortions, yet any shearing stress, however small, will suffice to produce any required amount of flow, however great, *provided that it be applied continuously for a sufficient length of time.*

The same statement applies to viscous plastic solids, under continuously applied shearing stress exceeding by however little their limit of solidity **S**.

This introduction of the element of *time* or *velocity* into the relations between shear and shearing stress is usually described, in the case of fluids, as constituting an imperfection in their fluidity; and it is obvious that a viscous plastic solid is *imperfectly plastic* in precisely the same sense that a viscous fluid is imperfectly fluid.

Apparently all fluids possess absolutely perfect elasticity of volume, and it is probable that plastic solids approach very nearly to this condition, at any rate up to a very high degree of pressure. We may say then that, while a perfect fluid or a perfectly plastic solid does not exist in nature, yet when in equilibrium or while undergoing changes of volume without distortion all fluids and plastic solids behave as if their fluidity or plasticity were perfect.

(*iii.*) The resistance offered to shear by viscosity is not of an elastic nature. The work done in overcoming it is not stored up as potential energy, but is entirely dissipated in the form of *molecular* kinetic energy or *heat* (§§ 2, 20).

Thus a viscous fluid has no resilience under distortion any more than a perfect fluid; and a viscous plastic solid has only so much distortional resilience as corresponds to the limit **S** of its solidity.

To make this distinction plain let us compare the behaviour of a perfectly elastic solid and of a viscous fluid under a simple distortion. The resistance of the solid is quite independent of the *rate* at which distortion takes place, and is simply a function of the *amount* of the distortion, continually increasing with that amount. None of the work done in overcoming this resistance is lost (the temperature being maintained constant and uniform), but it is all stored up as potential energy, ready at any instant to supply just as much work as will suffice to restore the body to its natural state from any condition of distortion in which it may be left. The fluid, on the other hand, offers a resistance which is quite independent of the *amount* of distortion existing at any moment, and depends only on the *rate* at which it is being produced. The work done in overcoming this resistance is transformed into heat, and if the fluid be maintained at a constant and uniform temperature this heat must be continually withdrawn [§§ 24, (*ii.*), 26], and the work is consequently lost both to the fluid and to the agent producing the strain. When the straining forces are removed, there is no tendency or power on the part of the fluid to reverse the strain, because no energy has been retained to be reconverted into mechanical work.

(*iv.*) It is sufficiently obvious that the principle of superposition of small strains (§§ 87, 88) must be equally applicable to

small rates of straining, which are simply the differential coefficients of small strains as to the time.

Also since the viscous resistances to finite rates of shearing are finite, the shearing stresses called into play by viscosity to resist small rates of shearing must be small stresses (§ 153), and therefore subject to the law of superposition (§ 155), equally with the elastic stresses.

Thus Hooke's Law (§ 197) is applicable to the viscous resistances offered by all bodies to distortion, when the rates of distortion are small quantities in the sense of § 58—that is to say, *the resistance is, within these limits, simply proportional to the rate of distortion.*

The coefficient or *modulus of viscosity* ν, for a given isotropic material at a given uniform temperature, is defined (*see* § 210) as the shearing stress required to produce the unit of shear per unit of time in any plane. For instance, if ν be given in the C.G.S. absolute measure, and any two parallel planes be taken in the interior of the body one centimetre apart, it will require a tangential stress of ν/ω dynes per square centimetre on each of these planes, in opposite directions, to produce a small relative velocity $1/\omega$ of a centimetre per second, in any direction parallel to themselves.

Sir William Thomson made experiments in 1865 on the viscosity of various metals, by observing the rate of diminution of the torsional vibrations of round wires (*see* § 16). The theory of such vibrations will be considered in Chapter X. The formulæ arrived at are very complicated, and it would require a considerable series of experiments to evaluate the modulus of viscosity.

The law by which the viscous resistance of solids depends upon the rate of distortion, when that rate is considerable, is unknown, as also is the law connecting stress with strain for finite values of the strain.

In the case of fluids however—as was first demonstrated on sound theoretical grounds by Stokes (1845), and subsequently verified in various ways by the experimental authorities named above—the proportional law holds for all rates of distortion (the restriction of § 210 does not apply to the flow of fluids, which, however great its extent, leaves all their mechanical properties unaltered).

The value of ν varies greatly for different fluids, and also depends largely on the temperature. Fluids may be arranged in four groups as follows, according to the magnitudes of their moduli of compressibility and viscosity, and the law of variation of the latter with the temperature.

(1.) *Gases and Vapours:* highly compressible fluids of small viscosity. Characterised by a tendency to indefinite expansion

and rarefaction which can only be restrained by the continued exercise of external pressure, or of impressed forces such as gravity: the hydrostatic pressure at every point in the interior of a gas or vapour is therefore essentially positive.

Another distinguishing property is that the viscosity *increases* as the temperature rises.

According to Clerk Maxwell the modulus of viscosity for atmospheric air at $t°$ Cent. is

$$\nu = \cdot 0001878(1 + \cdot 00366\, t)$$

in dynes per square centimetre.*

The value of ν for oxygen is rather greater, and for carbonic acid gas rather less, while for hydrogen it is less than half that of air.

All the remaining groups of fluids have a compressibility comparable with that of solids (*see* Table B *below*) and are characterised by the possession of a definite density and volume per unit mass at each temperature, when free from external pressure.

Their viscosity invariably diminishes as the temperature rises.

(2.) *The Mobile Liquids* (ether, alcohol, water, turpentine, mercury, etc.): viscosity much greater than that of gases but still very moderate. These liquids are therefore capable of flowing freely and *rapidly* under small shearing stresses, such as that of gravity; while a falling stream readily breaks up into separate drops.

Poiseuille found for water

At	0°	Cent., $\nu =$	·018	dynes per	sq. cent.
„	10°	„	·013	„	„
„	20°	„	·010	„	„

Helmholtz and Piotrowski found by another method that at 24°·5 Cent.

$$\nu = \cdot 014061 \text{ dynes per sq. cent.,}$$

while O. E. Meyer's results were about one sixth greater than Poiseuille's.

(3.) *The Viscid Liquids* (treacle, glycerine, Canada balsam, tar, etc.): largely increased viscosity at low and moderate temperatures, the flow under gravity being sluggish and very characteristic. A falling stream becomes excessively reduced in area before breaking up into drops. The viscosity diminishes with great rapidity as the temperature rises.

Schöttner found for glycerine

At	3°	Cent., $\nu =$	42	dynes per	sq. cent.
„	20°	„	8	„	„

* See reduction tables at end of this Appendix.

This enormous reduction of the viscidity* of glycerine may be easily demonstrated on a cold day. A bottle of pure glycerine which has been exposed to the air may be turned completely over before the liquid begins to run; but after being warmed for ten minutes before a fire the glycerine will become almost as mobile as turpentine.

(4.) *The Ultra-Viscous Fluids* (pitch, resin, cobbler's wax, sealing wax, etc.): possess enormous viscosity at ordinary temperatures, which however diminishes with quite startling rapidity as the temperature rises to the "melting point" after which they are merely more or less viscid.

Flow under gravity is in some cases imperceptible even in the course of years. The resinous seals, etc., in the Egyptian tombs are however often found to have flowed down to such an extent as to be reduced to mere shapeless masses.

A stick of sealing wax supported on two pegs near its extremities will bend till it drops between them, in two or three days unless the weather be very cold.

Sir W. Thomson fixed a cake of Canada pitch in the middle of a large vessel of water, to avoid rapid variations of temperature, and placed some bullets above it and some corks below. At the end of some months all were found to have forced their way through, merely by the effect of gravity, although it would have required enormous pressure to drive them through at a visible rate.

The viscosity of these fluids at low temperatures is in general so great that they are extremely brittle, and easily broken by very moderate forces suddenly applied.

The student will however find that with very cautious handling he can bend or twist a stick of good sealing wax to almost any extent. He will observe that the more it is manipulated *continuously* the easier manipulation becomes. The explanation of this is that the appreciable amount of heat generated by viscosity has not time to radiate, and the wax in consequence becomes warmer and less viscous. A good deal of warmth is also communicated by the hands. If the stick be laid aside for a short time it will be found to have recovered its original viscous properties.

It will also very probably be noticed that the stick when bent possesses a small amount of resilience, and straightens itself slightly when released. This is due to the fact that the viscous resistance to flow, even at such small rates of bending as can be imposed without danger of breaking the stick, is greater than the resistance to compression.

* *Viscidity* may be defined as the *visible* resistance of viscosity to gravitation.

Consequently in bending the stick the portion on the inner side of the curve is compressed longitudinally and the portion on the outer side elongated. If the stick were forced to remain in its bent form it would relieve itself gradually from this state of strain by *lateral* expansion of the concave side and lateral compression of the convex side, thus reducing the resultant strain to a mere distortion, and restoring the density at every point to its initial value.

As it is, when the stick is released immediately after having been bent, the resilience of volume of the wax expends itself in the only way which is unopposed by viscosity, namely in uniform cubical dilatation or compression of the parts that have been contracted or elongated. It is easy to see that the effect of this partial reversal of the strain is to reduce the curvature by an amount bearing a finite ratio to the whole.

The Plastic Solids (clay, wax, tallow, lead, etc.) display considerable viscosity, which increases much more rapidly with the rate of flow than does that of fluids, but which is probably less at very small velocities than that of any of the ultra-viscous fluids.

Thus, when a plastic body whose solidity is small compared with its rigidity is executing vibrations within the limits of its elasticity, the amplitude of the distortion is necessarily so minute that, even if the period be a very small fraction of a second, the *rate* of distortion is still very small, and the effects of viscosity are only observable by means of the gradual diminution of amplitude. But when the body is forced to *flow* at a finite rate the viscous resistance is enormously increased, and its heating effect may become very conspicuous.

The materials mentioned above possess very different degrees of solidity, from that of lead which, as we have already mentioned, was found by Tresca to be about 200,000 grammes or $19\frac{1}{2}$ millions of dynes per square centimetre, to that of clay the existence of which only rests upon a delicate experiment of Coulomb's.

A tallow candle laid on two pegs will not go on bending indefinitely until released from the stress caused by gravity, like a bar of sealing wax under the same circumstances, but will gradually (owing to its viscosity) assume a certain definite curve (*see* Chapter VII.) in which the *elastic* stresses called into play by flexure will maintain it in equilibrium. Under a distorting stress exceeding the limit of its solidity it can however be made in time to flow indefinitely.

B.—*Ductile Metals.*

This large and important class includes wrought iron, the softer qualities of steel, zinc, tin, copper, brass, gun metal, gold and silver, and in fact almost all those metals—cast iron and hard steel being the only important exceptions—which are most in request for purposes of construction, whether on a large or a small scale.

Ductility. A "perfectly ductile" material possesses perfect elasticity of bulk under all compressions and under all dilatations short of the limit of tenacity. It also possesses perfect elasticity of form under distorting stresses short of a certain limit **S**, beyond which flow begins, as with plastic bodies, The conditions of ductile flow differ however from those of plastic flow in the two following important particulars:—

(*i.*) The resistance **S** of the body to flow is not an absolutely fixed quantity depending only on the nature of the material, but continually increases with the amount of flow. Thus *to produce continuous flow in a ductile body it is necessary continuously to increase the distorting stress;* and if the maximum distorting stress applied to the body be greater than **S** but well within the strength of the material, the system will reach a state of equilibrium after the *definite* amount of flow which is required to equalise the resistance of the body with the applied stress **S′**.

(*ii.*) If the body be now released from stress, the elastic portion of the strain (including all cubical dilatation or compression) will be recovered, and the body will resume its original volume and density, the effect of the flow remaining as a permanent alteration of form.

If *the same type of stress* be now gradually reapplied, it is found that, within the elastic limits, the strain produced follows the same law as before. This proves that the elastic moduli, *as involved by this particular form of stress,* are unaffected by the *set.* [This point will be further considered presently, under head (*iii.*).]

The *limit* of perfect elasticity is however found to have been extended. Flow no longer begins at the original limit **S**, but is postponed until the distorting stress reaches the value **S′**—*i.e.,* the *maximum distorting stress under which the body has hitherto maintained equilibrium.* If the stress be carried beyond **S′** further flow will take place, and on release of the body and reapplication of the stress the limit of elasticity of form will be found to have been still further extended. This process may be continued until we reach the limits of the *strength* of the material.

Hardness. It is obvious that the elastic limit of a ductile body—depending as it does not only on the intrinsic qualities of

the material, but also on *its previous elastic history*—is distinct from the *solidity* of a plastic solid which, as we have seen, is unaltered by any straining process.

We shall therefore distinguish the resistance to flow of ductile solids by the usual term *Hardness.* The characteristic property of such solids is then that their hardness is increased by every process which produces a permanent set:—their volume and density when free from stress, as well as their elastic moduli, remaining unchanged.

The remark already made with reference to *solidity* may here be repeated with regard to *hardness* and *strength.* All these terms denote *limiting stresses,* and have no connection whatever with *rigidity* and *compressibility,* which are *modular* quantities —the first a stress, and the second the reciprocal of a stress.

The minimum value of the hardness of a given piece of ductile metal is that which it possesses when delivered by the manufacturer; the maximum value is equal to the *ultimate strength* (§ 222) of the material under shearing stress. When it has been hardened to this point, the material has lost all *malleability* and has become perfectly *brittle,* since its elastic limit now coincides with the point of rupture. It has, in fact, acquired the highest possible degree of *temper* (§ 15), solely through excessive straining.

(*iii.*) Every form of set affects the elastic symmetry of a ductile (though not of a plastic) material. For instance, as has already been remarked in § 207, an isotropic body drawn out or caused to flow by longitudinal stress in one given direction, with or without lateral stress symmetrical as to that direction, assumes to a greater or less extent the æolotropic condition of § 205 (*ii.*).

Metallic bodies also fall short of perfect ductility in the respect that set produces a condition of *imperfect elasticity* or constraint, which is apparently due to a residual interaction between the parts of the body, caused by the permanent deformation of the mean molecular configuration (§ 8). It is manifested within the *new* elastic limits in two ways—

First, by the incomplete recovery of the strain when the stress is removed:

Secondly, by the inexact reversal of the strain when the stress is reversed.

We shall see, however, when we come to consider the behaviour of ductile metals under tension that this *state of constraint* is only temporary, and is easily removed by a few successive reversals of the stress.

Viscosity. This property is displayed in very different degrees by ductile metals. In most of them it has very little effect (for

reasons already explained *) on small vibrations within the elastic limit, but if a bar of steel or iron be hammered, rolled or drawn out so as to flow visibly its temperature rises very rapidly.

Zinc and one or two other metals are however exceptional in the amount of their viscosity, which produces marked results even within the elastic limits.

In the curious state known as *elastic fatigue* (§ 16), the viscosity of metals is increased while their elastic properties are enfeebled.

We now proceed to exemplify the properties of ductile metals by a brief account of their behaviour, up to the point of crushing or rupture, under

(1) Cubical Compression.
(2) Longitudinal Extension.
(3) Longitudinal Compression.

(1.)—*Cubical Compression.*

All ductile metals possess perfect elasticity of bulk up to a high degree of pressure. The range of this elasticity seems to be associated with hardness, though there is no necessary reason why it should be so.

Thus it is improbable that any practicable amount of pressure would produce a permanent increase of density in iron or steel.

On the other hand the more malleable metals such as gold, silver and copper are known to undergo such permanent alteration under very considerable but still easily attainable pressures (see § 13).

Accurate experimental determinations are still wanting of the limits of elasticity in these cases, and of the point to which condensation may be pushed by the means at our disposal.

The experimental treatment of simple *dilatation* is encumbered by such great practical difficulties that our direct knowledge of this subject is almost *nil*.

In practice, it is found to be easiest to determine Young's modulus by longitudinal extension, and the rigidity by twisting† a bar, and then to deduce the modulus of compression by formula (38) of § 213

$$k = nq/(9n - 3q).$$

Since however (as we shall see in Chapter VII.) the bending

* See page 180.

† The twisting and bending of metal bars beyond the limits of their elasticity will be considered in Appendix V., to follow Chapter VII.

as well as the stretching of beams depends upon Young's modulus, the researches of experimental engineers have been chiefly directed to an accurate determination of q. Experiments on rigidity have been comparatively neglected, and consequently n and k are only known for a few materials.

It may however be taken for granted that k is sufficiently great in all solids to ensure that Hooke's law holds for compressions and dilatations within the elastic limit.

Experimental results show that when a ductile body is strained to any extent, in any manner which freely admits of change of form—as for instance in the cases of longitudinal elongation and compression to be presently considered—the cubical dilatation or compression of every portion may be assumed to be very small, and also to be almost entirely elastic or recoverable. Thus, in an experiment of Sir W. Thomson's, a permanent elongation of extreme amount ·1067, produced in a copper wire by gradual increase of tension, was accompanied by a permanent cubical dilatation of amount ·0085, or less than 8 per cent.

(2.)—*Longitudinal Extension.*

We now proceed to describe the phænomena exhibited by a bar of ductile metal, when very cautiously drawn out to the point of rupture: taking as examples the latest published experiments of Prof. A. B. W. Kennedy on bars of wrought iron and steel.

To begin with, the bar as obtained from the manufacturer has acquired considerable permanent *set* (§ 207) in the course of the different processes of rolling, hammering, drawing, etc., to which it has been submitted. It is in fact in the **state of constraint** described above under head (*iii.*), and consequently does not behave at first like a perfectly elastic body.

On the first application of any load **W** within the elastic limit (to be defined presently) a **total elongation** is produced which is indeed proportional to **W***; but on the removal of the load the bar does not return to its original length, but retains in the form of **set** a portion of the total elongation also proportional to **W**. Thus the **elastic elongation**, or that portion of the whole which is immediately recoverable, is likewise proportional to the load*.

If however the *same load* **W** be applied and removed several times in succession, it is found that the small residual set

* See note at foot of page 163.

gradually disappears, and ultimately the bar arrives at a condition which has been well termed by Prof. K. Pearson its **state of ease** for this particular load. In this state the bar behaves as a perfectly elastic solid *under all loads which do not exceed* W; the elongation being precisely proportional to the load, and the bar always returning to precisely the same length when released.

By repeating this process with gradually increasing loads, the state of ease may be extended up to a certain point, beyond which it cannot be produced. In this **ultimate** or **limiting state of ease** the bar really satisfies the definition of a perfectly elastic solid, so far at least as strains of this type are concerned. The whole of the initial constraint due to the processes of manufacture may be considered to have been removed, and the state to which the bar invariably returns on removal of the load may be regarded as its true **natural state** (§ 5).

The longitudinal stress produced by the maximum load consistent with this state of ease is the **mathematical limit of perfect elasticity**, or what we have called in § 222 the **elastic strength** of the material for longitudinal extension.

The maximum shearing stress—which is half the above—is what we have called the **natural hardness** of the material.

Figure 24 is a slightly diagrammatic representation of the straining of a bar of annealed basic steel of the softest quality, the natural dimensions of which were

Length	=	10 inches.
Breadth	=	1·508 „
Thickness	=	0·376 „
Sectional area	=	0·567 of a square inch.

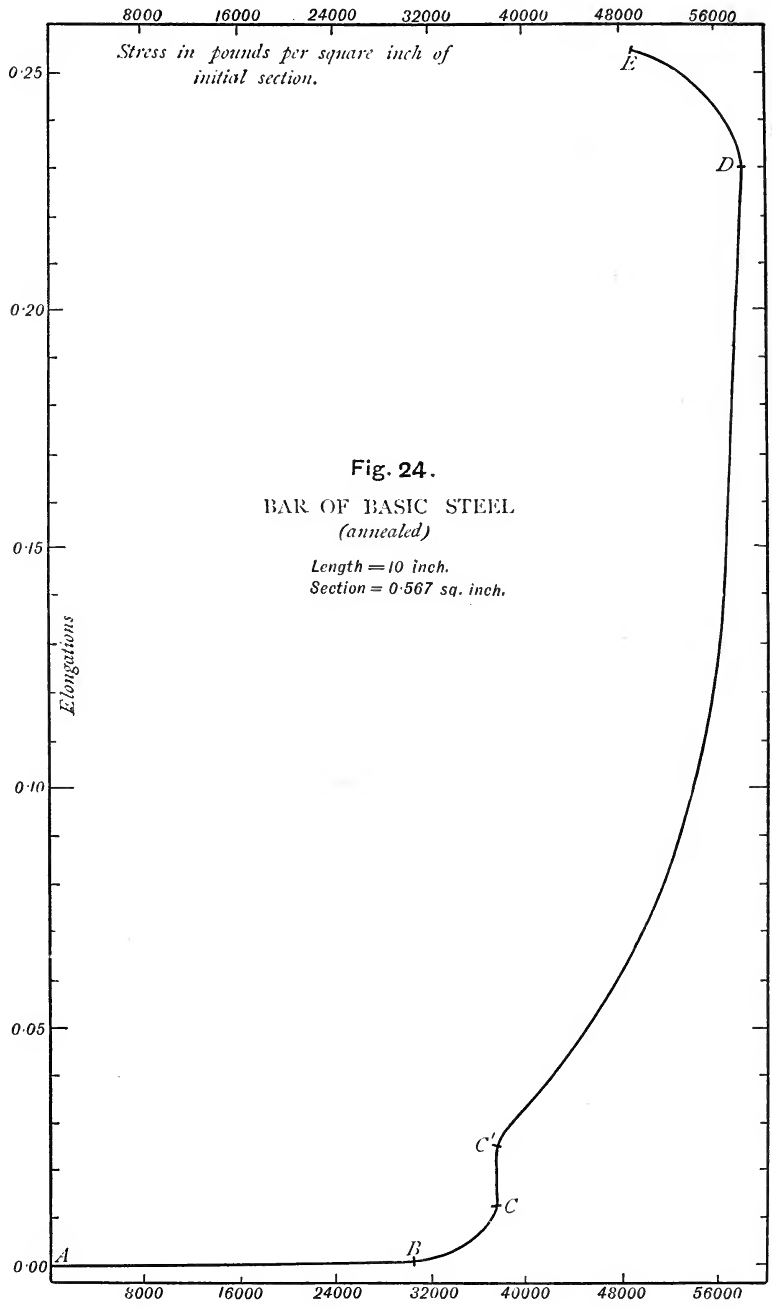
The vertical scale represents elongations, and the horizontal scale the corresponding *loads per unit of initial section* (pounds per square inch) which, up to this point at least, represent very approximately the actual longitudinal stresses.

The ultimate state of ease is represented by the portion *a* of this curve, the point *A* representing the natural state of the bar, and the point *B* its condition when strained to the limit of its elastic strength.

Hooke's law is found to hold throughout the whole of this stage, so that *AB* is a straight line.

The elongation at *B* is ·001, and the stress is 30,500 pounds, or 13·616 tons per square inch. Hence we deduce the following data:—

Elastic strength for longitudinal extension	=	13·616 tons per sq. inch.
Natural hardness . .	=	6·808 „ „
Young's modulus . .	=	13,616 „ „


8000
16000
24000
32000
40000
48000
56000
Stress in pounds per square inch of initial section.
0·25
0·20
0·15
0·10
0·05
0·00
Elongations
A
B
C
C′
D
E
Fig. 24.
BAR OF BASIC STEEL
(annealed)
Length = 10 inch.
Section = 0·567 sq. inch.

If the stress be increased beyond the limit *B*, more or less flow is produced, and a state of ease is no longer attainable. In this second stage (marked *b* in Figure 24) the major portion of the strain is still an *elastic elongation* proportional to the stress, and following the same modulus as before; but the *set*, being due to flow, no longer shows any tendency to disappear. It is still small, but its rate of increase (and therefore also that of the total elongation) increases with the stress applied. The stress-strain curve is therefore no longer a straight line.

Prof. Kennedy observes that "occasionally this stage does not occur at all, and both its higher and lower limits seem—more than any other points in the *life* of the material—to be susceptible of change depending on manipulation. Accidental shock will shorten the stage considerably, very gradual loading extends it somewhat." Prof. Kennedy therefore proposes that this should be called the **stage of unstable elastic equilibrium.**

We are now in fact approaching a very critical point. If the load be increased with extreme caution until a certain limit (represented by *C* in Figure 24) is reached, the resistance of the bar appears all at once to break down, and the elongation may be suddenly increased by many times its amount without any corresponding increase of load. Indeed, when once the "breaking down point" *C* has been passed, the bar may be held in equilibrium under considerably greater elongations by loads less than that required to bring it to the critical point. This fact is indicated by the slight backward curvature of the portion CC_1 of the curve in Figure 24, which however is much more marked in Figure 24A, reproduced with no alteration but that of scale from a curve traced automatically during an actual experiment. This point is now being submitted to further investigation by Prof. Kennedy.

"*C* is the point called *by engineers* the limit of elasticity, because it is the only one markedly visible without special apparatus."

In the case represented in Figure 24 the elongation at *C* is ·0[illegible]8, and the stress 37,250 pounds, or 15·45 tons per square inch, which is therefore the engineer's or **practical limit** of elastic strength.

At C_1 the elongation is increased to ·025 without any corresponding increase in the load, or (for practical purposes) in the stress. This is however not a very marked case. Prof. Kennedy mentions examples in which the elongation suddenly increases from ·003 at *C* to ·040 at C_1.

It is obvious that, in actual metals, the hardness is not quite such a definite quantity as we consider it to be theoretically. During the unstable stage *b* the distorting stress and the hardness of the body are struggling together, and the small amount

of flow which takes place is, as it were, tentative. At C the stress gains a conclusive advantage, and a sudden and rapid flow takes place, the body yielding during this short stage CC_1 as if it were plastic. I have therefore proposed (*Nature*, vol. xxxii., p. 76) to call the point C the **elastic crisis**. It is remarkable that during this stage the extension of the bar appears to be occurring at different parts of its length *successively*, and not simultaneously, as during the stages a and c.

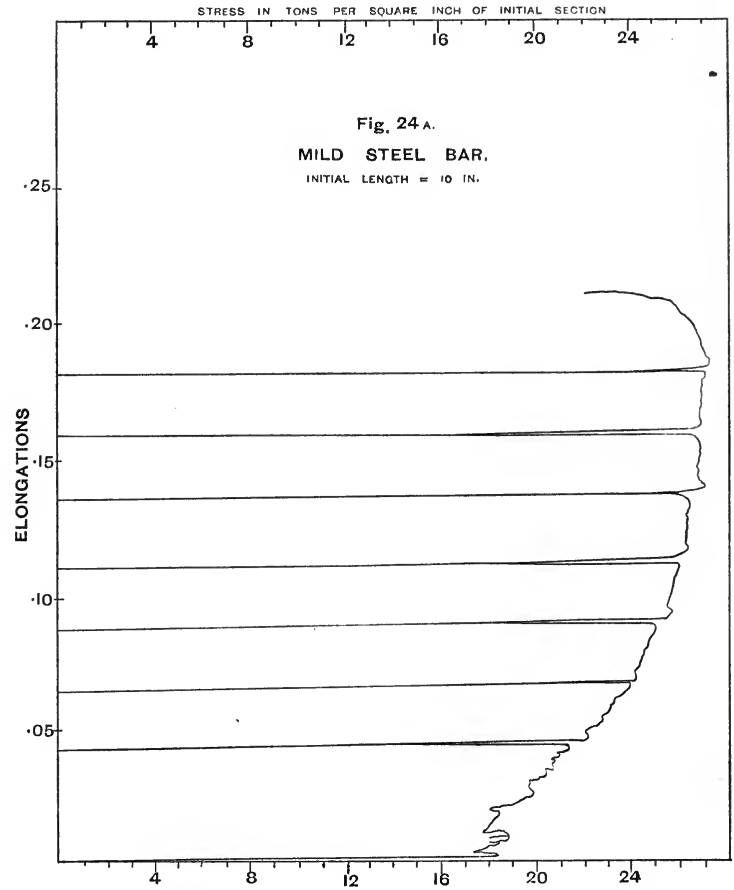


Fig. 24 A.
MILD STEEL BAR.
INITIAL LENGTH = 10 IN.

After once more attaining stable equilibrium at C_1, the bar passes into the stage c of regular ductile flow. Further increase of the stress produces increased elongation, a small portion of which is elastic, or recoverable on release: this elastic portion apparently still follows very closely the law of Young's modulus.

The strain however now consists almost entirely of a large and continually increasing flow, which remains as permanent set. In fact the bar now lengthens visibly and "thins down" uniformly throughout its length. This condition of the bar, from C_1 to D, is called the **stage of uniform flow**. The regular increase of hardness is shown by the continuous curvature of the line C_1D, which for a perfectly plastic solid would be straight.

Figure 24A exhibits very clearly the practical constancy of Young's modulus. In this case the load was gradually removed at several stages of the uniform flow, and then gradually reapplied. It is evident that the curves described by the bar in recovering the small elastic portion of the strain are practically straight lines parallel with that originally described in the stage a of perfect elasticity. These lines continually increase in length, the abscissæ of their extremities denoting at each stage twice the continuously increasing hardness of the bar, while the ordinates of the points where they meet the axis of elongation (or of zero stress) denote the increasing permanent elongations, due to flow, at the several stages. The slight protuberances of the curve at the extremities of the lines which represent the re-loading of the bar are a very interesting example of elastic fatigue (§ 16). After having been *for a few seconds* free from stress the elastic properties of the bar are slightly increased, and for a time it is able to bear a greater load without increase of elongation.

At D we reach the **point of maximum load**, which is also the **limit of uniform flow**. If this load be continued, the bar rapidly thins down *locally* (Figure 25) at some point of its length, until the cross section is so reduced that the stress across it reaches the **limit of tenacity**, and the bar breaks at that point.

In the case represented in Figure 24, the elongation at D was ·230, and the stress 57,600 pounds per square inch. Thus the elastic portion of the elongation was about

$$\frac{57,600}{q} = \frac{57,600}{30,500,000} = \cdot 00188,$$

and the remainder, or more than ·228, was permanent set. These figures are not quite accurate, because the area has now been sensibly reduced by flow, and the load per unit of initial area no longer represents the actual stress. We shall return to this point presently.

It is found possible, under favourable circumstances, to prolong the stage of local flow somewhat, by beginning to diminish the load immediately the first signs of its approach are observed. The elongation then continues to increase—almost entirely by local thinning—under diminishing *loads*, until at last the bar gives way, in its state of **maximum extension** (E in Figure 24), under a **terminal load** considerably less than the maximum.

Owing however to the rapid reduction of area at the weakest point of the bar, the *actual stress* experienced by the *constricted* portion increases more rapidly during the stage d of local flow than during any other, and the point E of terminal load is also the **point of maximum strength.**

In the case of Figure 24 the *ultimate extension* at E was ·255, and the terminal load per square inch of initial area 49,000 pounds. Since, however, the reduction of area, the constricted portion was ·548, this gives an actual terminal stress of

$$49{,}000 \times \frac{1}{1 - {\cdot}548} = \left\{\begin{array}{l} 108{,}290 \text{ pounds} \\ 48{\cdot}34 \text{ tons} \end{array}\right\} \text{per square inch.}$$

By gradual removal and re-application of the load, as in Figure 24A, Professor Kennedy has shown that even in this final stage the elastic part of the elongation follows the original value of Young's modulus (*Nature*, vol. xxxii., p. 270, Fig. 3).

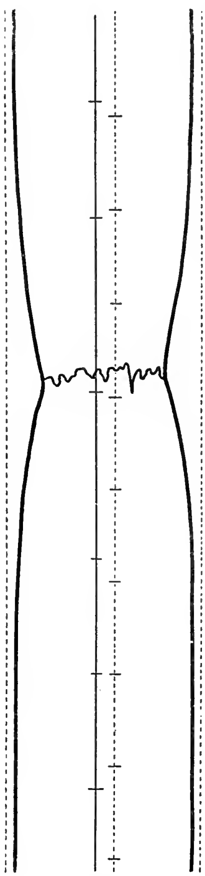


Fig. 25

Figure 25 represents the final stage of an actual experiment by Mr. Kirkcaldy on a bar of iron 1 inch in diameter. The ultimate elongation was ·300, and the ultimate constriction of sectional area was ·610. The terminal *load* was about 45,553 pounds, amounting to only 58,000 pounds per square inch of original section, but to as much as 146,000 pounds per square inch of the reduced section.

The dotted lines in the Figure represent the initial state of the bar, and the student will observe

(1.) The general reduction of diameter, due to uniform flow.

(2.) The excessive constriction of a limited portion, due to local flow.

(3.) The varying elongation, as shown by marks on the bar, originally at uniform distances apart, corresponding to this varying reduction of transverse dimensions.

Figure 25A consists of three curves obtained by consecutive measurements (not automatically), and exhibiting

I. The load per unit of original sectional area.

II. The load per unit of area of the non-constricted portion of the bar.

III. The load per unit of area of the section where the constriction is a maximum, and where fracture ultimately occurs.

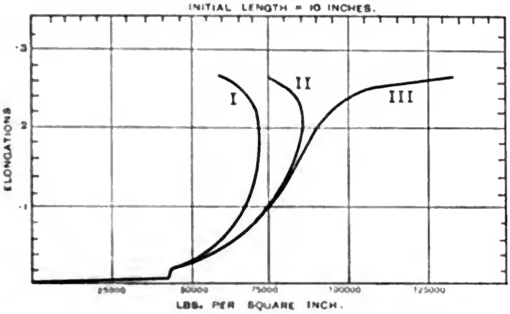
All three curves coincide till the stage of uniform flow is entered, when I. (which is the curve represented in Figures 24 and 24A) separates gradually from the others. When the point of maximum load is passed, and local flow begins, curve III. turns off more abruptly.

It is evident that while the termination of III. gives us the nearest approach to the ultimate strength or tenacity of the material, we cannot accept it as giving us any reliable information as to the relations between the strain and either the load or the stress, within this limit.

For all practical purposes, I. may be taken as the load-strain curve, and II. as the stress-strain curve.

It is a curious fact that, for bars of the same material and the same section, the form and dimensions of the conical constriction are almost invariable at the point of rupture. Consequently the apparent ultimate elongation, obtained by comparing the *whole*

Fig. 25A
LANDORE STEEL BAR.
INITIAL LENGTH = 10 INCHES.



length of the stretched bar (including the constricted portion) with its original length, depends very much on the latter, and is a very deceptive test of the quality of the bar. This is shown very clearly in Figure 26, in which the ordinates represent the "apparent" ultimate elongations of bars of three different materials, the lengths of which are given by the horizontal scale. The curve AA is for common wrought iron plate (tenacity 1[illegible] to 21 tons per square inch); BB for superior wrought iron (tenacity 21 to 25 tons); CC for very soft basic steel, the history of one sample of which is given in Figure 24 (tenacity between 2[illegible] and 27 tons).

The influence of the length of the bar on the apparent ultimate elongation in the last case is quite startling, as it diminishes from ·47 on a 2-inch to ·25 on a 10-inch bar. Two

methods have been suggested for obtaining a uniform experimental standard of ultimate extension: first, that all materials should be tested by means of bars of standard dimensions: secondly, that the length of the constricted portion should be subtracted from the total length in estimating the ultimate extension.

The stage from *D* to *E*, like that from *B* to *C*, requires special apparatus and excessive delicacy of manipulation to render its properties accurately measurable, and in consequence practical

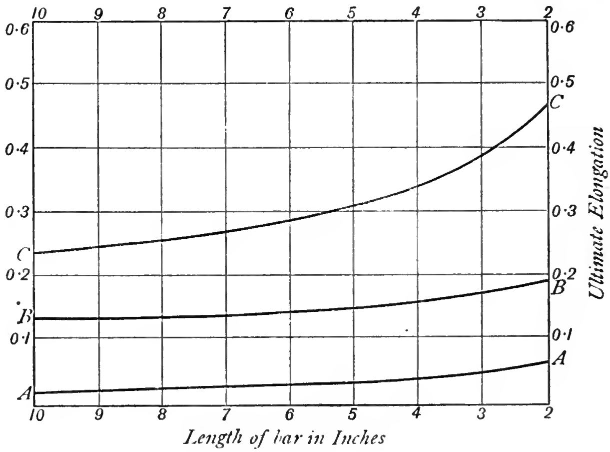


Fig. 26

men generally accept the elongation at *E* as the ultimate elongation, and the load at *D* (or maximum load) as the terminal or breaking load. There is no practical danger in this, so long as the error is avoided of taking *the load at D divided by the constricted area at E as the breaking stress.* This gives an entirely fallacious result, as will be seen by referring to the case of Figure 24 in which it would give

$$57{,}600 \times \frac{1}{1 - \cdot 548}$$

$$\left.\begin{array}{l} = 127{,}296 \text{ pounds} \\ = 56{\cdot}82 \text{ tons} \end{array}\right\} \text{per square inch.}$$

or 1·18 of its true value.

Taking into account that, as determined under any but the most favourable conditions, the limit of elasticity may be assigned to any point between *B* and *C*, or even C_1, and the maximum and breaking loads and stresses to any point between *D* and *E*: and

also the enormous differences in the quality of various specimens of the same metal (strictly speaking, totally different materials), due to the presence of impurities and the different processes of manufacture: the student will not be surprised to learn that the determinations of elastic constants published by different experimenters exhibit the most glaring discrepancies. The values given in the tables below can therefore only be considered as approximate averages.

On the other hand, the values of the moduli, at any rate when the material can be definitely specified, are probably very accurate.

(3.)—*Longitudinal Compression.*

It is impossible to perform experiments on the compression of a bar under longitudinal thrust with the same minute accuracy as those on its elongation under tension, as the following considerations sufficiently prove.

In theory, of course, we suppose that the load is always distributed uniformly over the end of the bar, whether as pressure or as traction, and that at the same time the ends are as free to contract or expand in area as any other portion of the bar. The longitudinal and lateral strains are then homogeneous or uniform throughout the length, and the stress across every transverse section is uniformly distributed over it, and has the same value for each.

Now, in practice, the load must either be fastened to one or more points of the terminal section—in which case the latter is free to alter in area, but the stress is not uniformly distributed; or rigidly attached to the entire face (*e.g.*, by soldering)—when the stress will be uniform, but the area of the face will be prevented from free variation; or, lastly, attached by clamps to the end of the bar—rendering it impossible for either supposed condition to be fulfilled.

In experiments on *extension* we may employ bars of considerable length, and so bring the effect of these terminal irregularities on the behaviour of the bar as a whole within the limit of small observational errors. But the equilibrium of a bar under considerable longitudinal *pressure* is in the highest degree unstable, and if the length of the bar be even a few multiples of its diameter, the slightest accidental shock will cause it to bend laterally.

Either then the bar must be enclosed in a trough to prevent flexure—which renders minute accuracy of observation impossible: or the experiments must be performed on very short blocks of material. In the latter case there is no escape from the difficulty

of attaching the load. If it be applied to portions only of the ends, the stress has (as it were) no room to equalise itself approximately over the intermediate sections; while if the block be placed on a rigid surface, and a rigid weight applied to its upper face, friction prevents its ends from expanding freely, and in consequence it bulges out considerably in the middle (Figure 27).

We must therefore look for considerable discrepancies in the results of experiments on longitudinal compression, even when made by the same observer on different blocks of the same material, and none of them can be accepted as more than approximate.

So far as we can judge, the value of Young's modulus and the limit of perfect elasticity seem to be about the same—in ductile materials—for compression as for extension. This is obviously what our theory would lead us to expect.

We may therefore suppose the straight line BA in Figure 24 produced for an equal distance AB' to represent the state of perfect elasticity under longitudinal compression (ultimate state of ease being, of course, presupposed).

The critical stage corresponding to b has not been observed, but at a point indistinguishable from the elastic limit ductile flow begins, with increasing hardness. From this point outwards, marked permanent set is visible in the form of longitudinal comparison and lateral bulging: fracture ultimately taking place by means of longitudinal cracks, due obviously to *lateral extension*.

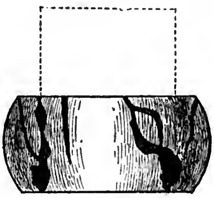


Fig.27

Figure 27 represents the mode of fracture of a short block of steel, and the amount of its ultimate compression. The dotted lines show its initial dimensions.

The following table gives the results of an experiment by Sir W. Fairbairn on a block of soft Bessemer steel, length ·997 of an inch, diameter ·720 of an inch.

Load in tons.	Height of block.	Approximate hardness in tons per square inch.
0·0	·997	11·0
* 8·3	·995	11·0*
16·7	·920	18·9
20·1	·865	21·5
23·3	·797	23·0
26·3	·731	23·7
29·5	·672	24·5
32·6	·613	24·7
35·8	·574	25·3
39·3	·535	26·0
41·0	·505	25·4

Young's modulus = 13,527 tons per sq. in.
Limit of elasticity (long) = 22 " "
Limit of tenacity = 88·5 " "
Ultimate contraction = ·59.

The first column gives the actual load applied, in tons: the second column the height of the block under these loads, in inches: the third column an approximation to the hardness or resistance to flow, obtained as follows.

The product of the initial section and height divided by the actual height at any moment gives the *mean section*. Dividing the load by this mean section we get approximately the mean value of the longitudinal stress throughout the block, and half of this gives the shearing stress.

The load marked with an asterisk corresponds to the limit of elasticity, and marks the point from which flow begins and the hardness increases. It is noticeable that after the load has reached a value of about 35 tons the hardness is practically stationary, and from that point to the moment of rupture the material behaves nearly as if it were plastic.

In the case of wrought iron set begins at about 10 tons per square inch, and rupture at about 20 tons per square inch.

C.—*Brittle Solids.*

This class may be divided into two groups, with reference to the relative magnitude of the rigidity and the strength under shearing stress (*i.e.*, of the *modulus* and the *limit*).

Group I., having a rigidity which is *very large* in comparison with its strength, includes cast iron and the harder varieties of steel and glass (the other qualities of which are ductile), as well as natural crystals of all kinds.

Group II., having a rigidity which is *very small* in comparison with its strength, includes the homogeneous jellies and indiarubber, etc.

A "perfectly brittle" solid is defined as being perfectly elastic up to the full limits of its ultimate strength, and consequently incapable of acquiring a *set* of any kind. The "elastic strength" and "ultimate strength" of such a solid are therefore identical under strain of any type.

It is probable that this definition is realised in perfection by crystals and jellies, and very approximately by those metals (such as soft steel) which are originally most malleable, after being tempered to the utmcst degree of hardness by straining beyond their original elastic limits.

Cast iron and indiarubber are capable of a certain amount of set, which is however a small fraction of the total strain.

Under extension the behaviour of the two groups differs only in the amount of deformation which can be produced before the limit of tenacity is reached.

In Group I. this is very small, and Hooke's law applies for all practical purposes up to the point of rupture.

In Group II. however a very considerable amount of *elastic* strain may be produced without rupturing the material—in the case of indiarubber an enormous amount, which the roughest experiments will show to be prolonged far beyond the limits of Hooke's law.

Cast iron is a very variable and irregular material, the elasticity of which is never perfect. It is impossible to bring it to a *state of ease*, so that a trifling set (very likely due to internal *constraint*) is visible from the very beginning of the straining. From .this point the percentage of set in the total elongation increases up to the point of rupture, but the maximum total elongation is itself so small (about the same as the maximum *perfectly elastic* elongation of soft steel—at B in Figure 24) that the set is not perceptible unless a very long bar be tested with delicate apparatus.

The following table gives the results of an experiment of Hodgkinson's on the stretching of a cast iron bar, length 600 inches, diameter 1·159 inches, which broke under a stress of 16,000 pounds to the square inch:—

CAST IRON BAR UNDER TENSION.

Stress in lbs. per square inch.	Total elongation.	Permanent set.	Percentage of set.
531	·0000400	perceptible.	......
1,062	·0000825	·0000025	3·0
1,592	·0001225	·0000033	2·7
2,123	·0001638	·0000075	4·6
3,185	·0002475	·0000175	7·1
4,246	·0003333	·0000258	7·7
5,308	·0004250	·0000367	8·6
6,370	·0005217	·0000467	9·0
7,431	·0006233	·0000617	9·9
8,493	·0007250	·0000767	10·6
9,554	·0008400	·0000933	11·1
10,616	·0009567	·0001117	11·7
11,678	·0010800	·0001325	12·3
12,739	·0012167	·0001583	13·0
13,801	·0013600	·0001858	13·7
14,863	·0015200	·0002200	14·5
15,924	·0016667	......	......
16,000	rupture.	......	......

Under longitudinal compression the two groups behave in very different ways.

Materials included in Group II., having little rigidity, expand freely in a lateral direction under moderate pressures, and are ultimately ruptured, like ductile metals under the same circumstances, by the lateral stress exceeding the limit of tenacity (Figure 27).

The materials of Group I., on the other hand, are too rigid to expand much laterally, so that the limit of tenacity is never approached; but since their hardness prevents them from flowing, they cannot relieve themselves from shearing stress, and they

are ultimately ruptured by tangential fracture or *cleavage* in one of the planes in which the shearing stress is a maximum: that is (*see* Example 4, page 117) in some plane inclined at an angle of 45°, or thereabouts, to the direction of pressure. This method of fracture is well shown in Figure 28, which represents the crushing of a cast iron bar.

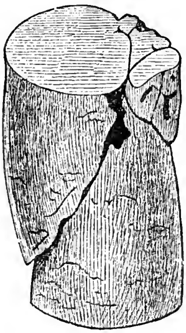


Fig. 28

The strength of these rigid materials under pressure therefore depends on their power of resisting shearing-cleavage, while their strength under tension depends, like that of all other materials, on their tenacity. These two strengths are thus quite independent, and it is characteristic of all this rigid group that the strength under compression is many times greater than that under tension:—in cast iron it is six times as great for ultimate strength, and three times for elastic strength. *See* Tables (C *bis*) and (D) *below.*

D.—*Timber.*

All kinds of wood are markedly heterogeneous and æolotropic in structure. But on the principle (§§ 1 and 43) of regarding only the relative magnitude of a body and its distinguishable components, we may look upon a long plank or bar, or a block of fair size, as being *as a whole* fairly homogeneous. We may also consider it to have three planes of æolotropic symmetry, depending upon the average direction of the "grain."

Many woods have very considerable tenacity in the direction corresponding to the length of the tree trunk—but most have very little indeed in the two perpendicular directions. Beams intended to resist compression, extension, and bending, or to display elasticity under such strains, are therefore always cut "with the grain," and the values of Young's modulus and the tenacities in the following tables must be taken to apply to that direction only.

NUMERICAL TABLES OF ELASTIC CONSTANTS, &c.

TABLE (A).

Reduction Factors.

pounds *to* grammes, - - - - - - -	453·593
grammes *to* pounds, - - - - - - -	0·002046
pounds *to* poundals, - - - - - - -	32·2
poundals *to* pounds, - - - - - - -	0·03106
grammes *to* dynes, - - - - - - -	981·4
dynes *to* grammes, - - - - - - -	0·001019
inches *to* centimetres, - - - - - -	2·54
centimetres *to* inches, - - - - - -	0·3937
square inches *to* square centimetres, - - -	6·4516
square centimetres *to* square inches, - - -	0·155
pounds per sq. in. *to* grammes per sq. cent., -	70·31
grammes per sq. cent. *to* pounds per sq. in., -	0·014223
tons per sq. in. *to* kilogrammes per sq. cent., -	157·494
kilogrammes per sq. cent. *to* tons per sq. in., -	0·00635
poundals per sq. in. *to* dynes per sq. cent., -	2143·21
dynes per square cent. *to* poundals per sq. in., -	0·0004667

TABLE (B).

Compressibility of Liquids.

Liquid.	Temp. Cent.	k in millions of dynes per square centimetre.	k in tons per square inch.
Ether, - - -	0°	9,300	60·177
,, - - -	14°	7,920	51·248
Alcohol, - - -	0°	12,100	78·295
,, - - -	15°	11,100	71·824
Carbon bisulphide, -	14°	16,000	103·531
Water, - - -	0°.0	20,200	130·707
,, - - -	1°.5	19,700	127·473
,, - - -	4°.1	20,300	......
,, - - -	10°.8	21,100	......
,, - - -	13°.4	21,300	137·826
,, - - -	18°.0	22,000	......
,, - - -	25°.0	22,200	......
,, - - -	34°.5	22,400	......
,, - - -	43°.0	22,900	......
,, - - -	53°.0	23,000	148·825
Mercury, - - -	15°.0	542,000	3507·092

Authority for water—Jamin, *Cours de Physique*, 2nd ed., t. i., pp. 168, 169: for the other liquids—Amaury and Descamps, *Comptes Rendus*, t. lxviii., p. 1564.

TABLE (C).

Elastic Constants of Solids.

Material.	ρ	$q\times10^{-6}$	$n\times10^{-6}$	$k\times10^{-6}$	E	$V\times10^{-4}$	T 10^{-4}
Flint glass, - -	2·942	615	243	437	......	......	......
,, ,, - -	2·935	585	240	347	......	......	......
Steel bars, - -	7·849	2120	834	1878	·00324	1310	809
Steel, cast, drawn,	7·717	1955	......	......	......	......	838
Steel wire, drawn, -	7·718	1881	......	......	·0050	22730	925
Steel piano wire, -	7·727	2049	......	......	·0115	135995	2362
Iron, cast, - -	7·235	1375	542	938	·00116	879	147
Iron, wrought, -	7·790	2040	784	1484	·00224	5120	457
Iron, wire, - -	7·553	1861	......	......	·0034	10952	638
Copper, cast, - -	......	......	......	......	......	......	134
Copper, drawn, -	8·893	1245	456	1717	·0033	6613	410
Copper, annealed, -	8·936	1052	......	......	·003	4745	316
Copper wire, - -	8·900	1185	445	1172	·0036	7480	422
Brass, cast, - -	......	645	......	......	·00198	1256	127
Brass, drawn, -	8·471	1096	373	1063	......	......	......
Brass wire, - -	......	1001	410	597	·00344	5905	343
Gun metal, - -	......	696	......	......	·00362	4562	252
Gold, drawn, - -	18·514	813	281	2538	·0034	4629	275
Silver, drawn, -	10·369	736	270	895	·0041	5962	296
Platinum, fine wire,	21·166	1593	622	1210	·0022	3852	350
Tin, cast, - -	7·400	417	......	......	·001	207	41
Zinc, drawn, - -	7·100	873	360	506	·0018	1448	158
Lead, - - -	11·215	177	......	......	·0012	135	22
Ash, - - - -	......	113	......	......	·0106	6370	120
Teak, - - -	......	169	......	......	·00621	3262	105
Oak, - - - -	0·750	103	......	......	·0102	5352	105
Red pine, - -	0·500	118	......	......	·00771	3510	91
Spruce, - - -	......	113	......	......	·0077	3347	87
Larch, - - -	......	79	......	......	·00861	2927	68

In the above table, ρ denotes the density in grammes per cubic centimetre; q, n, k the moduli in grammes weight per square centimetre; E the "practical" limit of elastic elongation (point C in Figure 24); V the resilience (§ 222) for longitudinal extension in gramme-centimetres per cubic centimetre; and T the tenacity in grammes per square centimetre.

This table is for the most part quoted from Sir William Thomson's article "Elasticity" in the *Encyclopædia Britannica*: the experimental authorities are Wertheim, Rankine, Everett, and Sir William Thomson.

Example.—For drawn copper:

Density - - = 8·893 grammes per cubic cent.
Young's modulus - = 1,245,000,000 gr. per sq. cent.
Rigidity - - = 456,000,000 ,,
Mod. of compression = 1,717,000,000 ,,
Elongation at "breaking-down point" . = ·0033
Resilience under tension = 66,130,000 gramme-centimetres per cubic cent.
Tenacity - = 4,100,000 grammes per square cent.

The *absolute* measures of the moduli, etc., can be deduced by reducing grammes to dynes, or multiplying the above values by 981·4.

The length-moduli and resilience in centimetres can also be deduced by dividing by 8·893, the density. (*See* §§ 221, 222.)

TABLE (C *bis*).

Practical Table in English Measure.

Material.	Elastic Strength.						Young's modulus and rigidity in tons per square inch.		Resilience under Tension.	
	Stress in tons per square inch.			Strain.					Foot-pounds per cubic foot.	Height per pound in feet.
	T.	C.	S.	T.	C.	S.	q	n		
Iron, cast, - -	3	9	...	·000375	·001125	...	8000	...	185	0·38
Iron, wrought, - -	9	9	7	·0007	·0007	·0014	13000	5000	1060	2·2
Steel, soft, - -	15	15	12	·0012	·0012	·0024	13000	5200	2900	6
Steel, hard, - -	25	25	20	·002	·002	·004	13000	5200	8000	16·5
Steel wire, strongest,	150	...	...	·0115	...	...	13000	...	276000	577
Fir, - - - - -	$1\frac{1}{2}$	...	...	·0021	...	...	700	35	2150	58
Oak, - - - -	2	...	...	·0028	...	...	700	35	4300	86

The above table is quoted from Prof. Cotterill's *Applied Mechanics.* The first six columns of figures give the "practical" elastic limits of stress and strain for tension (T.), compression (C.), and shear (S.).

TABLE (D).

Ultimate and Working Strength.

Material.	Ultimate Strength.				Working Strength.	
	Tons per square inch.			Ultimate Elongation.	Tons per square inch.	
	T.	C.	S.		T.	C.
Iron bars, - -	25	22	18	·20	4·5	4·5
Iron plates, - -	22	19	16	·10		
Steel, soft, - -	30	...	22½	·25	7	7
Steel, medium, -	35	...	27	·15	...	...
Steel, hard, - -	45	...	...	·80	...	...
Iron, cast, - -	7½	45	12	...	1·5	4·5
Lead, - - -	1½	...	...	...	...	...
Copper, sheet, -	13½	...	...	...	...	...
Copper, cast, - -	8½	...	...	...	...	...
Copper wire, -	...	...	...	...	4	...
Steel wire, common,	...	...	...	...	13	...
Oak, - - -	5½	...	1	...	0·75	0·45
Fir, - - -	5½	...	0 27	...	0·5	0·3

This table also is taken from Prof. Cotterill's *Applied Mechanics.* The "working strength" of a material is the maximum statical stress to which it is subjected in practice; the ratio which the full elastic strength bears to this constitutes the "factor of safety" always allowed to provide against unforeseen contingencies.

TABLE (E).

Effect on Young's modulus of change of temperature.

Material.	Density in grammes per cubic centim.	Young's modulus in millions of grammes per square centimetre, at		
		15°	100°	200°
Lead, - - -	11·232	173	163	...
Gold, - - - -	18·035	558	531	548
Silver, - - -	10·304	715	727	637
Copper, - - -	8·936	1052	938	786
Platinum, - - -	21·083	1552	1418	1296
Steel, drawn, English,	7·622	1728	2129	1928
Steel, cast, - - -	7·919	1956	1901	1792
Iron, Berry, - -	7·757	2079	2188	1770

Wertheim, *Annales de Chimie et de Physique*, tom. xii. (1844).

TABLE (F).

Effect on rigidity of change of temperature.

According to Kohlrausch

$$n = n_0(1 - \alpha t - \beta t^2)$$

where t is temperature Cent.

Material.	α	β
Iron, - - - - -	0·000447	0·00000052
Copper, - - -	0·000520	0·00000028
Brass, - - - -	0·000423	0·00000136

CHAPTER V.

CURVILINEAR COÖRDINATES.

230.] **Definitions and Notation.** Let any three orthogonal systems of surfaces in the body be defined by giving successive constant values to the parameters ξ, η, ζ in the equations

$$\left.\begin{array}{l} \chi_1(x, y, z) = \xi \\ \chi_2(x, y, z) = \eta \\ \chi_3(x, y, z) = \zeta \end{array}\right\} \begin{array}{r} \ldots\ldots(1) \\ \ldots\ldots(2) \\ \ldots\ldots(3) \end{array}$$

where χ_1, χ_2, χ_3 are continuous functions of the rectangular Cartesian coördinates x, y, z.

The position of any point P (x, y, z) in which these surfaces intersect will then be fully determined by the values of the parameters; and ξ, η, ζ may be called the **Curvilinear Coördinates** of P. We shall also speak of the three systems of surfaces defined by them as the corresponding **coördinate surfaces**.

Let $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, $(\lambda_3, \mu_3, \nu_3)$, be the direction-cosines, referred to Ox, Oy, Oz, of the normals to the three coördinate surfaces which meet in P—drawn in the directions in which the values of ξ, η, ζ *increase*. Then

$$\lambda_1 : \mu_1 : \nu_1 : 1 :: \frac{\partial\chi_1}{\partial x} : \frac{\partial\chi_1}{\partial y} : \frac{\partial\chi_1}{\partial z} : \sqrt{\left(\frac{\partial\chi_1}{\partial x}\right)^2 + \left(\frac{\partial\chi_1}{\partial y}\right)^2 + \left(\frac{\partial\chi_1}{\partial z}\right)^2},$$

where the proper value of ξ at P is to be substituted for χ_1 after differentiation.

We shall consequently always write these derivatives

$$\frac{\partial\xi}{\partial x}, \frac{\partial\xi}{\partial y}, \frac{\partial\xi}{\partial z},$$

and so for η and ζ; and we shall also assume that x, y and z have been eliminated from them by means of (1), (2) and (3), so that they are expressed as functions of ξ, η, ζ.

If now we write

$$\left.\begin{aligned} h_1^2 &= \left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2 + \left(\frac{\partial\xi}{\partial z}\right)^2 \\ h_2^2 &= \left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2 + \left(\frac{\partial\eta}{\partial z}\right)^2 \\ h_3^2 &= \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2 + \left(\frac{\partial\zeta}{\partial z}\right)^2 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(4)$$

we shall have

$$\left.\begin{aligned} \lambda_1 &= \frac{1}{h_1}\frac{\partial\xi}{\partial x}, \quad \mu_1 = \frac{1}{h_1}\frac{\partial\xi}{\partial y}, \quad \nu_1 = \frac{1}{h_1}\frac{\partial\xi}{\partial z} \\ \lambda_2 &= \frac{1}{h_2}\frac{\partial\eta}{\partial x}, \quad \mu_2 = \frac{1}{h_2}\frac{\partial\eta}{\partial y}, \quad \nu_2 = \frac{1}{h_2}\frac{\partial\eta}{\partial z} \\ \lambda_3 &= \frac{1}{h_3}\frac{\partial\zeta}{\partial x}, \quad \mu_3 = \frac{1}{h_3}\frac{\partial\zeta}{\partial y}, \quad \nu_3 = \frac{1}{h_3}\frac{\partial\zeta}{\partial z} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(5)$$

taking for h_1, h_2, h_3 the *positive* roots of (4). The conditions for orthogonalism are by (5)

$$\left.\begin{aligned} \frac{\partial\eta}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\eta}{\partial y}\frac{\partial\zeta}{\partial y} + \frac{\partial\eta}{\partial z}\frac{\partial\zeta}{\partial z} &= 0 \\ \frac{\partial\zeta}{\partial x}\frac{\partial\xi}{\partial x} + \frac{\partial\zeta}{\partial y}\frac{\partial\xi}{\partial y} + \frac{\partial\zeta}{\partial z}\frac{\partial\xi}{\partial z} &= 0 \\ \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial z}\frac{\partial\eta}{\partial z} &= 0 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(6)$$

If these conditions be satisfied (as we shall always suppose the case), equations (5) will also give the direction-cosines of the tangents at P to the three curves of intersection of the pairs of coördinate surfaces defined by η and ζ, ζ and ξ, ξ and η.

We know by Dupin's theorem (Frost's *Solid Geometry*, § 603) that the curve of intersection of any such pair of surfaces is a *Line of Curvature* on each.

Let ds_1, ds_2, ds_3 be the elements of these curves, measured from P in the directions of increase of ξ, η, ζ. Then in proceeding from P along s_1 we remain always on the same surface of system (2), and also on the same surface of system (3), describing a line of curvature on each; that is to say, ξ alone varies along s_1. Similarly η alone varies along s_2, and ζ alone varies along s_3.

The elementary (ultimately straight) lines ds_1, ds_2, ds_3 are in fact the three edges meeting in P of the element of volume (ultimately a rectangular parallelepiped) bounded by the surfaces whose parameters are

$$\xi,\ \eta,\ \zeta,\ \xi + d\xi,\ \eta + d\eta,\ \zeta + d\zeta.$$

The Cartesian coördinates of the further extremity of ds_1 are

$$x+\lambda_1 ds_1,\quad y+\mu_1 ds_1,\quad z+\nu_1 ds_1\,;$$

whence, by Taylor's Theorem,

$$d\xi = \lambda_1 ds_1 \frac{\partial \xi}{\partial x} + \mu_1 ds_1 \frac{\partial \xi}{\partial y} + \nu_1 ds_1 \frac{\partial \xi}{\partial z},$$

$$= h_1 ds_1 \text{ by (5) and (4).}$$

Thus we find

$$\left.\begin{aligned} ds_1 &= \frac{d\xi}{h_1} \\ ds_2 &= \frac{d\eta}{h_2} \\ ds_3 &= \frac{d\zeta}{h_3} \end{aligned}\right\} \quad \text{.........................} \quad (7)$$

for the lengths of those three edges of the element of volume which meet in $P(\xi, \eta, \zeta)$.

Ultimately, therefore, when this element approximates in form to a rectangular parallelepiped, its volume is

$$\frac{d\xi d\eta d\zeta}{h_1 h_2 h_3} \quad \text{.........................} \quad (8)$$

and the areas of the three faces which meet in P are

$$\frac{d\eta d\zeta}{h_2 h_3},\quad \frac{d\zeta d\xi}{h_3 h_1},\quad \frac{d\xi d\eta}{h_1 h_2} \quad \text{.........................} \quad (9)$$

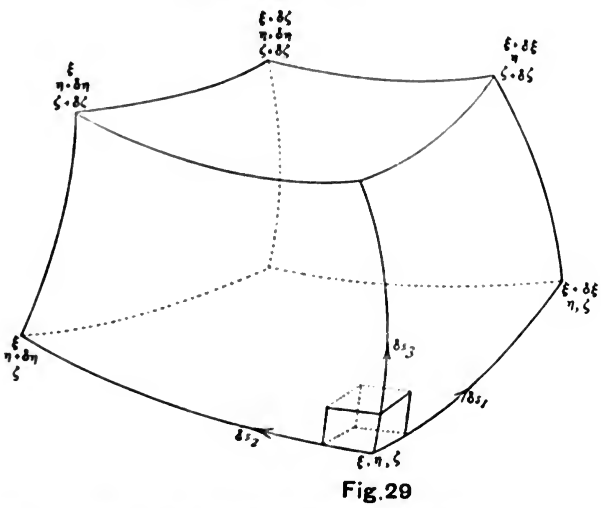


Fig. 29

n Figure 29 the six coördinate surfaces are of course really dra vn for finite differences of ξ, η, ζ, in order to exhibit the curva-

tures of the edges. The small figure at the corner (ξ, η, ζ) represents more approximately the rectangular parallelepiped into which it degenerates, when $d\xi$, $d\eta$, $d\zeta$ are truly elementary.

231.] **Formulæ of Differentiation.** Since ds_1 is an elementary line drawn in the direction $(\lambda_1, \mu_1, \nu_1)$, we have of course

$$\frac{\partial\Phi}{\partial s_1} = \lambda_1\frac{\partial\Phi}{\partial x} + \mu_1\frac{\partial\Phi}{\partial y} + \nu_1\frac{\partial\Phi}{\partial z}$$

where Φ is any function of x, y, z, and therefore also of ξ, η, ζ.

Similarly
$$\frac{\partial\Phi}{\partial x} = \lambda_1\frac{\partial\Phi}{\partial s_1} + \lambda_2\frac{\partial\Phi}{\partial s_2} + \lambda_3\frac{\partial\Phi}{\partial s_3}, \text{ etc.}$$

Now by (7)
$$\frac{\partial\Phi}{\partial s_1} = h_1\frac{\partial\Phi}{\partial \xi}, \text{ etc.}$$

Thus
$$\left.\begin{aligned}
h_1\frac{\partial\Phi}{\partial\xi} &= \lambda_1\frac{\partial\Phi}{\partial x} + \mu_1\frac{\partial\Phi}{\partial y} + \nu_1\frac{\partial\Phi}{\partial z}\\
h_2\frac{\partial\Phi}{\partial\eta} &= \lambda_2\frac{\partial\Phi}{\partial x} + \mu_2\frac{\partial\Phi}{\partial y} + \nu_2\frac{\partial\Phi}{\partial z}\\
h_3\frac{\partial\Phi}{\partial\zeta} &= \lambda_3\frac{\partial\Phi}{\partial x} + \mu_3\frac{\partial\Phi}{\partial y} + \nu_3\frac{\partial\Phi}{\partial z}\\
\frac{\partial\Phi}{\partial x} &= h_1\lambda_1\frac{\partial\Phi}{\partial\xi} + h_2\lambda_2\frac{\partial\Phi}{\partial\eta} + h_3\lambda_3\frac{\partial\Phi}{\partial\zeta}\\
\frac{\partial\Phi}{\partial y} &= h_1\mu_1\frac{\partial\Phi}{\partial\xi} + h_2\mu_2\frac{\partial\Phi}{\partial\eta} + h_3\mu_3\frac{\partial\Phi}{\partial\zeta}\\
\frac{\partial\Phi}{\partial z} &= h_1\nu_1\frac{\partial\Phi}{\partial\xi} + h_2\nu_2\frac{\partial\Phi}{\partial\eta} + h_3\nu_3\frac{\partial\Phi}{\partial\zeta}
\end{aligned}\right\} \text{.................. (10)}$$

Writing x, y, z successively for Φ in the first three of these equations we find

$$\left.\begin{aligned}
\lambda_1 &= h_1\frac{\partial x}{\partial\xi}, & \mu_1 &= h_1\frac{\partial y}{\partial\xi}, & \nu_1 &= h_1\frac{\partial z}{\partial\xi}\\
\lambda_2 &= h_2\frac{\partial x}{\partial\eta}, & \mu_2 &= h_2\frac{\partial y}{\partial\eta}, & \nu_2 &= h_2\frac{\partial z}{\partial\eta}\\
\lambda_3 &= h_3\frac{\partial x}{\partial\zeta}, & \mu_3 &= h_3\frac{\partial y}{\partial\zeta}, & \nu_3 &= h_3\frac{\partial z}{\partial\zeta}
\end{aligned}\right\} \text{..................... (11)}$$

Thus the conditions (6) for orthogonalism may also be written

$$\left.\begin{aligned}
\frac{\partial x}{\partial\eta}\frac{\partial x}{\partial\zeta} + \frac{\partial y}{\partial\eta}\frac{\partial y}{\partial\zeta} + \frac{\partial z}{\partial\eta}\frac{\partial z}{\partial\zeta} &= 0\\
\frac{\partial x}{\partial\zeta}\frac{\partial x}{\partial\xi} + \frac{\partial y}{\partial\zeta}\frac{\partial y}{\partial\xi} + \frac{\partial z}{\partial\zeta}\frac{\partial z}{\partial\xi} &= 0\\
\frac{\partial x}{\partial\xi}\frac{\partial x}{\partial\eta} + \frac{\partial y}{\partial\xi}\frac{\partial y}{\partial\eta} + \frac{\partial z}{\partial\xi}\frac{\partial z}{\partial\eta} &= 0
\end{aligned}\right\} \text{...................... (6a)}$$

We also deduce from (11)

$$\left.\begin{aligned}\frac{1}{h_1^2}&=\left(\frac{\partial x}{\partial\xi}\right)^2+\left(\frac{\partial y}{\partial\xi}\right)^2+\left(\frac{\partial z}{\partial\xi}\right)^2\\ \frac{1}{h_2^2}&=\left(\frac{\partial x}{\partial\eta}\right)^2+\left(\frac{\partial y}{\partial\eta}\right)^2+\left(\frac{\partial z}{\partial\eta}\right)^2\\ \frac{1}{h_3^2}&=\left(\frac{\partial x}{\partial\zeta}\right)^2+\left(\frac{\partial y}{\partial\zeta}\right)^2+\left(\frac{\partial z}{\partial\zeta}\right)^2\end{aligned}\right\}\qquad\qquad(12)$$

It frequently happens that, while equations (1), (2), (3) express ξ, η, ζ as more or less complex functions of x, y, z, they admit of very simple solutions for the latter coördinates as explicit functions of ξ, η, ζ. In such cases the formulæ (11), (6a), (12) may be used in preference to (5), (6), and (4). Equations (11) and (12) possess the further advantage that they admit of the elimination of x, y, z from the expressions for h_1, h_2, h_3 and the direction-cosines *before differentiation*.

Lastly from (5) and (11) we have

$$\left.\begin{aligned}h_1^2&=\frac{\partial\xi}{\partial x}\Big/\frac{\partial x}{\partial\xi}=\frac{\partial\xi}{\partial y}\Big/\frac{\partial y}{\partial\xi}=\frac{\partial\xi}{\partial z}\Big/\frac{\partial z}{\partial\xi}\\ h_2^2&=\frac{\partial\eta}{\partial x}\Big/\frac{\partial x}{\partial\eta}=\frac{\partial\eta}{\partial y}\Big/\frac{\partial y}{\partial\eta}=\frac{\partial\eta}{\partial z}\Big/\frac{\partial z}{\partial\eta}\\ h_3^2&=\frac{\partial\zeta}{\partial x}\Big/\frac{\partial x}{\partial\zeta}=\frac{\partial\zeta}{\partial y}\Big/\frac{\partial y}{\partial\zeta}=\frac{\partial\zeta}{\partial z}\Big/\frac{\partial z}{\partial\zeta}\end{aligned}\right\}\qquad\qquad(12a)$$

The transformation of $\nabla^2\Phi$, where Φ is any continuous function of position, from Cartesians to curvilinears, is most easily effected by an application of Green's theorem, by which we know that

$$\iiint\nabla^2\Phi\,.\,dV=\iint\frac{\partial\Phi}{\partial n}\,.\,dS,$$

where the triple integral is taken throughout any volume V, and the double integral over the whole of its surface S, dn being the element of *outward* normal.

Let us take for V the element of volume (8). The left-hand side of Green's equation then becomes simply

$$\nabla^2\Phi\,.\,\frac{d\xi d\eta d\zeta}{h_1h_2h_3}.$$

The right-hand side will be the sum of six terms, supplied by the six faces of the element. The three first terms, due to the faces which are elements of the surfaces ξ, η, ζ, are respectively

$$-\frac{\partial\Phi}{\partial\kappa_1}\cdot\frac{d\eta d\zeta}{h_2h_3};\quad-\frac{\partial\Phi}{\partial\kappa_2}\cdot\frac{d\zeta d\xi}{h_3h_1};\quad-\frac{\partial\Phi}{\partial\kappa_3}\cdot\frac{d\xi d\eta}{h_1h_2};$$

or, by (7)

$$\left.\begin{matrix} -\dfrac{h_1}{h_2h_3}\cdot\dfrac{\partial\Phi}{\partial\xi}\cdot d\eta d\zeta \\ -\dfrac{h_2}{h_3h_1}\cdot\dfrac{\partial\Phi}{\partial\eta}\cdot d\zeta d\xi \\ -\dfrac{h_3}{h_1h_2}\cdot\dfrac{\partial\Phi}{\partial\zeta}\cdot d\xi d\eta \end{matrix}\right\}.$$

Consequently the terms due to the opposite faces, which are elements of the surfaces $\xi+d\xi$, $\eta+d\eta$, $\zeta+d\zeta$, must be

$$\left.\begin{matrix} +\left[\dfrac{h_1}{h_2h_3}\cdot\dfrac{\partial\Phi}{\partial\xi}+d\xi\cdot\dfrac{\partial}{\partial\xi}\left(\dfrac{h_1}{h_2h_3}\cdot\dfrac{\partial\Phi}{\partial\xi}\right)\right]d\eta d\zeta, \\ +\left[\dfrac{h_2}{h_3h_1}\cdot\dfrac{\partial\Phi}{\partial\eta}+d\eta\cdot\dfrac{\partial}{\partial\eta}\left(\dfrac{h_2}{h_3h_1}\cdot\dfrac{\partial\Phi}{\partial\eta}\right)\right]d\zeta d\xi, \\ +\left[\dfrac{h_3}{h_1h_2}\cdot\dfrac{\partial\Phi}{\partial\zeta}+d\zeta\cdot\dfrac{\partial}{\partial\zeta}\left(\dfrac{h_3}{h_1h_2}\cdot\dfrac{\partial\Phi}{\partial\zeta}\right)\right]d\xi d\eta. \end{matrix}\right\}.$$

Thus the right-hand side of Green's equation becomes

$$\left\{\frac{\partial}{\partial\xi}\left(\frac{h_1}{h_2h_3}\frac{\partial\Phi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{h_2}{h_3h_1}\frac{\partial\Phi}{\partial\eta}\right)+\frac{\partial}{\partial\zeta}\left(\frac{h_3}{h_1h_2}\frac{\partial\Phi}{\partial\zeta}\right)\right\}d\xi d\eta d\zeta.$$

Equating the two expressions thus found, we have finally

$$\nabla^2\Phi=h_1h_2h_3\left\{\frac{\partial}{\partial\xi}\left(\frac{h_1}{h_2h_3}\frac{\partial\Phi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{h_2}{h_3h_1}\frac{\partial\Phi}{\partial\eta}\right)+\frac{\partial}{\partial\zeta}\left(\frac{h_3}{h_1h_2}\frac{\partial\Phi}{\partial\zeta}\right)\right\}\ldots(13)$$

Hence, in particular,

$$\left.\begin{aligned} \nabla^2\xi&=h_1h_2h_3\frac{\partial}{\partial\xi}\left(\frac{h_1}{h_2h_3}\right) \\ \nabla^2\eta&=h_1h_2h_3\frac{\partial}{\partial\eta}\left(\frac{h_2}{h_3h_1}\right) \\ \nabla^2\zeta&=h_1h_2h_3\frac{\partial}{\partial\zeta}\left(\frac{h_3}{h_1h_2}\right) \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

whence it follows that, if ξ, η, ζ are solutions of Laplace's equation

$$\nabla^2\Phi=0,$$

h_1/h_2h_3 must be independent of ξ, h_2/h_3h_1 independent of η, and h_3/h_1h_2 independent of ζ.

232.] **Principal Curvatures of the Coördinate Surfaces.** It has been remarked in § 230 that, by Dupin's theorem, each curve of the s_1 system is a line of curvature on each of the surfaces (belonging respectively to the η and ζ systems) which intersect along it; and so for the other systems.

The lines of curvature, at any point P, on the ξ surface pass-

ing through it are therefore the curves of the $\xi\eta$ and $\xi\zeta$ systems which intersect in P. We shall adopt the notation

$$_{\xi}\varpi_{\eta},\ _{\xi}\varpi_{\zeta}$$

for the curvatures of the normal sections of the ξ surface at P through the tangents to the $\xi\eta$ and $\xi\zeta$ curves, with a symmetrical notation for the other surfaces.

Thus

$$\left.\begin{array}{l} _{\xi}\varpi_{\eta},\ _{\xi}\varpi_{\zeta} \\ _{\eta}\varpi_{\zeta},\ _{\eta}\varpi_{\xi} \\ _{\zeta}\varpi_{\xi},\ _{\zeta}\varpi_{\eta} \end{array}\right\}$$

will denote the six principal curvatures of the coördinate surfaces at P.

By § 606 of Frost's *Solid Geometry* we have

$$\pm h_1 \cdot {}_{\xi}\varpi_{\eta} = \lambda_2^2 \frac{\partial^2\xi}{\partial x^2} + \mu_2^2 \frac{\partial^2\xi}{\partial y^2} + \nu_2^2 \frac{\partial^2\xi}{\partial z^2}$$

$$+ 2\mu_2\nu_2 \frac{\partial^2\xi}{\partial y \partial z} + 2\nu_2\lambda_2 \frac{\partial^2\xi}{\partial z \partial x} + 2\lambda_2\mu_2 \frac{\partial^2\xi}{\partial x \partial y} \qquad (15)$$

Now take the last of equations (6), and differentiate partially as to x: thus

$$\frac{\partial\eta}{\partial x}\frac{\partial^2\xi}{\partial x^2} + \frac{\partial\eta}{\partial y}\frac{\partial^2\xi}{\partial x \partial y} + \frac{\partial\eta}{\partial z}\frac{\partial^2\xi}{\partial z \partial x} = -\left[\frac{\partial\xi}{\partial x}\frac{\partial^2\eta}{\partial x^2} + \frac{\partial\xi}{\partial y}\frac{\partial^2\eta}{\partial x \partial y} + \frac{\partial\xi}{\partial z}\frac{\partial^2\eta}{\partial z \partial x}\right].$$

Next differentiate the same equation as to y: thus

$$\frac{\partial\eta}{\partial x}\frac{\partial^2\xi}{\partial x \partial y} + \frac{\partial\eta}{\partial y}\frac{\partial^2\xi}{\partial y^2} + \frac{\partial\eta}{\partial z}\frac{\partial^2\xi}{\partial y \partial z} = -\left[\frac{\partial\xi}{\partial x}\frac{\partial^2\eta}{\partial x \partial y} + \frac{\partial\xi}{\partial y}\frac{\partial^2\eta}{\partial y^2} + \frac{\partial\xi}{\partial z}\frac{\partial^2\eta}{\partial y \partial z}\right].$$

Finally, differentiating as to z,

$$\frac{\partial\eta}{\partial x}\frac{\partial^2\xi}{\partial z \partial x} + \frac{\partial\eta}{\partial y}\frac{\partial^2\xi}{\partial y \partial z} + \frac{\partial\eta}{\partial z}\frac{\partial^2\xi}{\partial z^2} = -\left[\frac{\partial\xi}{\partial x}\frac{\partial^2\eta}{\partial z \partial x} + \frac{\partial\xi}{\partial y}\frac{\partial^2\eta}{\partial y \partial z} + \frac{\partial\xi}{\partial z}\frac{\partial^2\eta}{\partial z^2}\right].$$

Multiply the first of these results by $\partial\eta/\partial x$, the second by $\partial\eta/\partial y$, and the third by $\partial\eta/\partial z$, and add.

Thus finally

$$\left(\frac{\partial\eta}{\partial x}\right)^2\frac{\partial^2\xi}{\partial x^2} + \left(\frac{\partial\eta}{\partial y}\right)^2\frac{\partial^2\xi}{\partial y^2} + \left(\frac{\partial\eta}{\partial z}\right)^2\frac{\partial^2\xi}{\partial z^2}$$

$$+ 2\frac{\partial\eta}{\partial y}\frac{\partial\eta}{\partial z}\frac{\partial^2\xi}{\partial y \partial z} + 2\frac{\partial\eta}{\partial z}\frac{\partial\eta}{\partial x}\frac{\partial^2\xi}{\partial z \partial x} + 2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}\frac{\partial^2\xi}{\partial x \partial y}$$

$$= -\frac{\partial\xi}{\partial x}\left[\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2} + \frac{\partial\eta}{\partial y}\frac{\partial^2\eta}{\partial x \partial y} + \frac{\partial\eta}{\partial z}\frac{\partial^2\eta}{\partial z \partial x}\right]$$

$$-\frac{\partial\xi}{\partial y}\left[\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x \partial y} + \frac{\partial\eta}{\partial y}\frac{\partial^2\eta}{\partial y^2} + \frac{\partial\eta}{\partial z}\frac{\partial^2\eta}{\partial y \partial z}\right]$$

$$-\frac{\partial\xi}{\partial z}\left[\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial z \partial x} + \frac{\partial\eta}{\partial y}\frac{\partial^2\eta}{\partial y \partial z} + \frac{\partial\eta}{\partial z}\frac{\partial^2\eta}{\partial z^2}\right].$$

By (5) this may be written

$$-h_2^2\left\{\lambda_2^2\frac{\partial^2\xi}{\partial x^2}+\mu_2^2\frac{\partial^2\xi}{\partial y^2}+\nu_2^2\frac{\partial^2\xi}{\partial z^2}+2\mu_2\nu_2\frac{\partial^2\xi}{\partial y\partial z}+2\nu_2\lambda_2\frac{\partial^2\xi}{\partial z\partial x}+2\lambda_2\mu_2\frac{\partial^2\xi}{\partial x\partial y}\right\}$$

$$=\frac{1}{2}\left(\frac{\partial\xi}{\partial x}\frac{\partial}{\partial x}+\frac{\partial\xi}{\partial y}\frac{\partial}{\partial y}+\frac{\partial\xi}{\partial z}\frac{\partial}{\partial z}\right)\left\{\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2+\left(\frac{\partial\eta}{\partial z}\right)^2\right\}$$

$$=\frac{1}{2}h_1\left(\lambda_1\frac{\partial}{\partial x}+\mu_1\frac{\partial}{\partial y}+\nu_1\frac{\partial}{\partial z}\right)h_2^2, \text{ by (4) and (5),}$$

$$=\frac{1}{2}h_1^2\frac{\partial}{\partial\xi}h_2^2, \text{ by (10).}$$

Thus by (15)

$$\mp h_2^2h_1\cdot{}_\xi\varpi_\eta=h_1^2h_2\frac{\partial h_2}{\partial\xi},$$

or

$${}_\xi\varpi_\eta=\mp\frac{h_1}{h_2}\frac{\partial h_2}{\partial\xi}.$$

According to the ordinary convention as to sign, we consider the curvature positive when the centre of curvature is situated in what we agree to reckon the positive direction of the normal. This we have taken (§ 230) to be the direction in which ξ increases; so that the curvatures must be reckoned *positive* when the surface turns its *concavity* in the direction in which ξ *increases.*

The easiest way to get rid of the ambiguity of sign in the above formula for the curvature, is by the following geometrical investigation, due to Lamé. It affords an independent proof of the formula, and has the advantage of absolutely determining the sign.

Let PS_1, PS_2, PS_3 in Figure 30 be the curves of intersection of the three coördinate surfaces at P, drawn—as usual—in the directions in which ξ, η, ζ increase; and let PQ_1, PQ_2 be the elementary arcs ds_1, ds_2. Let PC and Q_2C be consecutive normals to the ξ surface: then the plane CPQ_2 is that of the principal section of the ξ surface through the tangent at P to the curve PS_2. Thus C is the centre of curvature of this principal section, and we have

$$\frac{1}{CP}=\frac{1}{CQ_2}=\pm{}_\xi\varpi_\eta,$$

the upper or lower sign being taken according as the curvature is positive or negative, or according as C lies in the positive [A] or negative [B] direction of PS_1. Also PC is the tangent at P to the curve PS_1, and the elementary arc PQ_1 or ds_1 coincides in direction with PC or with CP produced.

With centre C and radius CQ_1 describe an elementary circular arc Q_1T, cutting CQ_2 (produced if necessary) at right angles in T; and from P draw PU parallel to CT to meet this arc in U.

Then Q_1T is the element of the s_2 curve drawn through Q_1,

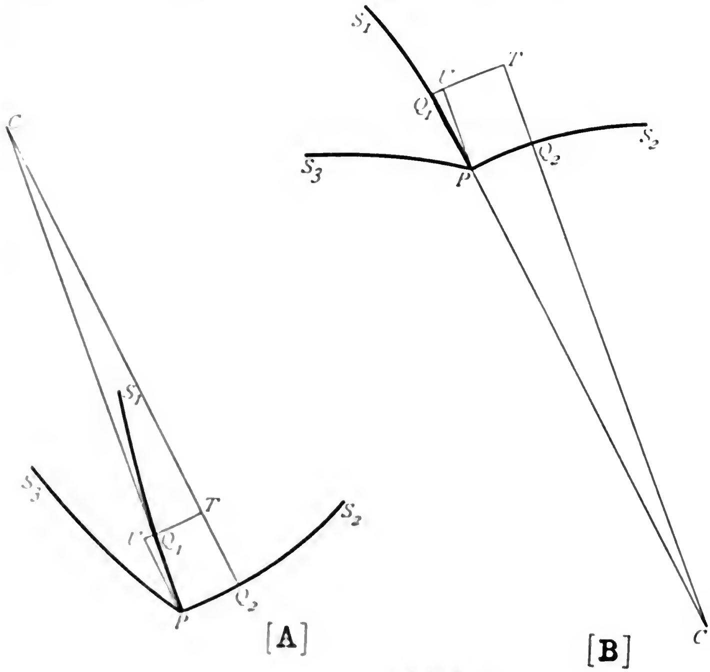


Fig. 30

and Q_2T is the element of the s_1 curve drawn through Q_2.

Since Q_1T is an elementary arc of the same system as PQ_2, and only differs from it in that the point from which it is drawn has for coördinates $(\xi + d\xi, \eta, \zeta)$ instead of (ξ, η, ζ), we must have

$$\begin{aligned} Q_1T &= \left(1 + d\xi\frac{\partial}{\partial\xi}\right)PQ_2, \\ &= \left(1 + d\xi\frac{\partial}{\partial\xi}\right)ds_2, \\ &= \left(1 + d\xi\frac{\partial}{\partial\xi}\right)\frac{d\eta}{h_2} \\ &= d\eta\left(1 + d\xi\frac{\partial}{\partial\xi}\right)\frac{1}{h_2}, \end{aligned}$$

since η is independent of ξ.

$$\therefore\ Q_1T = d\eta\left(\frac{1}{h_2} - \frac{d\xi}{h_2^2}\frac{\partial h_2}{d\xi}\right)$$

$$= ds_2\left(1 - \frac{h_1}{h_2}\cdot\frac{\partial h_2}{\partial\xi}\cdot ds_1\right).$$

Also, since PQ_2TU is approximately a rectangle, we must have

$$UT = PQ_2 = ds_2.$$

Now in the case [A] in which the ξ surface is *concave* in the direction in which ξ increases, Q_1 lies on the same side of P as C. Thus Q_1 lies between U and T, and

$$UQ_1 = UT - Q_1T$$

$$= +\frac{h_1}{h_2}\frac{\partial h_2}{\partial\xi}\cdot ds_1\, ds_2.$$

But in the case [B] in which the ξ surface is *convex* in the direction in which ξ increases, U lies between Q_1 and T, and

$$UQ_1 = Q_1T - UT$$

$$= -\frac{h_1}{h_2}\frac{\partial h_2}{\partial\xi}\cdot ds_1\, ds_2.$$

By construction, the triangles PUQ_1, CQ_2P are similar, so that

$$CP : PQ_1 :: PQ_2 : Q_1U$$

or

$$\frac{1}{CP} = \frac{Q_1U}{ds_1\,.\,ds_2}.$$

In the case [A] this gives us

$$+\,{}_\xi\varpi_\eta = +\frac{h_1}{h_2}\frac{\partial h_2}{\partial\xi},$$

and in the case [B]

$$-\,{}_\xi\varpi_\eta = -\frac{h_1}{h_2}\frac{\partial h_2}{\partial\xi}.$$

We have then definitely

$${}_\xi\varpi_\eta = \frac{h_1}{h_2}\frac{\partial h_2}{\partial\xi},$$

and similarly

$${}_\xi\varpi_\zeta = \frac{h_1}{h_3}\frac{\partial h_3}{\partial\xi};$$

the curvatures being considered positive when the ξ surface is concave in the direction of increase of ξ.

Writing down by symmetry the formulæ for the other two coördinate surfaces, we have for the six principal curvatures at (ξ, η, ζ)

$$\left.\begin{aligned} {}_\eta\varpi_\xi &= \frac{h_2}{h_1}\frac{\partial h_1}{\partial \eta}, & {}_\zeta\varpi_\xi &= \frac{h_3}{h_1}\frac{\partial h_1}{\partial \zeta} \\ {}_\zeta\varpi_\eta &= \frac{h_3}{h_2}\frac{\partial h_2}{\partial \zeta}, & {}_\xi\varpi_\eta &= \frac{h_1}{h_2}\frac{\partial h_2}{\partial \xi} \\ {}_\xi\varpi_\zeta &= \frac{h_1}{h_3}\frac{\partial h_3}{\partial \xi}, & {}_\eta\varpi_\zeta &= \frac{h_2}{h_3}\frac{\partial h_3}{\partial \eta} \end{aligned}\right\} \qquad (16)$$

Finally, if ϖ_1, ϖ_2, ϖ_3 be the absolute curvatures of the three curves of intersection s_1, s_2, s_3, in their osculating planes at P, we have (Frost's *Solid Geometry*, § 581)

$$\left.\begin{aligned} \varpi_1^2 &= {}_\eta\varpi_\xi^2 + {}_\zeta\varpi_\xi^2 \\ \varpi_2^2 &= {}_\zeta\varpi_\eta^2 + {}_\xi\varpi_\eta^2 \\ \varpi_3^2 &= {}_\xi\varpi_\zeta^2 + {}_\eta\varpi_\zeta^2 \end{aligned}\right\} \qquad (16a)$$

233.] **Surfaces in General.** Let any surface whatever be represented by the equation

$$\Phi(\xi, \eta, \zeta) = constant.$$

Then Φ can be expressed as a function of x, y, z, and if we write

$$\mathfrak{h}^2 = \left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2, \qquad (17)$$

the direction-cosines of the normal to the surface at any point, referred to Ox, Oy, Oz, will be

$$\frac{\partial\Phi}{\partial x}\Big/\mathfrak{h}, \quad \frac{\partial\Phi}{\partial y}\Big/\mathfrak{h}, \quad \frac{\partial\Phi}{\partial z}\Big/\mathfrak{h}.$$

Thus, if λ, μ, ν be the *cosines of the angles which this normal makes with the elementary lines* ds_1, ds_2, ds_3, *drawn from the same point*,

$$\lambda\mathfrak{h} = \lambda_1\frac{\partial\Phi}{\partial x} + \mu_1\frac{\partial\Phi}{\partial y} + \nu_1\frac{\partial\Phi}{\partial z}, \text{ etc.}$$

Hence, by (10),

$$\left.\begin{aligned} \lambda &= \frac{h_1}{\mathfrak{h}}\frac{\partial\Phi}{\partial\xi} \\ \mu &= \frac{h_2}{\mathfrak{h}}\frac{\partial\Phi}{\partial\eta} \\ \nu &= \frac{h_3}{\mathfrak{h}}\frac{\partial\Phi}{\partial\zeta} \end{aligned}\right\} \qquad (18)$$

from which we deduce that

$$\mathfrak{h}^2 = \left(h_1 \frac{\partial \Phi}{\partial \xi}\right)^2 + \left(h_2 \frac{\partial \Phi}{\partial \eta}\right)^2 + \left(h_3 \frac{\partial \Phi}{\partial \zeta}\right)^2 \qquad (19)$$

If dS be the element of the surface about the point (ξ, η, ζ), its projections upon the coördinate surfaces through the point are easily seen to be

$$\left.\begin{aligned} \lambda dS &= \frac{d\eta d\zeta}{h_2 h_3} \\ \mu dS &= \frac{d\zeta d\xi}{h_3 h_1} \\ \nu dS &= \frac{d\xi d\eta}{h_1 h_2} \end{aligned}\right\} \qquad (20)$$

234.] **Strain Components.** Now let us suppose the body to suffer a small strain, and let its effect on any point P in the body be to change its curvilinear coördinates from (ξ, η, ζ) to $(\xi+\alpha, \eta+\beta, \zeta+\gamma)$; α, β, γ being small quantities of the first order.

Let e, f, g denote the small elongations of the elementary lines ds_1, ds_2, ds_3, and a, b, c the small shears (§ 94) of the right angles between ds_2 and ds_3, ds_3 and ds_1, ds_1 and ds_2, respectively.

In general h_1, h_2, h_3 are functions of all three of the coördinates, and by Taylor's theorem we see that the effect upon them of a small strain will be represented, (to our order of approximation), by changing them into $h_1+\delta h_1$, $h_2+\delta h_2$, $h_3+\delta h_3$, where

$$\left.\begin{aligned} \delta h_1 &= \alpha\frac{\partial h_1}{\partial \xi} + \beta\frac{\partial h_1}{\partial \eta} + \gamma\frac{\partial h_1}{\partial \zeta} \\ \delta h_2 &= \alpha\frac{\partial h_2}{\partial \xi} + \beta\frac{\partial h_2}{\partial \eta} + \gamma\frac{\partial h_2}{\partial \zeta} \\ \delta h_3 &= \alpha\frac{\partial h_3}{\partial \xi} + \beta\frac{\partial h_3}{\partial \eta} + \gamma\frac{\partial h_3}{\partial \zeta} \end{aligned}\right\} \qquad (21)$$

now by (7)

$$ds_1 = \frac{d\xi}{h_1},$$

and therefore

$$(1+e)ds_1 = \frac{d\xi + d\alpha}{h_1 + \delta h_1} = \frac{d\xi}{h_1} + \frac{d\alpha}{h_1} - \frac{d\xi}{h_1{}^2}\delta h_1$$

$$\therefore \qquad e = \frac{\partial \alpha}{\partial \xi} - \frac{\delta h_1}{h_1},$$

and so for f and g.

Thus finally

$$\left.\begin{aligned} e &= \frac{\partial \alpha}{\partial \xi} - \frac{1}{h_1}\left[\alpha\frac{\partial h_1}{\partial \xi} + \beta\frac{\partial h_1}{\partial \eta} + \gamma\frac{\partial h_1}{\partial \zeta}\right] \\ f &= \frac{\partial \beta}{\partial \eta} - \frac{1}{h_2}\left[\alpha\frac{\partial h_2}{\partial \xi} + \beta\frac{\partial h_2}{\partial \eta} + \gamma\frac{\partial h_2}{\partial \zeta}\right] \\ g &= \frac{\partial \gamma}{\partial \zeta} - \frac{1}{h_3}\left[\alpha\frac{\partial h_3}{\partial \xi} + \beta\frac{\partial h_3}{\partial \eta} + \gamma\frac{\partial h_3}{\partial \zeta}\right] \end{aligned}\right\} \quad\dots\dots\dots\dots(22)$$

Substituting from (16) these equations may be put in the form

$$\left.\begin{aligned} e &= h_1\frac{\partial}{\partial \xi}\left(\frac{\alpha}{h_1}\right) - \left(\frac{\beta}{h_2}\right)_\eta \varpi_\xi - \left(\frac{\gamma}{h_3}\right)_\zeta \varpi_\xi \\ f &= h_2\frac{\partial}{\partial \eta}\left(\frac{\beta}{h_2}\right) - \left(\frac{\gamma}{h_3}\right)_\zeta \varpi_\eta - \left(\frac{\alpha}{h_1}\right)_\xi \varpi_\eta \\ g &= h_3\frac{\partial}{\partial \zeta}\left(\frac{\gamma}{h_3}\right) - \left(\frac{\alpha}{h_1}\right)_\xi \varpi_\zeta - \left(\frac{\beta}{h_2}\right)_\eta \varpi_\zeta \end{aligned}\right\} \quad\dots\dots\dots\dots(23)$$

Again, as in § 94, the small shear a is simply the cosine of the angle between the altered directions of ds_2 and ds_3. Thus

$$\begin{aligned} a &= (\lambda_2 + \delta\lambda_2)(\lambda_3 + \delta\lambda_3) + (\mu_2 + \delta\mu_2)(\mu_3 + \delta\mu_3) + (\nu_2 + \delta\nu_2)(\nu_3 + \delta\nu_3) \\ &= \lambda_2\delta\lambda_3 + \mu_2\delta\mu_3 + \nu_2\delta\nu_3 + \lambda_3\delta\lambda_2 + \mu_3\delta\mu_2 + \nu_3\delta\nu_2 \end{aligned}$$

by (6).

Now, by (5),

$$\begin{aligned} \lambda_3 + \delta\lambda_3 &= \frac{1}{h_3 + \delta h_3}\frac{\partial}{\partial x}(\zeta + \gamma) \\ &= \frac{1}{h_3}\frac{\partial \zeta}{\partial x}\left(1 - \frac{\delta h_3}{h_3}\right) + \frac{1}{h_3}\frac{\partial \gamma}{\partial x}. \end{aligned}$$

$$\therefore \quad \delta\lambda_3 = \frac{1}{h_3}\frac{\partial \gamma}{\partial x} - \lambda_3\frac{\delta h_3}{h_3},$$

and so for the others. Thus by (5) and (6)

$$a = \frac{1}{h_3}\left(\lambda_2\frac{\partial \gamma}{\partial x} + \mu_2\frac{\partial \gamma}{\partial y} + \nu_2\frac{\partial \gamma}{\partial z}\right) + \frac{1}{h_2}\left(\lambda_3\frac{\partial \beta}{\partial x} + \mu_3\frac{\partial \beta}{\partial y} + \nu_3\frac{\partial \beta}{\partial z}\right),$$

and finally, by (10),

$$\left.\begin{aligned} a &= \frac{h_2}{h_3}\frac{\partial \gamma}{\partial \eta} + \frac{h_3}{h_2}\frac{\partial \beta}{\partial \zeta} \\ b &= \frac{h_3}{h_1}\frac{\partial \alpha}{\partial \zeta} + \frac{h_1}{h_3}\frac{\partial \gamma}{\partial \xi} \\ c &= \frac{h_1}{h_2}\frac{\partial \beta}{\partial \xi} + \frac{h_2}{h_1}\frac{\partial \alpha}{\partial \eta} \end{aligned}\right\} \quad\dots\dots\dots\dots(24)$$

Lastly, if Δ be the cubical dilatation at (ξ, η, ζ) we of course have $\Delta = e+f+g$; and by (22) this is easily put into the form

$$\Delta = h_1 h_2 h_3 \left\{ \frac{\partial}{\partial\xi}\left(\frac{\alpha}{h_1 h_2 h_3}\right) + \frac{\partial}{\partial\eta}\left(\frac{\beta}{h_1 h_2 h_3}\right) + \frac{\partial}{\partial\zeta}\left(\frac{\gamma}{h_1 h_2 h_3}\right) \right\} \quad \text{.........(25)}$$

235.] **The Component Displacements.** Let u, v, w be the components of the displacement of any point P (ξ, η, ζ), resolved along the elementary lines ds_1, ds_2, ds_3:—or, more exactly, along the normals to the three coördinate surfaces which meet in the point.

We proceed to find expressions for the six small component strains in terms of u, v, w. These expressions will not be so simple as in the case of Cartesian coördinates, because now, instead of resolving the displacement of each point in three *fixed* orthogonal directions, we resolve it along the tangents to three orthogonal curves whose directions vary continuously from point to point. We must therefore expect any expressions which involve the *variations* of the component displacements along these curves to involve also the curvatures of the coördinate surfaces; and this we shall find to be the case.

In Figure 31, PQ represents the edge ds_1 of the element of volume (8), represented complete in Figure 29; its size being supposed so reduced that its edges are practically straight lines. QR is the consecutive element of the s_1 curve.

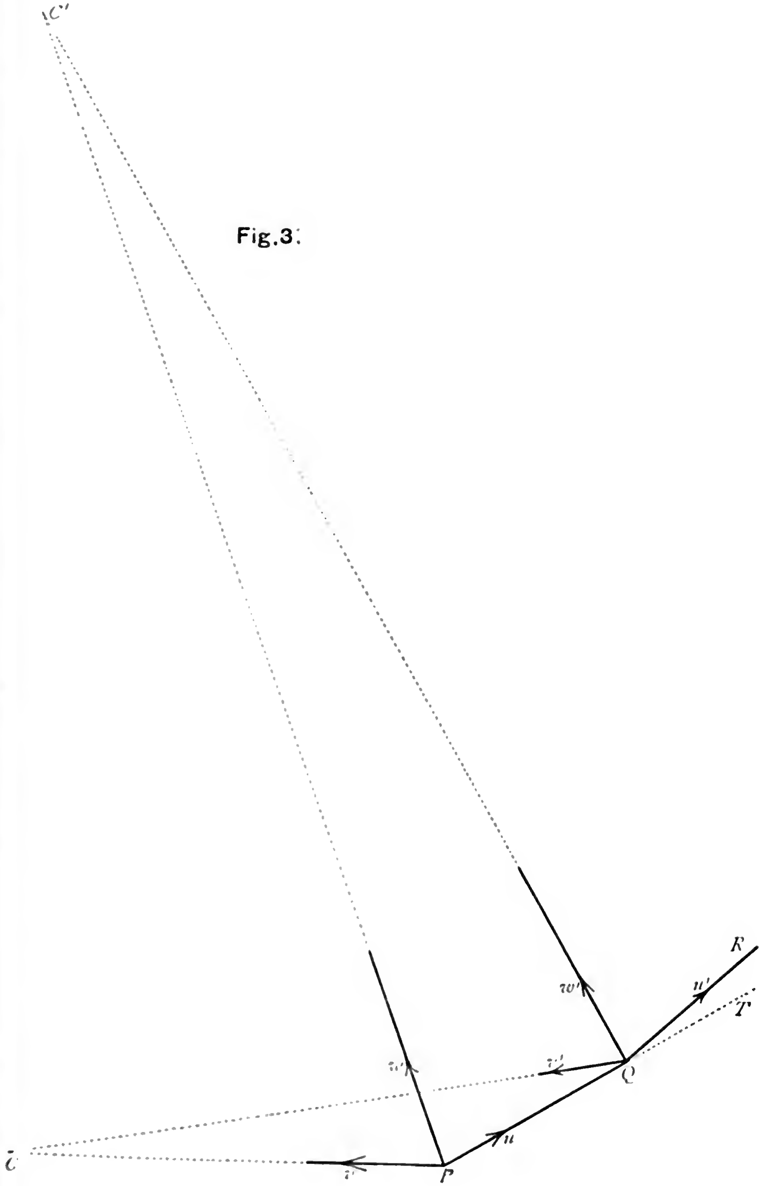
PC and PC' are drawn in the directions of the elements ds_2 and ds_3, and QC and QC' in the directions of the corresponding elements at Q. Thus PQ, PC, PC' are mutually perpendicular, and so are QR, QC, QC'.

Since PQ and QR are consecutive elements of a line of curvature on both the η and ζ surfaces through P, C will be the centre of curvature of the principal section of the η surface through that curve, and C' will be the corresponding centre for the ζ surface. [The changes of direction of the elementary lines, in passing from P to Q, are of course enormously exaggerated, in order to bring C and C' within the compass of the Figure.]

Again, if PQ be produced onwards towards T, the plane TQR is the osculating plane of the s_1 curve at P; and if we denote the absolute curvature of that curve by ϖ_1, and adopt the notation of § 232 for the principal curvatures of the coördinate surfaces, we shall have

$$\left.\begin{aligned} \text{angle } RQT &= \varpi_1 \,.\, PQ \\ \text{angle } PCQ &= {}_\eta\varpi_\xi \,.\, PQ \\ \text{angle } PC'Q &= {}_\zeta\varpi_\xi \,.\, PQ \end{aligned}\right\}.$$

Fig.3.



If u', v', w' be the component displacements of Q, on the system above described, the component of its displacement in the direction QT will obviously be

$$u'\,.\cos RQT - v'\,.\cos PQC - w'\,.\cos PQC'$$
$$= u'\,.\cos RQT - v'\,.\sin PCQ - w'\,.\sin PC'Q\,;$$

and, to the first power of the element PQ or ds_1 this is

$$u' - (v'\,.\,{}_\eta\varpi_\xi + w'\,.\,{}_\zeta\varpi_\xi)ds_1.$$

Now

$$\left.\begin{aligned} u' &= u + ds_1\frac{\partial u}{\partial s_1} \\ v' &= v + ds_1\frac{\partial v}{\partial s_1} \\ w' &= w + ds_1\frac{\partial w}{\partial s_1} \end{aligned}\right\}.$$

Hence, to the first power of ds_1, the displacement of Q resolved along QT is

$$u + ds_1\left[\frac{\partial u}{\partial s_1} - v\,.\,{}_\eta\varpi_\xi - w\,.\,{}_\zeta\varpi_\xi\right].$$

The displacement of P in the same direction is simply u; so that the increase of length gained by ds_1 is

$$ds_1\left[\frac{\partial u}{\partial s_1} - v\,.\,{}_\eta\varpi_\xi - w\,.\,{}_\zeta\varpi_\xi\right].$$

This gain of length is of course equal to $e\,.\,ds_1$. Equating these two values, and applying the same process to ds_2 and ds_3, we have finally

$$\left.\begin{aligned} e &= \frac{\partial u}{\partial s_1} - v\,.\,{}_\eta\varpi_\xi - w\,.\,{}_\zeta\varpi_\xi \\ f &= \frac{\partial v}{\partial s_2} - w\,.\,{}_\zeta\varpi_\eta - u\,.\,{}_\xi\varpi_\eta \\ g &= \frac{\partial w}{\partial s_3} - u\,.\,{}_\xi\varpi_\zeta - v\,.\,{}_\eta\varpi_\zeta \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(26)$$

Substituting from (7) and (16), we can easily show that

$$\Delta = h_1h_2h_3\left[\frac{\partial}{\partial\xi}\left(\frac{u}{h_2h_3}\right) + \frac{\partial}{\partial\eta}\left(\frac{v}{h_3h_1}\right) + \frac{\partial}{\partial\zeta}\left(\frac{w}{h_1h_2}\right)\right]\ldots\ldots\ldots(27)$$

Again the displacement of Q parallel to PC' is very approximately

$$w' \,.\, \cos QC'P + u' \sin QC'P,$$

or
$$w + \left(\frac{\partial w}{\partial s_1} + u \,.\, {}_{\zeta}\varpi_{\xi}\right) ds_1.$$

The displacement of P in the same direction being simply w, it is clear that the relative displacement of P and Q parallel to PC' will diminish the right angle QPC' by the small amount

$$\frac{\left(\dfrac{\partial w}{\partial s_1} + u \,.\, {}_{\zeta}\varpi_{\xi}\right) ds_1}{(1+e) ds_1}$$

which, to our order of approximation, is

$$\frac{\partial w}{\partial s_1} + u \,.\, {}_{\zeta}\varpi_{\xi}.$$

Similarly, if R be the further extremity of the arc ds_3, the relative displacement of P and R parallel to PQ will diminish the same right angle by the small amount

$$\frac{\partial u}{\partial s_3} + w \,.\, {}_{\xi}\varpi_{\zeta}.$$

The sum of these two, or the total decrement of the original right angle between ds_3 and ds_1, is by the last Article equal to the small component shear b.

The values of a and c may be calculated with equal ease or deduced by symmetry, and finally we have

$$\left.\begin{aligned} a &= \frac{\partial w}{\partial s_2} + \frac{\partial v}{\partial s_3} + v \,.\, {}_{\zeta}\varpi_{\eta} + w \,.\, {}_{\eta}\varpi_{\zeta} \\ b &= \frac{\partial u}{\partial s_3} + \frac{\partial w}{\partial s_1} + w \,.\, {}_{\xi}\varpi_{\zeta} + u \,.\, {}_{\zeta}\varpi_{\xi} \\ c &= \frac{\partial v}{\partial s_1} + \frac{\partial u}{\partial s_2} + u \,.\, {}_{\eta}\varpi_{\xi} + v \,.\, {}_{\xi}\varpi_{\eta} \end{aligned}\right\} \quad \text{.....................(28)}$$

Substituting from (16), these may be put in the form

$$\left.\begin{aligned} a &= \frac{h_2}{h_3}\frac{\partial}{\partial \eta}(wh_3) + \frac{h_3}{h_2}\frac{\partial}{\partial \zeta}(vh_2) \\ b &= \frac{h_3}{h_1}\frac{\partial}{\partial \zeta}(uh_1) + \frac{h_1}{h_3}\frac{\partial}{\partial \xi}(wh_3) \\ c &= \frac{h_1}{h_2}\frac{\partial}{\partial \xi}(vh_2) + \frac{h_2}{h_1}\frac{\partial}{\partial \eta}(uh_1) \end{aligned}\right\} \quad \text{.....................(29)}$$

If we compare equations (26) with (23), (29) with (24), and (27) with (25) we see at once that

$$\left.\begin{aligned} \alpha &= uh_1 \\ \beta &= vh_2 \\ \gamma &= wh_3 \end{aligned}\right\} \qquad (30)$$

which we might have *inferred* from (7), all the six quantities u, v, w, α, β, γ being very small.

Lastly, we have seen that the edge PR rotates about ds_2 towards PQ through the angle

$$\frac{\partial u}{\partial s_3} + w \,.\, {}_\xi\varpi_\zeta;$$

and that PQ rotates about ds_2 towards PR through the angle

$$\frac{\partial w}{\partial s_1} + u \,.\, {}_\zeta\varpi_\xi.$$

Exactly as in equations (59) of § 123, half the difference of these quantities measures the rotation (as distinguished from strain) of the element of volume as a whole about ds_2.

If then Θ_1, Θ_2, Θ_3 be taken to represent the three component rotations of the element about the normals to the three coördinate surfaces through its centre, we have

$$2\Theta_2 = \frac{\partial u}{\partial s_3} + w \,.\, {}_\xi\varpi_\zeta - \frac{\partial w}{\partial s_1} - u \,.\, {}_\zeta\varpi_\xi.$$

Writing down the symmetrical formulæ for Θ_3 and Θ_1, and eliminating the curvatures by (16), we have finally

$$\left.\begin{aligned} 2\Theta_1 &= h_2h_3\left[\frac{\partial}{\partial\eta}\left(\frac{w}{h_3}\right) - \frac{\partial}{\partial\zeta}\left(\frac{v}{h_2}\right)\right] \\ 2\Theta_2 &= h_3h_1\left[\frac{\partial}{\partial\zeta}\left(\frac{u}{h_1}\right) - \frac{\partial}{\partial\xi}\left(\frac{w}{h_3}\right)\right] \\ 2\Theta_3 &= h_1h_2\left[\frac{\partial}{\partial\xi}\left(\frac{v}{h_2}\right) - \frac{\partial}{\partial\eta}\left(\frac{u}{h_1}\right)\right] \end{aligned}\right\} \qquad (31)$$

These equations may also be deduced directly by transforming the corresponding Cartesian equations of § 123. Thus, with the notation of the present Chapter

$$\Theta_1 = \lambda_1\theta_1 + \mu_1\theta_2 + \nu_1\theta_3,$$

etc., etc.,

$$2\theta_1 = \frac{\partial}{\partial y}(\nu_1 u + \nu_2 v + \nu_3 w) - \frac{\partial}{\partial z}(\mu_1 u + \mu_2 v + \mu_3 w),$$

etc., etc.

236.] **Irrotational Strain.** If the strain be pure or irrotational (§§ 124-127) there will of course be a displacement potential ϕ. In this case (§ 124) the component displacements parallel to Ox, Oy, Oz are

$$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}.$$

Thus, with our present notation,

$$\left.\begin{aligned} \lambda_1 u + \lambda_2 v + \lambda_3 w &= \frac{\partial\phi}{\partial x} \\ \mu_1 u + \mu_2 v + \mu_3 w &= \frac{\partial\phi}{\partial y} \\ \nu_1 u + \nu_2 v + \nu_3 w &= \frac{\partial\phi}{\partial z} \end{aligned}\right\},$$

and consequently

$$u = \lambda_1 \frac{\partial\phi}{\partial x} + \mu_1 \frac{\partial\phi}{\partial y} + \nu_1 \frac{\partial\phi}{\partial z}, \text{ etc.}$$

Hence, by equations (10),

$$\left.\begin{aligned} u &= h_1 \frac{\partial\phi}{\partial\xi} \\ v &= h_2 \frac{\partial\phi}{\partial\eta} \\ w &= h_3 \frac{\partial\phi}{\partial\zeta} \end{aligned}\right\} \quad\text{..........}(32)$$

and consequently by (30)

$$\left.\begin{aligned} \alpha &= h_1^2 \frac{\partial\phi}{\partial\xi} \\ \beta &= h_2^2 \frac{\partial\phi}{\partial\eta} \\ \gamma &= h_3^2 \frac{\partial\phi}{\partial\zeta} \end{aligned}\right\} \quad\text{..........}(33)$$

Substituting from (33) in (25), or from (32) in (27), and comparing the result with (13), we see that

$$\Delta = \nabla^2\phi \quad\text{..........}(34)$$

which agrees with § 124.

The conditions that a given strain may be irrotational are seen from (32) to be

$$\left.\begin{aligned}\frac{\partial}{\partial\eta}\left(\frac{w}{h_3}\right)-\frac{\partial}{\partial\zeta}\left(\frac{v}{h_2}\right)=0\\ \frac{\partial}{\partial\zeta}\left(\frac{u}{h_1}\right)-\frac{\partial}{\partial\xi}\left(\frac{w}{h_3}\right)=0\\ \frac{\partial}{\partial\xi}\left(\frac{v}{h_2}\right)-\frac{\partial}{\partial\eta}\left(\frac{u}{h_1}\right)=0\end{aligned}\right\}\qquad\text{(35)}$$

which indeed, by (31), are simply equivalent to $\Theta_1=\Theta_2=\Theta_3=0$.

237.] **Stress and Applied Force.** We shall employ the same notation for stress as hitherto, writing now

$$\left.\begin{matrix}P, U, T\\ U, Q, S\\ T, S, R\end{matrix}\right\}$$

for the components parallel to ds_1, ds_2, ds_3 of the stresses across the elementary areas described about (ξ, η, ζ) in each of the three coördinate surfaces through that point.

The components, in the same directions, of the Applied Force per unit mass on the elementary mass of which P is the centre will be denoted by Ξ, H, Z; and the density by ρ as before.

Just as in §§ 138-143, we obtain the equations of equilibrium and of motion by considering an element surrounding the point (ξ, η, ζ) and bounded by the six coördinate surfaces

$$\xi\pm\tfrac{1}{2}d\xi,\ \eta\pm\tfrac{1}{2}d\eta,\ \zeta\pm\tfrac{1}{2}d\zeta.$$

This element is cut up by the surfaces ξ, η, ζ into eight such elements as that of Figure 29, having P for a common corner; the lengths of the edges which meet in P being

$$\tfrac{1}{2}ds_1,\ \tfrac{1}{2}ds_2,\ \tfrac{1}{2}ds_3,$$

respectively.

Figure 32 represents this divided element (the curvatures being much exaggerated, as before), and corresponds to Figure 8 in every respect; the three faces turned towards the eye being the concave or positive faces (§ 232).

The areas of the sections $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$ are ultimately given by (9), and the volume of the element by (8).

Let us now resolve the total stress across each face parallel to the tangent at P to the s_1 curve. This component of the total stress, together with the applied force

$$\frac{\rho\Xi d\xi d\eta d\zeta}{h_1h_2h_3},\qquad\text{(36)}$$

must be equal to the effective force

$$\frac{\rho\ddot{u}\,d\xi d\eta d\zeta}{h_1 h_2 h_3} \quad\ldots\ldots\ldots\ldots\ldots\ldots(37)$$

in the same direction.

Take first the ξ faces, the coördinates of whose centres are

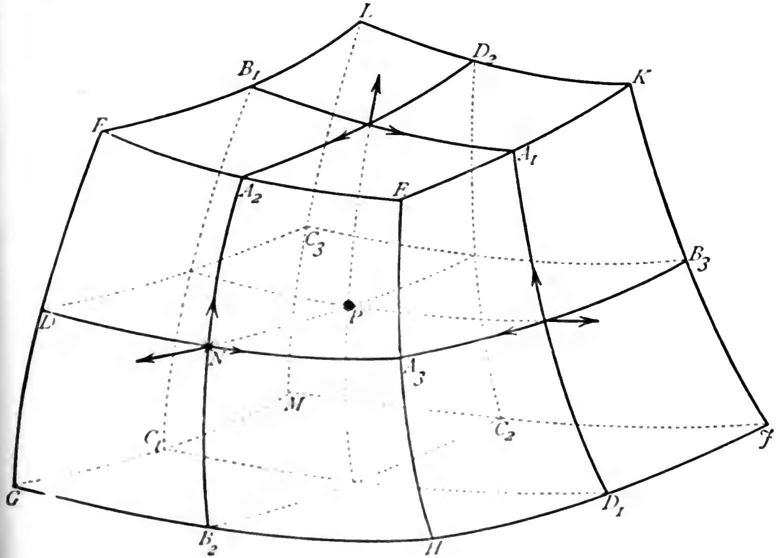


Fig. 32.

$(\xi \pm \frac{1}{2}d\xi, \eta, \zeta)$. The components along the tangents to the s_1, s_2, s_3 curves at P of the total stress across $A_1B_1C_1D_1$ are ultimately

$$\frac{P d\eta d\zeta}{h_2 h_3}, \quad \frac{U d\eta d\zeta}{h_2 h_3}, \quad \frac{T d\eta d\zeta}{h_2 h_3}.$$

Hence the components of the total stress across the positive ξ face $EFGH$, *along the tangents at N to PN, NA_3, NA_2*, are

$$\left.\begin{array}{l} \left[\dfrac{P}{h_2h_3} + \frac{1}{2}d\xi\dfrac{\partial}{\partial\xi}\left(\dfrac{P}{h_2h_3}\right)\right]d\eta d\zeta \\ \left[\dfrac{U}{h_2h_3} + \frac{1}{2}d\xi\dfrac{\partial}{\partial\xi}\left(\dfrac{U}{h_2h_3}\right)\right]d\eta d\zeta \\ \left[\dfrac{T}{h_2h_3} + \frac{1}{2}d\xi\dfrac{\partial}{\partial\xi}\left(\dfrac{T}{h_2h_3}\right)\right]d\eta d\zeta \end{array}\right\}.$$

Resolving these parallel to the tangent at P to the s_1 curve, exactly as we resolved the displacements in § 235, we have for

the required component of the total stress across the positive ξ face,

$$\left\{\left[\frac{P}{h_2h_3}+\tfrac{1}{2}d\xi\frac{\partial}{\partial\xi}\left(\frac{P}{h_2h_3}\right)\right]\cos\left(\tfrac{1}{2}\,.\,\varpi_1\,.\,ds_1\right)\right.$$
$$-\left[\frac{U}{h_2h_3}+\tfrac{1}{2}d\xi\frac{\partial}{\partial\xi}\left(\frac{U}{h_2h_3}\right)\right]\sin\left(\tfrac{1}{2}\,.\,{}_\eta\varpi_\xi\,.\,ds_1\right)$$
$$\left.-\left[\frac{T}{h_2h_3}+\tfrac{1}{2}d\xi\frac{\partial}{\partial\xi}\left(\frac{T}{h_2h_3}\right)\right]\sin\left(\tfrac{1}{2}\,.\,{}_\zeta\varpi_\xi\,.\,ds_1\right)\right\}d\eta d\zeta.$$

The corresponding component of the total stress across the negative ξ face $JKLM$, reckoned in the positive direction, is of course

$$-\left\{\left[\frac{P}{h_2h_3}-\tfrac{1}{2}d\xi\frac{\partial}{\partial\xi}\left(\frac{P}{h_2h_3}\right)\right]\cos\left(\tfrac{1}{2}\,.\,\varpi_1\,.\,ds_1\right)\right.$$
$$+\left[\frac{U}{h_2h_3}-\tfrac{1}{2}d\xi\frac{\partial}{\partial\xi}\left(\frac{U}{h_2h_3}\right)\right]\sin\left(\tfrac{1}{2}\,.\,{}_\eta\varpi_\xi\,.\,ds_1\right)$$
$$\left.+\left[\frac{T}{h_2h_3}-\tfrac{1}{2}d\xi\frac{\partial}{\partial\xi}\left(\frac{T}{h_2h_3}\right)\right]\sin\left(\tfrac{1}{2}\,.\,{}_\zeta\varpi_\xi\,.\,ds_1\right)\right\}d\eta d\zeta.$$

Substituting from (7), and neglecting squares of small quantities, these two faces together give a component total stress

$$\left[\frac{\partial}{\partial\xi}\left(\frac{P}{h_2h_3}\right)-\frac{U\,.\,{}_\eta\varpi_\xi+T\,.\,{}_\zeta\varpi_\xi}{h_1h_2h_3}\right]d\xi d\eta d\zeta\,\ldots\ldots\ldots\ldots\ldots(38)$$

Again, the component due to the positive η face $EHJK$ is approximately

$$\left\{\left[\frac{Q}{h_3h_1}+\tfrac{1}{2}d\eta\frac{\partial}{\partial\eta}\left(\frac{Q}{h_3h_1}\right)\right]\sin\left(\tfrac{1}{2}\,.\,{}_\xi\varpi_\eta\,.\,ds_2\right)\right.$$
$$\left.+\left[\frac{U}{h_3h_1}+\tfrac{1}{2}d\eta\frac{\partial}{\partial\eta}\left(\frac{U}{h_3h_1}\right)\right]\cos\left(\tfrac{1}{2}\,.\,{}_\xi\varpi_\eta\,.\,ds_2\right)\right\}d\zeta d\xi,$$

and that due to the negative η face $FGML$ is

$$\left\{\left[\frac{Q}{h_3h_1}-\tfrac{1}{2}d\eta\frac{\partial}{\partial\eta}\left(\frac{Q}{h_3h_1}\right)\right]\sin\left(\tfrac{1}{2}\,.\,{}_\xi\varpi_\eta\,.\,ds_2\right)\right.$$
$$\left.-\left[\frac{U}{h_3h_1}-\tfrac{1}{2}d\eta\frac{\partial}{\partial\eta}\left(\frac{U}{h_3h_1}\right)\right]\cos\left(\tfrac{1}{2}\,.\,{}_\xi\varpi_\eta\,.\,ds_2\right)\right\}d\zeta d\xi.$$

These two faces therefore contribute

$$\left[\frac{\partial}{\partial\eta}\left(\frac{U}{h_3h_1}\right)+\frac{Q\,.\,{}_\xi\varpi_\eta}{h_1h_2h_3}\right]d\xi d\eta d\zeta\,\ldots\ldots\ldots\ldots\ldots\ldots(39)$$

to the required component.

Finally, the positive and negative ζ faces, $EFLK$, $GHJM$ contribute

$$\left\{\left[\frac{R}{h_1h_2}+\tfrac{1}{2}d\zeta\frac{\partial}{\partial\zeta}\left(\frac{R}{h_1h_2}\right)\right]\sin\left(\tfrac{1}{2}\,.\,{}_\xi\varpi_\zeta\,.\,ds_3\right)\right.$$
$$\left.+\left[\frac{T}{h_1h_2}+\tfrac{1}{2}d\zeta\frac{\partial}{\partial\zeta}\left(\frac{T}{h_1h_2}\right)\right]\cos\left(\tfrac{1}{2}\,.\,{}_\xi\varpi_\zeta\,.\,ds_3\right)\right\}d\xi d\eta,$$
$$+\left\{\left[\frac{R}{h_1h_2}-\tfrac{1}{2}d\zeta\frac{\partial}{\partial\zeta}\left(\frac{R}{h_1h_2}\right)\right]\sin\left(\tfrac{1}{2}\,.\,{}_\xi\varpi_\zeta\,.\,ds_3\right)\right.$$
$$\left.-\left[\frac{T}{h_1h_2}-\tfrac{1}{2}d\zeta\frac{\partial}{\partial\zeta}\left(\frac{T}{h_1h_2}\right)\right]\cos\left(\tfrac{1}{2}\,.\,{}_\xi\varpi_\zeta\,.\,ds_3\right)\right\}d\xi d\eta\,;$$

or, on reduction,

$$\left[\frac{\partial}{\partial\zeta}\left(\frac{T}{h_1h_2}\right)+\frac{R\,.\,{}_\xi\varpi_\zeta}{h_1h_2h_3}\right]d\xi d\eta d\zeta \qquad\qquad (40)$$

Equating the sum of (36), (38), (39), and (40) to (37), we deduce the first of the three equations of motion

$$\frac{\partial}{\partial\xi}\left(\frac{P}{h_2h_3}\right)+\frac{\partial}{\partial\eta}\left(\frac{U}{h_3h_1}\right)-\frac{U}{h_3h_1^2}\frac{\partial h_1}{\partial\eta}+\frac{\partial}{\partial\zeta}\left(\frac{T}{h_1h_2}\right)$$
$$-\frac{T}{h_1^2h_2}\frac{\partial h_1}{\partial\zeta}+\frac{1}{h_1h_2h_3}\left[Q\,.\,{}_\xi\varpi_\eta+R\,.\,{}_\xi\varpi_\zeta+(\Xi-\ddot{u})\right]=0.$$

Rearranging the terms which involve T and U, and writing down the other two equations from symmetry, we have finally

$$\left.\begin{aligned}
&\frac{\partial}{\partial\xi}\left(\frac{P}{h_2h_3}\right)+h_1\frac{\partial}{\partial\eta}\left(\frac{U}{h_3h_1^2}\right)+h_1\frac{\partial}{\partial\zeta}\left(\frac{T}{h_1^2h_2}\right)\\
&\qquad+\frac{1}{h_1h_2h_3}\left[Q\,.\,{}_\xi\varpi_\eta+R\,.\,{}_\xi\varpi_\zeta+\rho(\Xi-\ddot{u})\right]=0\\
&h_2\frac{\partial}{\partial\xi}\left(\frac{U}{h_2^2h_3}\right)+\frac{\partial}{\partial\eta}\left(\frac{Q}{h_3h_1}\right)+h_2\frac{\partial}{\partial\zeta}\left(\frac{S}{h_1h_2^2}\right)\\
&\qquad+\frac{1}{h_1h_2h_3}\left[R\,.\,{}_\eta\varpi_\zeta+P\,.\,{}_\eta\varpi_\xi+\rho(\mathrm{H}-\ddot{v})\right]=0\\
&h_3\frac{\partial}{\partial\xi}\left(\frac{T}{h_2h_3^2}\right)+h_3\frac{\partial}{\partial\eta}\left(\frac{S}{h_3^2h_1}\right)+\frac{\partial}{\partial\zeta}\left(\frac{R}{h_1h_2}\right)\\
&\qquad+\frac{1}{h_1h_2h_3}\left[P\,.\,{}_\zeta\varpi_\xi+Q\,.\,{}_\zeta\varpi_\eta+\rho(\mathrm{Z}-\ddot{w})\right]=0
\end{aligned}\right\} \qquad (41)$$

where by (30)

$$\left.\begin{aligned}
\ddot{u}&=\ddot{\alpha}/h_1\\
\ddot{v}&=\ddot{\beta}/h_2\\
\ddot{w}&=\ddot{\gamma}/h_3
\end{aligned}\right\} \qquad (42)$$

By means of (16) we can put equations (41) into Lamé's form

$$\left.\begin{aligned}&\frac{\partial P}{\partial s_1}+\frac{\partial U}{\partial s_2}+\frac{\partial T}{\partial s_3}+\rho(\Xi-\ddot{u})=(P-Q)\,.\,{}_\xi\varpi_\eta\\&\qquad+(P-R)\,.\,{}_\xi\varpi_\zeta+U(2\,.\,{}_\eta\varpi_\xi+{}_\eta\varpi_\zeta)+T(2\,.\,{}_\zeta\varpi_\xi+{}_\zeta\varpi_\eta)\\&\frac{\partial U}{\partial s_1}+\frac{\partial Q}{\partial s_2}+\frac{\partial S}{\partial s_3}+\rho(\mathrm{H}-\ddot{v})=(Q-R)\,.\,{}_\eta\varpi_\zeta\\&\qquad+(Q-P)\,.\,{}_\eta\varpi_\xi+S(2\,.\,{}_\zeta\varpi_\eta+{}_\zeta\varpi_\xi)+U(2\,.\,{}_\xi\varpi_\eta+{}_\xi\varpi_\zeta)\\&\frac{\partial T}{\partial s_1}+\frac{\partial S}{\partial s_2}+\frac{\partial R}{\partial s_3}+\rho(\mathrm{Z}-\ddot{w})=(R-P)\,.\,{}_\zeta\varpi_\xi\\&\qquad+(R-Q)\,.\,{}_\zeta\varpi_\eta+T(2\,.\,{}_\xi\varpi_\zeta+{}_\xi\varpi_\eta)+S(2\,.\,{}_\eta\varpi_\zeta+{}_\eta\varpi_\xi)\end{aligned}\right\}\quad\ldots\ldots\ldots(43)$$

in which their analogy with equations (4) of § 143 is sufficiently obvious.

238.] **Stress and Surface Traction.** Precisely as in §§ 144, 145, we may show that the components of the traction across the element dS of any surface (§ 233) passing through the point (ξ, η, ζ) must be

$$\left.\begin{aligned}&P\lambda+U\mu+T\nu\\&U\lambda+Q\mu+S\nu\\&T\lambda+S\mu+R\nu\end{aligned}\right\},$$

where λ, μ, ν are given by (18) and (19).

Hence if the bounding surface of the body be represented by

$$\Phi(\xi, \eta, \zeta)=constant\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(44)$$

and if Ξ', H', Z' be the components of the surface traction at the point (ξ, η, ζ) of the surface, we must have at every such point the relations

$$\left.\begin{aligned}Ph_1\frac{\partial\Phi}{\partial\xi}+Uh_2\frac{\partial\Phi}{\partial\eta}+Th_3\frac{\partial\Phi}{\partial\zeta}&=\mathfrak{h}\Xi'\\Uh_1\frac{\partial\Phi}{\partial\xi}+Qh_2\frac{\partial\Phi}{\partial\eta}+Sh_3\frac{\partial\Phi}{\partial\zeta}&=\mathfrak{h}\mathrm{H}'\\Th_1\frac{\partial\Phi}{\partial\xi}+Sh_2\frac{\partial\Phi}{\partial\eta}+Rh_3\frac{\partial\Phi}{\partial\zeta}&=\mathfrak{h}\mathrm{Z}'\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(45)$$

where $\mathfrak{h}$ is given by

$$\mathfrak{h}^2=\left(h_1\frac{\partial\Phi}{\partial\xi}\right)^2+\left(h_2\frac{\partial\Phi}{\partial\eta}\right)^2+\left(h_3\frac{\partial\Phi}{\partial\zeta}\right)^2\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

It is often advisable to choose the system of coördinates so that the surface of the body shall be a coördinate surface—belonging, we will suppose, to the ξ system. In this case (44) can be put in the form $\xi=constant$, and we may take $\Phi=\xi$.

Thus by (19) $\mathrm{h}=h_1$, and by (18) $\lambda=1$, $\mu=\nu=0$; as of course it should be.

The boundary conditions (45) then reduce to

$$\left.\begin{aligned} P &= \Xi' \\ U &= \mathrm{H}' \\ T &= \mathrm{Z}' \end{aligned}\right\} \qquad (45a)$$

The corresponding conditions, when the surface belongs to either of the other systems, may be written down by inspection.

239. Strain and Stress. Equations of Motion in terms of Strain. If the body be isotropic, the relations between Strain and Stress will of course be, as in the last Chapter,

$$\left.\begin{aligned} P &= (m+n)e + (m-n)(f+g) \\ Q &= (m+n)f + (m-n)(g+e) \\ R &= (m+n)g + (m-n)(e+f) \\ S &= na \\ T &= nb \\ U &= nc \end{aligned}\right\} \qquad (46)$$

The potential energy V, per unit of unstrained volume, is also given by formulæ (33) or (34) of Chapter IV.; and the total potential energy W of the strain by

$$W = \iiint V \frac{d\xi d\eta d\zeta}{h_1 h_2 h_3} \qquad (47)$$

To obtain the equations of motion in terms of u, v, w, or of a, β, γ, by direct substitution from (22), (24), (25), or from (26), (27), (28) in (46), and thence in (41) or (43), is in the general case of curvilinears an excessively tedious operation, and it is not easy to put them into a symmetrical form.

Lamé has shown, by direct transformation of the Cartesian equations, that they may be written

$$\left.\begin{aligned} (m+n)\frac{\partial\Delta}{\partial s_1} - 2n\left[h_3\frac{\partial}{\partial s_2}\left(\frac{\Theta_3}{h_3}\right) - h_2\frac{\partial}{\partial s_3}\left(\frac{\Theta_2}{h_2}\right)\right] + \rho(\Xi - \ddot{u}) = 0 \\ (m+n)\frac{\partial\Delta}{\partial s_2} - 2n\left[h_1\frac{\partial}{\partial s_3}\left(\frac{\Theta_1}{h_1}\right) - h_3\frac{\partial}{\partial s_1}\left(\frac{\Theta_3}{h_3}\right)\right] + \rho(\mathrm{H} - \ddot{v}) = 0 \\ (m+n)\frac{\partial\Delta}{\partial s_3} - 2n\left[h_2\frac{\partial}{\partial s_1}\left(\frac{\Theta_2}{h_2}\right) - h_1\frac{\partial}{\partial s_2}\left(\frac{\Theta_1}{h_1}\right)\right] + \rho(\mathrm{Z} - \ddot{w}) = 0 \end{aligned}\right\} \qquad (48)$$

where Δ is given by (27), and Θ_1, Θ_2, Θ_3 by (31).

In this form they present a striking analogy to equations (52a) of § 218: from which they were derived by Lamé, as stated above.

Special Application of Curvilinears.

240.] **Equipotential Surfaces.** If the strain is pure (§ 236), the resultant displacement is at each point normal to one of a system of continuous equipotential surfaces, defined by giving constant values to the displacement potential, and the Lines of Displacement (§ 127) are a system of continuous curves, cutting these surfaces everywhere orthogonally.

There is obviously no reason why we should not take the ξ surfaces for the equipotentials, when we can thus simplify our formulæ. In this case the s_1 curves will be the Lines of Displacement; ϕ will be a function of ξ only, and we shall have at every point

$$\left.\begin{aligned} \beta = \gamma = 0 \\ v = w = 0 \end{aligned}\right\} \qquad (49)$$

Also, by (31) and (32),

$$a = h_1 u = h_1{}^2 \frac{d\phi}{d\xi} \qquad (50)$$

Equations (23), (24), (25) now become

$$\left.\begin{aligned}
e &= h_1 \frac{\partial}{\partial\xi}\left(h_1 \frac{d\phi}{d\xi}\right) \\
f &= -h_1 \cdot {}_\xi\varpi_\eta \cdot \frac{d\phi}{d\xi} \\
g &= -h_1 \cdot {}_\xi\varpi_\zeta \cdot \frac{d\phi}{d\xi} \\
\Delta &= h_1 h_2 h_3 \frac{\partial}{\partial\xi}\left(\frac{h_1}{h_2 h_3}\frac{d\phi}{d\xi}\right) \\
a &= 0 \\
b &= 2h_1 \frac{d\phi}{d\xi} \cdot {}_\zeta\varpi_\xi \\
c &= 2h_1 \frac{d\phi}{d\xi} \cdot {}_\eta\varpi_\xi
\end{aligned}\right\} \qquad (51)$$

from which we can substitute in (46) and thence in (41) or (43) and (42).

Since the conditions (35) are in this case necessarily fulfilled, equations (48) reduce to

$$\left.\begin{aligned}
&(m+n)h_1 \frac{\partial}{\partial\xi}\left[h_1 h_2 h_3 \frac{\partial}{\partial\xi}\left(\frac{h_1}{h_2 h_3}\frac{d\phi}{d\xi}\right)\right] + \rho\left(\Xi - h_1 \frac{d\ddot{\phi}}{d\xi}\right) = 0 \\
&(m+n)h_2 \left\{ 2\frac{\partial}{\partial\xi}\left(h_1 \frac{\partial h_1}{\partial\eta}\frac{d\phi}{d\xi}\right) - \frac{d\phi}{d\xi}\frac{\partial}{\partial\eta}\left[\frac{h_1}{h_2 h_3}\frac{\partial}{\partial\xi}(h_1 h_2 h_3)\right]\right\} + \rho \mathrm{H} = 0 \\
&(m+n)h_3 \left\{ 2\frac{\partial}{\partial\xi}\left(h_1 \frac{\partial h_1}{\partial\eta}\frac{d\phi}{d\xi}\right) - \frac{d\phi}{d\xi}\frac{\partial}{\partial\zeta}\left[\frac{h_1}{h_2 h_3}\frac{\partial}{\partial\xi}(h_1 h_2 h_3)\right]\right\} + \rho \mathrm{Z} = 0
\end{aligned}\right\} \qquad (52)$$

The second and third of these equations are reduced from forms symmetrical with the first, by the consideration that ϕ is independent of η and ζ.

241.] **Principal Surfaces of the Strain.** Let us next suppose that equations (1), (2), (3) represent the three systems of Principal Surfaces (§ 216). The curves of intersection, s_1, s_2, s_3, will then be the Lines of Stress.

In this case ξ, η, ζ may be called the *Principal Coördinates of the Strain*. Lamé applied to the coördinate surfaces under these conditions the term *Isostatic*.

If ϵ_1, ϵ_2, ϵ_3 denote the principal elongations, and N_1, N_2, N_3 the principal normal stresses, we must have

$$e = \epsilon_1,\ f = \epsilon_2,\ g = \epsilon_3;\ a = b = c = 0:$$
$$P = N_1,\ Q = N_2,\ R = N_3;\ S = T = U = 0.$$

The conditions that ξ, η, ζ may be the principal coördinates are therefore, by (24)

$$\left.\begin{aligned} h_2{}^2\frac{\partial\gamma}{\partial\eta} + h_3{}^2\frac{\partial\beta}{\partial\zeta} &= 0 \\ h_3{}^2\frac{\partial a}{\partial\zeta} + h_1{}^2\frac{\partial\gamma}{\partial\xi} &= 0 \\ h_1{}^2\frac{\partial\beta}{\partial\xi} + h_2{}^2\frac{\partial a}{\partial\eta} &= 0 \end{aligned}\right\} \tag{53}$$

If these are satisfied we have by (26)

$$\left.\begin{aligned} \epsilon_1 &= h_1\frac{\partial u}{\partial\xi} - v\,.\,{}_\eta\varpi_\xi - w\,.\,{}_\zeta\varpi_\xi \\ \epsilon_2 &= h_2\frac{\partial v}{\partial\eta} - w\,.\,{}_\zeta\varpi_\eta - u\,.\,{}_\xi\varpi_\eta \\ \epsilon_3 &= h_3\frac{\partial w}{\partial\zeta} - u\,.\,{}_\xi\varpi_\zeta - v\,.\,{}_\eta\varpi_\zeta \end{aligned}\right\} \tag{54}$$

and by (46)

$$\left.\begin{aligned} N_1 &= (m+n)\epsilon_1 + (m-n)(\epsilon_2+\epsilon_3) \\ N_2 &= (m+n)\epsilon_2 + (m-n)(\epsilon_3+\epsilon_1) \\ N_3 &= (m+n)\epsilon_3 + (m-n)(\epsilon_1+\epsilon_2) \end{aligned}\right\} \tag{55}$$

Equations (41) reduce to

$$\left.\begin{aligned} h_1h_2h_3\frac{\partial}{\partial\xi}\left(\frac{N_1}{h_2h_3}\right) + N_2\,.\,{}_\xi\varpi_\eta + N_3\,.\,{}_\xi\varpi_\zeta + \rho(\Xi - \ddot{u}) &= 0 \\ h_1h_2h_3\frac{\partial}{\partial\eta}\left(\frac{N_2}{h_3h_1}\right) + N_3\,.\,{}_\eta\varpi_\zeta + N_1\,.\,{}_\eta\varpi_\xi + \rho(\mathrm{H} - \ddot{v}) &= 0 \\ h_1h_2h_3\frac{\partial}{\partial\zeta}\left(\frac{N_3}{h_1h_2}\right) + N_1\,.\,{}_\zeta\varpi_\xi + N_2\,.\,{}_\zeta\varpi_\eta + \rho(\mathrm{Z} - \ddot{w}) &= 0 \end{aligned}\right\} \tag{56}$$

while (43) give us Lamé's standard form

$$\left.\begin{aligned}\frac{\partial N_1}{\partial s_1}+\rho(\Xi-\ddot{u})&=(N_1-N_2)\cdot{}_{\xi}\varpi_{\eta}+(N_1-N_3)\cdot{}_{\xi}\varpi_{\zeta}\\ \frac{\partial N_2}{\partial s_2}+\rho(\mathrm{H}-\ddot{v})&=(N_2-N_3)\cdot{}_{\eta}\varpi_{\zeta}+(N_2-N_1)\cdot{}_{\eta}\varpi_{\xi}\\ \frac{\partial N_3}{\partial s_3}+\rho(\mathrm{Z}-\ddot{w})&=(N_3-N_1)\cdot{}_{\zeta}\varpi_{\xi}+(N_3-N_2)\cdot{}_{\zeta}\varpi_{\eta}\end{aligned}\right\}\quad\text{.........(57)}$$

Finally, the boundary conditions (45) become simply

$$\left.\begin{aligned}N_1h_1\frac{\partial\Phi}{\partial\xi}&=\mathfrak{h}\Xi'\\ N_2h_2\frac{\partial\Phi}{\partial\eta}&=\mathfrak{h}\mathrm{H}'\\ N_3h_3\frac{\partial\Phi}{\partial\zeta}&=\mathfrak{h}\mathrm{Z}'\end{aligned}\right\}\quad\text{......... ....(58)}$$

If the surface of the body be one of the Principal Surfaces—belonging, let us say, to the ξ system—these conditions reduce further, by (45a) to

$$\left.\begin{aligned}N_1&=\Xi'\\ \mathrm{H}'=\mathrm{Z}'&=0\end{aligned}\right\}\quad\text{..............................(58}a\text{)}$$

the Surface Traction being in this case necessarily normal.

242.] **Case in which all the Principal Surfaces remain such.** There is one interesting case of the last article in which the strain is such that each of the principal surfaces is altered into another—very slightly different—member of the same family.

The requisite conditions obviously are:—α independent of η and ζ, β independent of ζ and ξ, and γ independent of ξ and η.

It should be noted that, although these conditions always satisfy (53), they do not in general satisfy (35), so that a strain of this character is a *rotational strain*—except for certain particular systems of coördinates. In fact, by eliminating α, β, γ between (34), we obtain the condition

$$\left.\begin{aligned}&\frac{\partial h_2}{\partial\xi}\cdot\frac{\partial h_3}{\partial\eta}\cdot\frac{\partial h_1}{\partial\zeta}=\frac{\partial h_3}{\partial\xi}\cdot\frac{\partial h_1}{\partial\eta}\cdot\frac{\partial h_2}{\partial\zeta}\\ &\text{or, by (16),}\\ &{}_{\eta}\varpi_{\zeta}\cdot{}_{\zeta}\varpi_{\xi}\cdot{}_{\xi}\varpi_{\eta}={}_{\zeta}\varpi_{\eta}\cdot{}_{\eta}\varpi_{\xi}\cdot{}_{\xi}\varpi_{\zeta}\end{aligned}\right\}\quad\text{.................. (59)}$$

which is not satisfied by all coördinate systems.

Various Systems of Curvilinears.

We now proceed to express our general formulæ in terms of some of the most important systems of orthogonal curvilinears. As a preliminary exercise, the student will do well to convince himself that, on making

$$\xi = x, \quad \eta = y, \quad \zeta = z,$$

they reduce immediately to the Cartesian formulæ obtained in the last three Chapters.

243.] **Spherical Polars.** In this system we write

$$\left.\begin{aligned} \xi &= r = [x^2 + y^2 + z^2]^{\frac{1}{2}} \\ \eta &= \theta = \tan^{-1}[(x^2 + y^2)^{\frac{1}{2}}/z] \\ \zeta &= \omega = \tan^{-1}[y/x] \end{aligned}\right\} \quad \text{.........................(60)}$$

whence

$$\left.\begin{aligned} x &= r \sin\theta\cos\omega \\ y &= r \sin\theta\sin\omega \\ z &= r\cos\theta \end{aligned}\right\} \quad \text{.............................(61)}$$

The surfaces for which r is constant are spheres with O for centre and r for radius; the θ surfaces are right circular cones with vertex O, axis Oz, and semi-vertical angle θ; and the ω surfaces are planes through Oz making angle ω with zx.

Substituting from (61) in (12), we find

$$h_1 = 1, \; h_2 = \frac{1}{r}, \; h_3 = \frac{1}{r \sin\theta} \quad \text{.......................(62)}$$

and consequently by (7)

$$ds_1 = dr, \; ds_2 = rd\theta, \; ds_3 = r\sin\theta d\omega \; ;$$

while the elements of volume (8) become

$$r^2 \sin\theta dr d\theta d\omega.$$

The cosines λ, μ, ν of the angles made by the normal to the surface $\Phi(r, \theta, \omega) = constant$ with the normals to the coördinate surfaces are by (18)

$$\left.\begin{aligned} \lambda &= \frac{1}{\mathfrak{h}}\frac{\partial\Phi}{\partial r} \\ \mu &= \frac{1}{\mathfrak{h}r}\frac{\partial\Phi}{\partial\theta} \\ \nu &= \frac{1}{\mathfrak{h}r\sin\theta}\frac{\partial\Phi}{\partial\omega} \end{aligned}\right\} \quad \text{.............................(63)}$$

where, by (19),

$$\mathrm{h}^2 = \left(\frac{\partial \Phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \omega}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots(64)$$

Formula (13) becomes

$$\nabla^2\Phi = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2\Phi}{\partial \omega^2} \quad \ldots(65)$$

Substituting from (62) in (16)

$$\left.\begin{array}{ll} {}_\theta\varpi_r = 0, & {}_\omega\varpi_r = 0 \\ {}_\omega\varpi_\theta = 0, & {}_r\varpi_\theta = -\dfrac{1}{r} \\ {}_r\varpi_\omega = -\dfrac{1}{r}, & {}_\theta\varpi_\omega = -\dfrac{\cot\theta}{r} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots(66)$$

while by (30)

$$\left.\begin{array}{l} u = \alpha \\ v = r\beta \\ w = r\gamma\sin\theta \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots(67)$$

It is evident from (66) that the s_1 curves on this system become straight lines, and from (67) that α is linear and identical with u. We shall therefore retain the latter symbol only.

Equations (22), (24), (25), (26), (27), (28) give us

$$\left.\begin{array}{l} e = \dfrac{\partial u}{\partial r} \\ f = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{u}{r} = \dfrac{\partial \beta}{\partial \theta} + \dfrac{u}{r} \\ g = \dfrac{1}{r\sin\theta}\dfrac{\partial w}{\partial \omega} + \dfrac{u}{r} + \dfrac{v}{r}\cot\theta = \dfrac{\partial \gamma}{\partial \omega} + \dfrac{u}{r} + \beta\cot\theta \\ \Delta = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(ur^2) + \dfrac{1}{r\sin\theta}\left[\dfrac{\partial}{\partial \theta}(v\sin\theta) + \dfrac{\partial w}{\partial \omega}\right] \\ \quad = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(ur^2) + \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial \theta}(\beta\sin\theta) + \dfrac{\partial \gamma}{\partial \omega} \\ a = \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial \theta}\left(\dfrac{w}{\sin\theta}\right) + \dfrac{1}{r\sin\theta}\dfrac{\partial v}{\partial \omega} = \sin\theta\dfrac{\partial \gamma}{\partial \theta} + \dfrac{1}{\sin\theta}\dfrac{\partial \beta}{\partial \omega} \\ b = \dfrac{1}{r\sin\theta}\dfrac{\partial u}{\partial \omega} + r\dfrac{\partial}{\partial r}\left(\dfrac{w}{r}\right) = \dfrac{1}{r\sin\theta}\dfrac{\partial u}{\partial \omega} + r\sin\theta\dfrac{\partial \gamma}{\partial r} \\ c = r\dfrac{\partial}{\partial r}\left(\dfrac{v}{r}\right) + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = r\dfrac{\partial \beta}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \end{array}\right\} \quad \ldots\ldots(68)$$

while by (31)

$$\left.\begin{aligned}
2\Theta_1 &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(w\sin\theta) - \frac{\partial v}{\partial\omega}\right] \\
&= \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\gamma\sin^2\theta) - \frac{\partial\beta}{\partial\omega}\right] \\
2\Theta_2 &= \frac{1}{r\sin\theta}\left[\frac{\partial u}{\partial\omega} - \sin\theta\frac{\partial}{\partial r}(wr)\right] \\
&= \frac{1}{r\sin\theta}\left[\frac{\partial u}{\partial\omega} - \sin^2\theta\frac{\partial}{\partial r}(\gamma r^2)\right] \\
2\Theta_3 &= \frac{1}{r}\left[\frac{\partial}{\partial r}(vr) - \frac{\partial u}{\partial\theta}\right] \\
&= \frac{1}{r}\left[\frac{\partial}{\partial r}(\beta r^2) - \frac{\partial u}{\partial\theta}\right]
\end{aligned}\right\} \quad \text{.....(69)}$$

The equations of motion (41) become

$$\left.\begin{aligned}
&\frac{1}{r^2}\frac{\partial}{\partial r}(Pr^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(U\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\omega} \\
&\qquad - \frac{1}{r}(Q+R) = \rho(\ddot{u} - \Xi) \\
&\frac{1}{r^3}\frac{\partial}{\partial r}(Ur^3) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(Q\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial S}{\partial\omega} \\
&\qquad - \frac{R\cos\theta}{r\sin\theta} = \rho(\ddot{v} - \mathrm{H}) = \rho(r\ddot{\beta} - \mathrm{H}) \\
&\frac{1}{r^3}\frac{\partial}{\partial r}(Tr^3) + \frac{1}{r\sin^2\theta}\frac{\partial}{\partial\theta}(S\sin^2\theta) + \frac{1}{r\sin\theta}\frac{\partial R}{\partial\omega} \\
&\qquad = \rho(\ddot{w} - Z) = \rho(r\sin\theta\ddot{\gamma} - Z)
\end{aligned}\right\} \quad \text{.....(70)}$$

where P, Q, R, S, T, U are given by (46) and (68).

By Lamé's transformation (48) we have

$$\left.\begin{aligned}
&(m+n)\frac{\partial\Delta}{\partial r} - \frac{2n}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\Theta_3\sin\theta) - \frac{\partial\Theta}{\partial\omega}\right] = \rho(\ddot{u} - \Xi) \\
&\frac{m+n}{r}\frac{\partial\Delta}{\partial\theta} - \frac{2n}{r}\left[\frac{1}{\sin\theta}\frac{\partial\Theta_1}{\partial\omega} - \frac{\partial}{\partial r}(\Theta_3 r)\right] = \rho(\ddot{v} - \mathrm{H}) \\
&\frac{m+n}{r\sin\theta}\frac{\partial\Delta}{\partial\omega} - \frac{2n}{r}\left[\frac{\partial}{\partial r}(\Theta_2 r) - \frac{\partial\Theta_1}{\partial\theta}\right] = \rho(\ddot{w} - Z)
\end{aligned}\right\} \quad \text{.....(71)}$$

If Φ be the bounding surface of the body, the boundary conditions (45) take the form

$$\left.\begin{aligned}
P\frac{\partial\Phi}{\partial r} + \frac{U}{r}\frac{\partial\Phi}{\partial\theta} + \frac{T}{r\sin\theta}\frac{\partial\Phi}{\partial\omega} &= \mathfrak{h}\Xi' \\
U\frac{\partial\Phi}{\partial r} + \frac{Q}{r}\frac{\partial\Phi}{\partial\theta} + \frac{S}{r\sin\theta}\frac{\partial\Phi}{\partial\omega} &= \mathfrak{h}\mathrm{H}' \\
T\frac{\partial\Phi}{\partial r} + \frac{S}{r}\frac{\partial\Phi}{\partial\theta} + \frac{R}{r\sin\theta}\frac{\partial\Phi}{\partial\omega} &= \mathfrak{h}Z'
\end{aligned}\right\} \quad \text{.....(72)}$$

$\mathfrak{h}$ being given by (64).

The conditions (35) that the strain may be pure follow at once from (69), on making

$$\Theta_1 = 0,\ \ \Theta_2 = 0,\ \ \Theta_3 = 0\ ;$$

and in this case, by (32) and (33), if ϕ be the displacement potential,

$$\left.\begin{aligned} u &= \frac{\partial\phi}{\partial r} \\ v &= r\beta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \\ w &= r\gamma\sin\theta = \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\omega} \end{aligned}\right\} \text{......................(73)}$$

If r, θ, ω be the principal coördinates of the strain, we must have, by (53),

$$\left.\begin{aligned} \sin^2\theta\frac{\partial}{\partial\theta}\left(\frac{w}{\sin\theta}\right) + \frac{\partial v}{\partial\omega} &= 0 \\ \frac{\partial u}{\partial\omega} + r^2\sin\theta\frac{\partial}{\partial r}\left(\frac{w}{r}\right) &= 0 \\ r^2\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial u}{\partial\theta} &= 0 \end{aligned}\right\} \text{......................(74)}$$

or the equivalent conditions

$$\left.\begin{aligned} \sin^2\theta\frac{\partial\gamma}{\partial\theta} + \frac{\partial\beta}{\partial\omega} &= 0 \\ \frac{\partial u}{\partial\omega} + r^2\sin^2\theta\frac{\partial\gamma}{\partial r} &= 0 \\ r^2\frac{\partial\beta}{\partial r} + \frac{\partial u}{\partial\theta} &= 0 \end{aligned}\right\} \text{......................(75)}$$

If these conditions be fulfilled, e, f, g, as given by (68), are the principal elongations ϵ_1, ϵ_2, ϵ_3; and the principal normal stresses N_1, N_2, N_3 are then given by (55).

Lamé's equations (57) then take the form

$$\left.\begin{aligned} \frac{1}{r^2}\frac{\partial}{\partial r}(N_1 r^2) - \frac{1}{r}(N_2 + N_3) &= \rho(\ddot{u} - \Xi) \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(N_2\sin\theta) - \frac{N_3\cos\theta}{r\sin\theta} &= \rho(\ddot{v} - \mathrm{H}) \\ \frac{1}{r\sin\theta}\frac{\partial N_3}{\partial\omega} &= \rho(\ddot{w} - \mathrm{Z}) \end{aligned}\right\} \text{..............(76)}$$

while the surface conditions (58) become

$$\left.\begin{aligned} N_1\frac{\partial\Phi}{\partial r} &= \mathrm{h}\Xi' \\ N_2\frac{\partial\Phi}{\partial\theta} &= \mathrm{h}r\mathrm{H}' \\ N_3\frac{\partial\Phi}{\partial\omega} &= \mathrm{h}r\sin\theta Z' \end{aligned}\right\}\qquad(77)$$

244.] **Cylindrical Polars.** In this system

$$\left.\begin{aligned} \xi &= r = (x^2+y^2)^{\frac{1}{2}} \\ \eta &= \theta = \tan^{-1}(y/x) \\ \zeta &= z \end{aligned}\right\}\qquad(78)$$

whence $x = r\cos\theta,\ \ y = r\sin\theta.$

The r surfaces are right circular cylinders with Oz for axis, and r for radius; the θ surfaces are planes through Oz.

Here we have

$$h_1 = 1,\ h_2 = \frac{1}{r},\ h_3 = 1\qquad(79)$$

and consequently by (7)

$$ds_1 = dr,\ ds_2 = rd\theta,\ ds_3 = dz\,;$$

the element of volume (8) becoming

$$rdrd\theta dz.$$

The cosines (λ, μ, ν) of the angles made by the normal to any surface $\Phi(r, \theta, z) = constant$ with the normals to the three coördinate surfaces at the same point are, by (18),

$$\left.\begin{aligned} \lambda &= \frac{1}{\mathrm{h}}\frac{\partial\Phi}{\partial r} \\ \mu &= \frac{1}{\mathrm{h}r}\frac{\partial\Phi}{\partial\theta} \\ \nu &= \frac{1}{\mathrm{h}}\frac{\partial\Phi}{\partial z} \end{aligned}\right\}\qquad(80)$$

where, by (19),

$$\mathrm{h}^2 = \left(\frac{\partial\Phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\qquad(81)$$

Formula (13) now takes the form

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}\qquad(82)$$

Substituting from (79) in (16), we get for the curvatures of the coördinate surfaces

$$\left.\begin{aligned} {}_\theta\varpi_r &= 0, \quad {}_z\varpi_r = 0 \\ {}_z\varpi_\theta &= 0, \quad {}_r\varpi_\theta = -\frac{1}{r} \\ {}_r\varpi_z &= 0, \quad {}_\theta\varpi_z = 0 \end{aligned}\right\} \quad \text{.........................(83)}$$

and by (30)

$$\left.\begin{aligned} u &= \alpha \\ v &= r\beta \\ w &= \gamma \end{aligned}\right\} \quad \text{..................................(84)}$$

Obviously on this system the s_1 and s_3 curves become straight lines, and α and γ are linear, and identical with u and w. We shall therefore retain the latter symbols only.

By equations (22), (24), (25), (26), (27), (28) we have

$$\left.\begin{aligned} e &= \frac{\partial u}{\partial r} \\ f &= \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} = \frac{\partial \beta}{\partial \theta} + \frac{u}{r} \\ g &= \frac{\partial w}{\partial z} \\ \Delta &= \frac{1}{r}\frac{\partial}{\partial r}(ur) + \frac{\partial \beta}{\partial \theta} + \frac{\partial w}{\partial z} \\ a &= \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} = \frac{1}{r}\frac{\partial w}{\partial \theta} + r\frac{\partial \beta}{\partial z} \\ b &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ c &= r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \theta} = r\frac{\partial \beta}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} \end{aligned}\right\} \quad \text{..................(85)}$$

while by (31)

$$\left.\begin{aligned} 2\Theta_1 &= \frac{1}{r}\frac{\partial w}{\partial \theta} - r\frac{\partial \beta}{\partial z} \\ 2\Theta_2 &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \\ 2\Theta_3 &= \frac{1}{r}\frac{\partial}{\partial r}(\beta r^2) - \frac{1}{r}\frac{\partial u}{\partial \theta} \end{aligned}\right\} \quad \text{........................(86)}$$

The equations of motion (41) reduce to

$$\left.\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}(Pr)+\frac{1}{r}\frac{\partial U}{\partial\theta}+\frac{\partial T}{\partial z}-\frac{Q}{r}&=\rho(\ddot{u}-\Xi)\\ \frac{1}{r^2}\frac{\partial}{\partial r}(Ur^2)+\frac{1}{r}\frac{\partial Q}{\partial\theta}+\frac{\partial S}{\partial z}\qquad&=\rho(\ddot{v}-\mathrm{H})\\ \frac{1}{r}\frac{\partial}{\partial r}(Tr)+\frac{1}{r}\frac{\partial S}{\partial\theta}+\frac{\partial R}{\partial z}\qquad&=\rho(\ddot{w}-\mathrm{Z})\end{aligned}\right\}\text{..............}(87)$$

where P, Q, R, S, T, U are given by (46) and (85).

Lamé's transformation (48) becomes

$$\left.\begin{aligned}(m+n)\frac{\partial\Delta}{\partial r}-2n\left[\frac{1}{r}\frac{\partial\Theta_3}{\partial\theta}-\frac{\partial\Theta_2}{\partial z}\right]\quad&=\rho(\ddot{u}-\Xi)\\ \frac{m+n}{r}\frac{\partial\Delta}{\partial\theta}-2n\left[\frac{\partial\Theta_1}{\partial z}-\frac{\partial\Theta_3}{\partial r}\right]\qquad&=\rho(\ddot{v}-\mathrm{H})\\ (m+n)\frac{\partial\Delta}{\partial z}-\frac{2n}{r}\left[\frac{\partial}{\partial r}(\Theta_2 r)-\frac{\partial\Theta_1}{\partial\theta}\right]&=\rho(\ddot{w}-\mathrm{Z})\end{aligned}\right\}\text{..........}(88)$$

If Φ be the bounding surface of the body, we have for the boundary conditions (45)

$$\left.\begin{aligned}P\frac{\partial\Phi}{\partial r}+\frac{U}{r}\frac{\partial\Phi}{\partial\theta}+T\frac{\partial\Phi}{\partial z}&=\mathfrak{h}\Xi'\\ U\frac{\partial\Phi}{\partial r}+\frac{Q}{r}\frac{\partial\Phi}{\partial\theta}+S\frac{\partial\Phi}{\partial z}&=\mathfrak{h}'\mathrm{H}\\ T\frac{\partial\Phi}{\partial r}+\frac{S}{r}\frac{\partial\Phi}{\partial\theta}+R\frac{\partial\Phi}{\partial z}&=\mathfrak{h}\mathrm{Z}'\end{aligned}\right\}\text{....................}(89)$$

$\mathfrak{h}$ being given by (81).

The conditions that the strain may be pure are of course

$$\Theta_1=0,\ \Theta_2=0,\ \Theta_3=0;$$

and in this case, if ϕ be the displacement potential,

$$\left.\begin{aligned}u&=\frac{\partial\phi}{\partial r}\\ v=r\beta&=\frac{1}{r}\frac{\partial\phi}{\partial\theta}\\ w&=\frac{\partial\phi}{\partial z}\end{aligned}\right\}\text{..............................}(90)$$

The conditions that r, θ, z may be the principal coördinates of the strain are, by (53)

$$\left.\begin{aligned}\frac{1}{r}\frac{\partial w}{\partial\theta}+\frac{\partial v}{\partial z}&=0\\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}&=0\\ r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{1}{r}\frac{\partial u}{\partial\theta}&=0\end{aligned}\right\}\text{.......................}(91)$$

If these be satisfied, e, f, g, as given by (85) are the principal elongations $\epsilon_1, \epsilon_2, \epsilon_3$; and the principal normal stresses N_1, N_2, N_3 are then given by (55).

Lamé's equations (57) reduce to

$$\left.\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}(N_1 r) - \frac{N_2}{r} &= \rho(\ddot{u} - \Xi)\\ \frac{1}{r}\frac{\partial N_2}{\partial \theta} &= \rho(\ddot{v} - \mathrm{H})\\ \frac{\partial N_3}{\partial z} &= \rho(\ddot{w} - \mathrm{Z})\end{aligned}\right\}\qquad(92)$$

while the boundary conditions (58) become

$$\left.\begin{aligned}N_1\frac{\partial \Phi}{\partial r} &= \mathfrak{h}\Xi'\\ N_2\frac{\partial \Phi}{\partial \theta} &= \mathfrak{h}r\mathrm{H}'\\ N_3\frac{\partial \Phi}{\partial z} &= \mathfrak{h}\mathrm{Z}'\end{aligned}\right\}\qquad(93)$$

245.] **Conjugate Cylindrics.** In this system

$$\left.\begin{aligned}\xi + \iota\eta &= F(x + \iota y)\\ \zeta &= z\end{aligned}\right\}\qquad(94)$$

where ι denotes $\sqrt{-1}$; F being any function whatever. This relation constitutes ξ and η conjugate functions of x and y. Some of the most important properties of these functions will be found collected in the examples at the end of this chapter. The student will find no difficulty in proving them for himself.

Differentiating the first of equations (94), we find

$$\left.\begin{aligned}\frac{\partial \xi}{\partial x} + \iota\frac{\partial \eta}{\partial x} &= F'(x + \iota y)\\ \frac{\partial \xi}{\partial y} + \iota\frac{\partial \eta}{\partial y} &= \iota F'(x + \iota y)\end{aligned}\right\}.$$

Hence, eliminating $F'(x+\iota y)$,

$$\frac{\partial \xi}{\partial y} + \iota\frac{\partial \eta}{\partial y} = \iota\left(\frac{\partial \xi}{\partial x} + \iota\frac{\partial \eta}{\partial x}\right);$$

and, on equating real and imaginary parts,

$$\left.\begin{aligned}\frac{\partial \xi}{\partial x} &= \frac{\partial \eta}{\partial y}\\ \frac{\partial \xi}{\partial y} &= -\frac{\partial \eta}{\partial x}\end{aligned}\right\}\qquad(95)$$

Similarly, by differentiating (94) as to ξ and η, we find

$$\left.\begin{aligned}\frac{\partial x}{\partial \xi} &= \frac{\partial y}{\partial \eta}\\ \frac{\partial x}{\partial \eta} &= -\frac{\partial y}{\partial \xi}\end{aligned}\right\} \qquad (95a)$$

From (95) we deduce by differentiation

$$\left.\begin{aligned}\nabla^2 \xi &= 0\\ \nabla^2 \eta &= 0\end{aligned}\right\} \qquad (96)$$

and in fact Conjugate Functions are sometimes defined as solutions of (96) which also satisfy (95).

It further follows from (95) that the coördinate surfaces satisfy the conditions (6) of orthogonalism. The ξ and η surfaces are in fact two orthogonal systems of cylinders with their generators parallel to Oz.

Again, by (95) we see that

$$\left.\begin{aligned}h_1 = h_2 &= h \text{ (say)}\\ h_3 &= 1\end{aligned}\right\} \qquad (97)$$

whence

$$ds_1 = \frac{d\xi}{h}, \quad ds_2 = \frac{d\eta}{h}, \quad ds_3 = dz,$$

and the element of volume becomes

$$\frac{1}{h^2} d\xi d\eta dz.$$

The formula (13) reduces to

$$\nabla^2 \Phi = h^2 \left(\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} \right) + \frac{\partial^2 \Phi}{\partial z^2} \qquad (98)$$

while equations (18) and (19) give us

$$\left.\begin{aligned}\lambda &= \frac{h}{\mathrm{h}} \frac{\partial \Phi}{\partial \xi}\\ \mu &= \frac{h}{\mathrm{h}} \frac{\partial \Phi}{\partial \eta}\\ \nu &= \frac{1}{\mathrm{h}} \frac{\partial \Phi}{\partial z}\end{aligned}\right\} \qquad (99)$$

and $$\mathrm{h}^2 = h^2 \left[\left(\frac{\partial \Phi}{\partial \xi} \right)^2 + \left(\frac{\partial \Phi}{\partial \eta} \right)^2 \right] + \left(\frac{\partial \Phi}{\partial z} \right)^2 \qquad (100)$$

Substituting from (97) in (16), we find for the curvatures

$$\left.\begin{aligned} {}_{\eta}\varpi_{\xi} &= \frac{\partial h}{\partial \eta}, & {}_{z}\varpi_{\xi} &= 0 \\ {}_{z}\varpi_{\eta} &= 0, & {}_{\xi}\varpi_{\eta} &= \frac{\partial h}{\partial \xi} \\ {}_{\xi}\varpi_{z} &= 0, & {}_{\eta}\varpi_{z} &= 0 \end{aligned}\right\} \qquad (101)$$

and by (30)

$$\left.\begin{aligned} u &= \alpha/h \\ v &= \beta/h \\ w &= \gamma \end{aligned}\right\} \qquad (102)$$

On this system the s_3 curves become straight lines (parallel to Oz), and γ is linear, and identical with w.

The strain components are now given by

$$\left.\begin{aligned} e &= h\frac{\partial u}{\partial \xi} - v\frac{\partial h}{\partial \eta} \\ f &= h\frac{\partial v}{\partial \eta} - u\frac{\partial h}{\partial \xi} \\ g &= \frac{\partial w}{\partial z} \\ \Delta &= h^2\left[\frac{\partial}{\partial \xi}\left(\frac{u}{h}\right) + \frac{\partial}{\partial \eta}\left(\frac{v}{h}\right)\right] + \frac{\partial w}{\partial z} \\ a &= h\frac{\partial w}{\partial \eta} + \frac{\partial v}{\partial z} \\ b &= \frac{\partial u}{\partial z} + h\frac{\partial w}{\partial \xi} \\ c &= \frac{\partial}{\partial \xi}(vh) + \frac{\partial}{\partial \eta}(uh) \end{aligned}\right\} \qquad (103)$$

and the component rotations by

$$\left.\begin{aligned} 2\Theta_1 &= h\frac{\partial w}{\partial \eta} - \frac{\partial v}{\partial z} \\ 2\Theta_2 &= \frac{\partial u}{\partial z} - h\frac{\partial w}{\partial \xi} \\ 2\Theta_3 &= h^2\left[\frac{\partial}{\partial \xi}\left(\frac{v}{h}\right) - \frac{\partial}{\partial \eta}\left(\frac{u}{h}\right)\right] \end{aligned}\right\} \qquad (104)$$

The equations of motion (41) become

$$\left.\begin{aligned}
h^2\frac{\partial}{\partial\xi}\left(\frac{P}{h}\right)+h^3\frac{\partial}{\partial\eta}\left(\frac{U}{h^2}\right)+\frac{\partial T}{\partial z}+Q\frac{\partial h}{\partial\xi}&=\rho(\ddot{u}-\Xi)\\
h^3\frac{\partial}{\partial\xi}\left(\frac{U}{h^2}\right)+h^2\frac{\partial}{\partial\eta}\left(\frac{Q}{h}\right)+\frac{\partial S}{\partial z}+P\frac{\partial h}{\partial\eta}&=\rho(\ddot{v}-\mathrm{H})\\
h^2\frac{\partial}{\partial\xi}\left(\frac{T}{h}\right)+h^2\frac{\partial}{\partial\eta}\left(\frac{S}{h}\right)+\frac{\partial R}{\partial z}&=\rho(\ddot{w}-\mathrm{Z})
\end{aligned}\right\}\quad\text{.............(105)}$$

where P, Q, R, S, T, U are given by (46) and (103).

By Lamé's transformation (48)

$$\left.\begin{aligned}
(m+n)h\frac{\partial\Delta}{\partial\xi}-2n\left[h\frac{\partial\Theta_3}{\partial\eta}-\frac{\partial\Theta_2}{\partial z}\right]&=\rho(\ddot{u}-\Xi)\\
(m+n)h\frac{\partial\Delta}{\partial\eta}-2n\left[\frac{\partial\Theta_1}{\partial z}-h\frac{\partial\Theta_3}{\partial\xi}\right]&=\rho(\ddot{v}-\mathrm{H})\\
(m+n)\frac{\partial\Delta}{\partial z}-2nh^2\left[\frac{\partial}{\partial\xi}\left(\frac{\Theta_2}{h}\right)-\frac{\partial}{\partial\eta}\left(\frac{\Theta_1}{h}\right)\right]&=\rho(\ddot{w}-\mathrm{Z})
\end{aligned}\right\}\quad\text{.......(106)}$$

If Φ be the surface of the body, the boundary conditions (45) become

$$\left.\begin{aligned}
h\left[P\frac{\partial\Phi}{\partial\xi}+U\frac{\partial\Phi}{\partial\eta}\right]+T\frac{\partial\Phi}{\partial z}&=\mathrm{h}\Xi'\\
h\left[U\frac{\partial\Phi}{\partial\xi}+Q\frac{\partial\Phi}{\partial\eta}\right]+S\frac{\partial\Phi}{\partial z}&=\mathrm{h}\mathrm{H}'\\
h\left[T\frac{\partial\Phi}{\partial\xi}+S\frac{\partial\Phi}{\partial\eta}\right]+R\frac{\partial\Phi}{\partial z}&=\mathrm{h}\mathrm{Z}'
\end{aligned}\right\}\quad\text{..................(107)}$$

where h is given by (100).

The conditions that the strain may be pure follow at once from (104), on making

$$\Theta_1=0,\ \ \Theta_2=0,\ \ \Theta_3=0;$$

and in this case, by (32) and (33), if ϕ be the displacement potential,

$$\left.\begin{aligned}
u&=\frac{a}{h}=h\frac{\partial\phi}{\partial\xi}\\
v&=\frac{\beta}{h}=h\frac{\partial\phi}{\partial\eta}\\
w&=\frac{\partial\phi}{\partial z}
\end{aligned}\right\}\quad\text{.......................(108)}$$

If ξ, η, z be the principal coördinates of the strain, we have by (103)

$$\left.\begin{aligned} h\frac{\partial w}{\partial \eta}+\frac{\partial v}{\partial z}=0 \\ \frac{\partial u}{\partial z}+h\frac{\partial w}{\partial \xi}=0 \\ \frac{\partial}{\partial \xi}(vh)+\frac{\partial}{\partial \eta}(uh)=0 \end{aligned}\right\} \qquad (109)$$

The e, f, g of equations (103) will then be equal respectively to ϵ_1, ϵ_2, ϵ_3; and N_1, N_2, N_3 will be given by (55).

Lamé's equations (57) will take the form

$$\left.\begin{aligned} h^2\frac{\partial}{\partial \xi}\left(\frac{N_1}{h}\right)+N_2\frac{\partial h}{\partial \xi}=\rho(\ddot{u}-\Xi) \\ h^2\frac{\partial}{\partial \eta}\left(\frac{N_2}{h}\right)+N_1\frac{\partial h}{\partial \eta}=\rho(\ddot{v}-\mathrm{H}) \\ \frac{\partial N_3}{\partial z}=\rho(\ddot{w}-\mathrm{Z}) \end{aligned}\right\} \qquad (110)$$

and the boundary conditions (58) reduce to

$$\left.\begin{aligned} N_1 h\frac{\partial \Phi}{\partial \xi}=\mathfrak{h}\Xi' \\ N_2 h\frac{\partial \Phi}{\partial \eta}=\mathfrak{h}\mathrm{H}' \\ N_3\frac{\partial \Phi}{\partial z}=\mathfrak{h}\mathrm{Z}' \end{aligned}\right\} \qquad (111)$$

246.] As an example of conjugate cylinders, let ξ and η be given by the equation

$$x+\iota y=C\cosh(\xi+\iota\eta).$$

Then it is easily shown that

$$\left.\begin{aligned} x=C\cosh\xi\,.\,\cos\eta \\ y=C\sinh\xi\,.\,\sin\eta \end{aligned}\right\}.$$

Thus

$$\left.\begin{aligned} \frac{x^2}{C^2\cosh^2\xi}+\frac{y^2}{C^2\sinh^2\xi}=1 \\ \frac{x^2}{C^2\cos^2\eta}-\frac{y^2}{C^2\sin^2\eta}=1 \end{aligned}\right\}.$$

The ξ cylinders have for their transverse sections a system of confocal ellipses, and the η cylinders a system of confocal hyperbolas: the common foci of the two systems being situated on the axis of x at equal distances C on either side of the origin. These confocal conics are represented in Figure 33, § 250.

From (12) we deduce

$$h = \frac{1}{C\sqrt{\cosh^2\xi - \cos^2\eta}} = \frac{1}{C\sqrt{\sinh^2\xi + \sin^2\eta}},$$

and from (101)

$$\left.\begin{aligned} {}_\eta\varpi_\xi &= -\frac{\sin\eta \,.\, \cos\eta}{C(\sinh^2\xi + \sin^2\eta)^{\frac{3}{2}}} \\ {}_\xi\varpi_\eta &= -\frac{\sinh\xi \,.\, \cosh\xi}{C(\sinh^2\xi + \sin^2\eta)^{\frac{3}{2}}} \end{aligned}\right\}.$$

The geometrical interpretation of these is as follows. If A, B; A', B' be the transverse and conjugate semi-axes of the ellipse and hyperbola intersecting at any point P, (ξ, η) or (x, y), the curvatures at P of the ellipse and hyperbola respectively are

$$\left.\begin{aligned} {}_\xi\varpi_\eta &= -\frac{AB}{(A^2 - A'^2)^{\frac{3}{2}}} = -\frac{AB}{(B^2 + B'^2)^{\frac{3}{2}}} \\ {}_\eta\varpi_\xi &= -\frac{A'B'}{(A^2 - A'^2)^{\frac{3}{2}}} = -\frac{A'B'}{(B^2 + B'^2)^{\frac{3}{2}}} \end{aligned}\right\}.$$

247.] **Surfaces of Revolution.** All the more important systems of orthogonal cylindrical surfaces (including conjugate cylinders) have one plane of symmetry through Oz, and many of them have two such planes, mutually perpendicular. It is clear that if the plane of zx be a plane of symmetry for the ξ and η cylinders, Ox will be an axis of symmetry of their normal sections by the plane of xy, which we may call the ξ and η curves (strictly the ξz and ηz curves).

If then we suppose the plane of xy to rotate about Ox, the two orthogonal systems of curves traced upon it will describe two orthogonal systems of surfaces of revolution, having Ox for their common axis. Adding to these the system of planes through the axis of revolution, we have a complete set of new orthogonal coordinate surfaces.

Let

$$\left.\begin{aligned} \xi_1 &= \chi_1(x, y) \\ \eta_1 &= \chi_2(x, y) \\ \zeta_1 &= z \end{aligned}\right\}.$$

be the original cylindrical system. Then the transformed system will obviously be

$$\left.\begin{aligned}\xi &= \chi_1(x,\ \sqrt{y^2+z^2})\\ \eta &= \chi_2(x,\ \sqrt{y^2+z^2})\\ \zeta &= \tan^{-1}(z/y)\end{aligned}\right\}.$$

Now the only quantities involved in the equations of this Chapter which depend in the least on x, y, z are h_1, h_2, h_3: and these are supposed to be expressed, before insertion in the equations, as explicit functions of ξ, η, ζ (§ 230).

But the symmetrical form of (4) or (12) shows that x, y, z may be interchanged in any way without in the least affecting the forms of h_1, h_2, h_3, when expressed as functions of ξ, η, ζ.

We may therefore take the axis of revolution for the axis of z in our new system, and take for Ox and Oy any axes whatever, perpendicular to Oz and to one another.

This amounts to transforming the cylindrical system

$$\left.\begin{aligned}\xi_1 &= \chi_1(x_1,\ y_1)\\ \eta_1 &= \chi_2(x_1,\ y_1)\\ \zeta_1 &= z_1\end{aligned}\right\}\quad\text{.........................(112)}$$

into the system

$$\left.\begin{aligned}\xi &= \chi_1(z,\ \sqrt{x^2+y^2})\\ \eta &= \chi_2(z,\ \sqrt{x^2+y^2})\\ \zeta &= \tan^{-1}(y/x)\end{aligned}\right\}\quad\text{.........................(113)}$$

Ox_1 being the axis of symmetry of the old system, and Oz the axis of revolution of the new system.

Similarly, if Oy_1 be an axis of symmetry, we may construct a second system of surfaces of revolution, defined by the functions

$$\left.\begin{aligned}\xi &= \chi_1(\sqrt{x^2+y^2},\ z)\\ \eta &= \chi_2(\sqrt{x^2+y^2},\ z)\\ \zeta &= \tan^{-1}(y/x)\end{aligned}\right\}\quad\text{.........................(114)}$$

Suppose that equations (112) can be solved so as to give x_1, y_1 explicitly in terms of ξ_1, η_1: let the solution be

$$\left.\begin{aligned}x_1 &= F_1(\xi_1,\ \eta_1)\\ y_1 &= F_2(\xi_1,\ \eta_1)\\ z_1 &= \zeta_1\end{aligned}\right\}\quad\text{.........................(112}a\text{)}$$

Then the solution of (113) will obviously be

$$\left.\begin{aligned} x &= \cos\zeta \,.\, F_2(\xi,\eta) \\ y &= \sin\zeta \,.\, F_2(\xi,\eta) \\ z &= F_1(\xi,\eta) \end{aligned}\right\} \qquad\qquad (113a)$$

and the solution of (114) will be

$$\left.\begin{aligned} x &= \cos\zeta \,.\, F_1(\xi,\eta) \\ y &= \sin\zeta \,.\, F_1(\xi,\eta) \\ z &= F_2(\xi,\eta) \end{aligned}\right\} \qquad\qquad (114a)$$

By formulæ (12) we have, for the original system (112),

$$\left.\begin{aligned} \frac{1}{h_1} &= \sqrt{\left(\frac{\partial F_1}{\partial \xi_1}\right)^2 + \left(\frac{\partial F_2}{\partial \xi_1}\right)^2} \\ \frac{1}{h_2} &= \sqrt{\left(\frac{\partial F_1}{\partial \eta_1}\right)^2 + \left(\frac{\partial F_2}{\partial \eta_1}\right)^2} \\ \frac{1}{h_3} &= 1 \end{aligned}\right\} \qquad\qquad (115)$$

for the first transformed system (113)

$$\left.\begin{aligned} \frac{1}{h_1} &= \sqrt{\left(\frac{\partial F_1}{\partial \xi}\right)^2 + \left(\frac{\partial F_2}{\partial \xi}\right)^2} \\ \frac{1}{h_2} &= \sqrt{\left(\frac{\partial F_1}{\partial \eta}\right)^2 + \left(\frac{\partial F_2}{\partial \eta}\right)^2} \\ \frac{1}{h_3} &= F_2(\xi,\eta) \end{aligned}\right\} \qquad\qquad (116)$$

and for the second transformed system (114)

$$\left.\begin{aligned} \frac{1}{h_1} &= \sqrt{\left(\frac{\partial F_1}{\partial \xi}\right)^2 + \left(\frac{\partial F_2}{\partial \xi}\right)^2} \\ \frac{1}{h_2} &= \sqrt{\left(\frac{\partial F_1}{\partial \eta}\right)^2 + \left(\frac{\partial F_2}{\partial \eta}\right)^2} \\ \frac{1}{h_3} &= F_1(\xi,\eta) \end{aligned}\right\} \qquad\qquad (117)$$

Thus neither transformation affects the forms of h_1 and h_2 as functions of ξ and η. But, on the other hand, they both make h_3 dependent on ξ and η. Considerations of symmetry alone are sufficient to shew that all three must be independent of ζ.

248.] As a simple example of the results of this Article, let us transform from the cylindrical polars of § 244 to the spherical polars of § 243.

The original system is

$$\left.\begin{aligned} \xi_1 &= \sqrt{x_1^2 + y_1^2} \\ \eta_1 &= \tan^{-1}(y_1/x_1) \\ \zeta_1 &= z_1 \end{aligned}\right\},$$

whence

$$\left.\begin{aligned} x_1 &= \xi_1 \,.\, \cos\eta_1 \\ y_1 &= \xi_1 \,.\, \sin\eta_1 \\ z_1 &= \zeta_1 \end{aligned}\right\},$$

and therefore

$$\frac{1}{h_1} = 1, \quad \frac{1}{h_2} = \xi_1, \quad \frac{1}{h_3} = 1.$$

This system is perfectly symmetrical about *any* axis perpendicular to Oz_1, and consequently the two transformed systems are identical. They are given by

$$\left.\begin{aligned} \xi &= \sqrt{x^2 + y^2 + z^2} \\ \eta &= \tan^{-1}[\sqrt{x^2 + y^2}/z] \\ \zeta &= \tan^{-1}(y/x) \end{aligned}\right\},$$

whence

$$\left.\begin{aligned} x &= \xi \,.\, \sin\eta \,.\, \cos\zeta \\ y &= \xi \,.\, \sin\eta \,.\, \sin\zeta \\ z &= \xi \,.\, \cos\eta \end{aligned}\right\},$$

and

$$\frac{1}{h_1} = 1, \quad \frac{1}{h_2} = \xi, \quad \frac{1}{h_3} = \xi \sin\eta.$$

Comparing these results with the general formulæ, it will be seen that they correspond in every respect.

249.] **Conjugate Surfaces of Revolution.** Let the original system of cylindrical surfaces be given, as in § 245, by

$$\xi_1 + \iota\eta_1 = F(x_1 + \iota y_1) \qquad\qquad (94)$$

Then the transformed systems of surfaces of revolution will be given by

$$\xi + \iota\eta = F(z + \iota\sqrt{x^2 + y^2}) \qquad\qquad (118)$$

and

$$\xi + \iota\eta = F(\sqrt{x^2 + y^2} + \iota z) \qquad\qquad (119)$$

respectively, according as the axis of revolution Oz coincides with Ox_1 or Oy_1.

If the solutions of these equations be given as before by (112a), (113a), (114a), we have by substitution in (95a)

$$\left.\begin{aligned}\frac{\partial F_1}{\partial\xi} &= \frac{\partial F_2}{\partial\eta}\\ \frac{\partial F_1}{\partial\eta} &= -\frac{\partial F_2}{\partial\xi}\end{aligned}\right\},$$

whence, by (116) and (117),

$$\frac{1}{h_1} = \frac{1}{h_2} = \sqrt{\left(\frac{\partial F_1}{\partial\xi}\right)^2 + \left(\frac{\partial F_1}{\partial\eta}\right)^2} = \sqrt{\left(\frac{\partial F_2}{\partial\xi}\right)^2 + \left(\frac{\partial F_2}{\partial\eta}\right)^2}.$$

Thus if we write

$$\left.\begin{aligned}h_1 = h_2 &= h\\ h_3 &= h'\end{aligned}\right\} \qquad\qquad (120)$$

we have in either of the transformed systems

$$\left.\begin{aligned}\frac{1}{h'} &= \sqrt{x^2+y^2}\\ \frac{1}{h} &= \frac{1}{h'^2}\sqrt{\left(\frac{\partial h'}{\partial\xi}\right)^2 + \left(\frac{\partial h'}{\partial\eta}\right)^2}\end{aligned}\right\} \qquad\qquad (121)$$

both h and h' being functions of ξ and η, but independent of ζ. Writing θ for ζ, so that θ has the same meaning as in § 244, we have now for the principal curvatures, instead of the values given in (101),

$$\left.\begin{aligned}{}_\eta\varpi_\xi &= \frac{\partial h}{\partial\eta}, & {}_\theta\varpi_\xi &= 0\\ {}_\theta\varpi_\eta &= 0, & {}_\xi\varpi_\eta &= \frac{\partial h}{\partial\xi}\\ {}_\xi\varpi_\theta &= \frac{h}{h'}\frac{\partial h'}{\partial\xi}, & {}_\eta\varpi_\theta &= \frac{h}{h'}\frac{\partial h'}{\partial\eta}\end{aligned}\right\} \qquad\qquad (122)$$

Thus if, as in § 232, ϖ_3 denote the absolute curvature of the s_3 or $\xi\eta$ curves, we have by (16a)

$$\begin{aligned}\varpi_3^2 = {}_\xi\varpi_\theta^2 + {}_\eta\varpi_\theta^2 &= \frac{h^2}{h'^2}\left[\left(\frac{\partial h'}{\partial\xi}\right)^2 + \left(\frac{\partial h'}{\partial\eta}\right)^2\right]\\ &= h'^2\\ &= \frac{1}{x^2+y^2}, \text{ by (121).}\end{aligned}$$

This is obvious geometrically, for these curves are circles in planes parallel to xy and having their centres in Oz, the axis of revolution.

The elementary arcs are

$$ds_1 = \frac{d\xi}{h},\ ds_2 = \frac{d\eta}{h},\ ds_3 = \frac{d\theta}{h'},$$

and the element of volume is

$$\frac{1}{h^2 h'} d\xi d\eta d\theta.$$

The formula (13) becomes

$$\nabla^2\Phi = h^2 h'\left[\frac{\partial}{\partial\xi}\left(\frac{1}{h'}\frac{\partial\Phi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{h'}\frac{\partial\Phi}{\partial\eta}\right)\right] + h'^2\frac{\partial^2\Phi}{\partial\theta^2} \qquad (123)$$

and equations (18) and (19) reduce to

$$\left.\begin{aligned} \lambda &= \frac{h}{\mathfrak{h}}\frac{\partial\Phi}{\partial\xi} \\ \mu &= \frac{h}{\mathfrak{h}}\frac{\partial\Phi}{\partial\eta} \\ \nu &= \frac{h'}{\mathfrak{h}}\frac{\partial\Phi}{\partial\theta} \end{aligned}\right\} \qquad (124)$$

and

$$\mathfrak{h}^2 = h^2\left[\left(\frac{\partial\Phi}{\partial\xi}\right)^2 + \left(\frac{\partial\Phi}{\partial\eta}\right)^2\right] + h'^2\left(\frac{\partial\Phi}{\partial\theta}\right)^2 \qquad (125)$$

also, by (30),

$$\left.\begin{aligned} u &= \alpha/h \\ v &= \beta/h \\ w &= \gamma/h' \end{aligned}\right\} \qquad (126)$$

The strain components are

$$\left.\begin{aligned} e &= h\frac{\partial u}{\partial\xi} - v\frac{\partial h}{\partial\eta} \\ f &= h\frac{\partial v}{\partial\eta} - u\frac{\partial h}{\partial\xi} \\ g &= h'\frac{\partial w}{\partial\theta} - u\frac{h}{h'}\frac{\partial h'}{\partial\xi} - v\frac{h}{h'}\frac{\partial h'}{\partial\eta} \\ \Delta &= h^2 h'\left[\frac{\partial}{\partial\xi}\left(\frac{u}{hh'}\right) + \frac{\partial}{\partial\eta}\left(\frac{v}{hh'}\right)\right] + h'\frac{\partial w}{\partial\theta} \\ a &= \frac{h}{h'}\frac{\partial}{\partial\eta}(wh') + h'\frac{\partial v}{\partial\theta} \\ b &= h'\frac{\partial u}{\partial\theta} + \frac{h}{h'}\frac{\partial}{\partial\xi}(wh') \\ c &= \frac{\partial}{\partial\xi}(vh) + \frac{\partial}{\partial\eta}(uh) \end{aligned}\right\} \qquad (127)$$

and the component rotations

$$\left.\begin{aligned} 2\Theta_1 &= h'\left[h\frac{\partial}{\partial\eta}\left(\frac{w}{h'}\right) - \frac{\partial v}{\partial\theta}\right] \\ 2\Theta_2 &= h'\left[\frac{\partial u}{\partial\theta} - h\frac{\partial}{\partial\xi}\left(\frac{w}{h'}\right)\right] \\ 2\Theta_3 &= h^2\left[\frac{\partial}{\partial\xi}\left(\frac{v}{h}\right) - \frac{\partial}{\partial\eta}\left(\frac{u}{h}\right)\right] \end{aligned}\right\} \quad \text{.....................(128)}$$

Lamé's equations (48) become

$$\left.\begin{aligned} (m+n)h\frac{\partial\Delta}{\partial\xi} - 2nh'\left[h\frac{\partial}{\partial\eta}\left(\frac{\Theta_3}{h'}\right) - \frac{\partial\Theta_2}{\partial\theta}\right] &= \rho(\ddot{u} - \Xi) \\ (m+n)h\frac{\partial\Delta}{\partial\eta} - 2nh'\left[\frac{\partial\Theta_1}{\partial\theta} - h\frac{\partial}{\partial\xi}\left(\frac{\Theta_3}{h'}\right)\right] &= \rho(\ddot{v} - \mathrm{H}) \\ (m+n)h'\frac{\partial\Delta}{\partial\theta} - 2nh^2\left[\frac{\partial}{\partial\xi}\left(\frac{\Theta_2}{h}\right) - \frac{\partial}{\partial\eta}\left(\frac{\Theta_1}{h}\right)\right] &= \rho(\ddot{w} - \mathrm{Z}) \end{aligned}\right\} \quad \text{......(129)}$$

The conditions that the strain may be pure are

$$\Theta_1 = 0,\ \Theta_2 = 0,\ \Theta_3 = 0\ ;$$

and the conditions that ξ, η, θ may be the principal coördinates of the strain are

$$a = 0,\ b = 0,\ c = 0.$$

In the latter case, equations (56) and (58) reduce to

$$\left.\begin{aligned} h^2h'\frac{\partial}{\partial\xi}\left(\frac{N_1}{hh'}\right) + N_2\frac{\partial h}{\partial\xi} + N_3\frac{h}{h'}\frac{\partial h'}{\partial\xi} &= \rho(\ddot{u} - \Xi) \\ h^2h'\frac{\partial}{\partial\eta}\left(\frac{N_2}{hh'}\right) + N_3\frac{h}{h'}\frac{\partial h'}{\partial\eta} + N_1\frac{\partial h}{\partial\eta} &= \rho(\ddot{v} - \mathrm{H}) \\ h'\frac{\partial N_3}{\partial\theta} &= \rho(\ddot{w} - \mathrm{Z}) \end{aligned}\right\} \quad \text{..............(130)}$$

and

$$\left.\begin{aligned} N_1 h\frac{\partial\Phi}{\partial\xi} &= h\Xi' \\ N_2 h\frac{\partial\Phi}{\partial\eta} &= h\mathrm{H}' \\ N_3 h'\frac{\partial\Phi}{\partial\theta} &= h\mathrm{Z}' \end{aligned}\right\} \quad \text{..............................(131)}$$

The student will find no difficulty in adapting equations (41) and (42): and, in the case of pure strain, (32) and (33).

It should be carefully borne in mind that ξ and η are conjugate functions of r and z, where $r=\sqrt{x^2+y^2}$, as in § 244, and that they only satisfy the equation corresponding *in form* to the equation

$$\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}=0$$

of § 245. That is, they satisfy

$$\left.\begin{aligned}\frac{\partial^2\xi}{\partial r^2}+\frac{\partial^2\xi}{\partial z^2}=0\\ \frac{\partial^2\eta}{\partial r^2}+\frac{\partial^2\eta}{\partial z^2}=0\end{aligned}\right\},$$

and consequently by (82)

$$\left.\begin{aligned}\nabla^2\xi=\frac{1}{r}\frac{\partial\xi}{\partial r}\\ \nabla^2\eta=\frac{1}{r}\frac{\partial\eta}{\partial r}\end{aligned}\right\}.$$

They are not therefore solutions of Laplace's equation, $\nabla^2\Phi=0$, as are the conjugate cylindrics of § 245.

250.] As an example of Conjugate Surfaces of revolution, let us transform the cylindrical system of § 246 by the method of § 247. We see from Figure 33 that this system is symmetrical about both Ox_1 and Oy_1. We therefore have the two transformed systems

$$z+\iota\sqrt{x^2+y^2}=C\cosh(\xi+\iota\eta)$$

and

$$\sqrt{x^2+y^2}+\iota z=C\cosh(\xi+\iota\eta).$$

(*i.*) The first system gives us

$$\left.\begin{aligned}\frac{x^2+y^2}{C^2\sinh^2\xi}+\frac{z^2}{C^2\cosh^2\xi}=1\\ \frac{z^2}{C^2\cos^2\eta}-\frac{x^2+y^2}{C^2\sin^2\eta}=1\end{aligned}\right\},$$

the ξ surfaces being confocal prolate spheroids, and the η surfaces confocal hyperboloids of revolution of two sheets. These surfaces will be described by the rotation of Figure 33 about the transverse axis.

We have

$$x^2+y^2=C^2\sinh^2\xi\,.\,\sin^2\eta,$$

and thus by (121)

$$\left.\begin{aligned}h'=\frac{1}{C\sinh\xi\sin\eta}\\ h=\frac{1}{C\sqrt{\sinh^2\xi+\sin^2\eta}}\end{aligned}\right\}.$$

(*ii.*) The second system gives us

$$\left.\begin{aligned}\frac{x^2+y^2}{C^2\cosh^2\xi}+\frac{z^2}{C^2\sinh^2\xi}=1\\ \frac{x^2+y^2}{C^2\cos^2\eta}-\frac{z^2}{C^2\sin^2\eta}=1\end{aligned}\right\}.$$

Here the ξ surfaces are confocal oblate spheroids, and the η surfaces confocal hyperboloids of revolution of one sheet. These

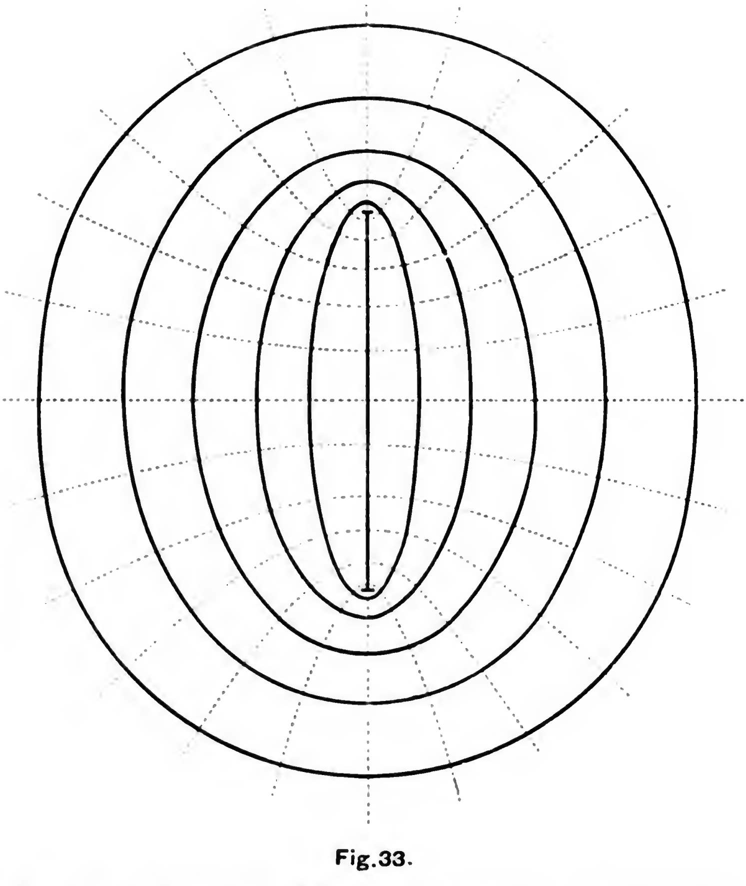


Fig.33.

surfaces will be described if Figure 33 is made to rotate about its conjugate axis.

We have in this case

$$x^2+y^2=C^2\cosh^2\xi\,.\,\cos^2\eta,$$

whence by (121)

$$\left.\begin{aligned} h' &= \frac{1}{C \cosh \xi \cos \eta} \\ h &= \frac{1}{C\sqrt{\sinh^2\xi + \sin^2\eta}} \end{aligned}\right\}.$$

251.] **Spheroidals.** The system of the last Article might be employed in dealing with bodies whose bounding surface is a spheroid of revolution: but, as a matter of fact, the formulæ may be much simplified by a further transformation.

Let the bounding surface of the body be

$$\frac{x^2+y^2}{A^2} + \frac{z^2}{B^2} = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots(132)$$

then any confocal quadric must be of the form

$$\frac{x^2+y^2}{A^2-p} + \frac{z^2}{B^2-p} \quad\ldots\ldots\ldots\ldots\ldots\ldots(133)$$

Let ξ and η be the lesser and greater roots of (133), considered as a quadratic in p.

(*i.*) When the bounding surface (132) is a *prolate* spheroid,

$$B^2 > \eta > A^2 > \xi > -\infty.$$

The ξ surfaces are the *prolate* spheroids

$$\frac{x^2+y^2}{A^2-\xi} + \frac{z^2}{B^2-\xi} = 1, \quad\ldots\ldots\ldots\ldots\ldots\ldots(134)$$

confocal with the bounding surface (132), which is represented by

$$\xi = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(135)$$

For positive values of ξ these spheroids lie within (135), and for negative values of ξ without it.

The η surfaces are the hyperboloids of two sheets

$$\frac{z^2}{B^2-\eta} - \frac{x^2+y^2}{\eta - A^2} = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots(136)$$

which are also confocal with (132) or (135).

Taking, as before,

$$\zeta = \theta = \tan^{-1}(y/x),$$

it is easily shewn that

$$\left.\begin{aligned} x^2 &= \cos^2\theta \,.\, \frac{(A^2-\xi)(\eta-A^2)}{B^2-A^2} \\ y^2 &= \sin^2\theta \,.\, \frac{(A^2-\xi)(\eta-A^2)}{B^2-A^2} \\ z^2 &= \frac{(B^2-\xi)(B^2-\eta)}{B^2-A^2} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(137)$$

$$\left.\begin{aligned} h_1 &= 2\sqrt{\frac{(A^2-\xi)(B^2-\xi)}{\eta-\xi}} \\ h_2 &= 2\sqrt{\frac{(\eta-A^2)(B^2-\eta)}{\eta-\xi}} \\ h_3 &= \sqrt{\frac{B^2-A^2}{(A^2-\xi)(\eta-A^2)}} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(138)$$

Distinguishing the prolate system of the last Article by the suffix 1, we have evidently

$$\left.\begin{aligned} C^2\sinh^2\xi_1 &= A^2-\xi \\ C^2\cosh^2\xi_1 &= B^2-\xi \\ C_2\cos^2\eta_1 &= B^2-\eta \\ C^2\sin^2\eta_1 &= \eta-A^2 \end{aligned}\right\},$$

$$\therefore \quad \left.\begin{aligned} &B^2-A^2=C^2 \\ &\xi = A^2\cosh^2\xi_1 - B^2\sinh^2\xi_1 \\ &\eta = A^2\cos^2\eta_1 + B^2\sin^2\eta_1 \end{aligned}\right\}.$$

(*ii.*) When the bounding surface is *oblate*,

$$A^2>\eta>B^2>\xi>-\infty.$$

The ξ surfaces are the confocal *oblate* spheroids (134), and the η surfaces are the confocal hyperboloids of one sheet

$$\frac{x^2+y^2}{A^2-\eta}-\frac{z^2}{\eta-B^2}=1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (139)$$

In this case

$$\left.\begin{aligned} x^2 &= \cos^2\theta\,.\,\frac{(A^2-\xi)(A^2-\eta)}{A^2-B^2} \\ y^2 &= \sin^2\theta\,.\,\frac{(A^2-\xi)(A^2-\eta)}{A^2-B^2} \\ z^2 &= \frac{(B^2-\xi)(\eta-B^2)}{A^2-B^2} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (140)$$

while

$$\left.\begin{aligned} h_1 &= 2\sqrt{\frac{(A^2-\xi)(B^2-\xi)}{\eta-\xi}} \\ h_2 &= 2\sqrt{\frac{(A^2-\eta)(\eta-B^2)}{\eta-\xi}} \\ h_3 &= \sqrt{\frac{A^2-B^2}{(A^2-\xi)(A^2-\eta)}} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (141)$$

If we distinguish the oblate system of § 250 by the suffix 2, we find for the relations connecting it with our present system,

$$\left.\begin{aligned} C^2\cosh^2\xi_2 &= A^2-\xi \\ C^2\sinh^2\xi_2 &= B^2-\xi \\ C^2\cos^2\eta_2 &= A^2-\eta \\ C^2\sin^2\eta_2 &= \eta-B^2 \end{aligned}\right\}.$$

$$\therefore \quad \left.\begin{aligned} A^2-B^2 &= C^2 \\ \xi &= B^2\cosh^2\xi_2 - A^2\sinh^2\xi_2 \\ \eta &= B^2\cos^2\eta_2 + A^2\sin^2\eta_2 \end{aligned}\right\}.$$

In both the systems of the present Article, the bounding surface is given by $\xi=0$; hence $\mathfrak{h}=h_1$, $\lambda=1$, $\mu=0$, $\nu=0$, at every point of the surface; and the boundary conditions reduce to

$$\left.\begin{aligned} P &= \Xi' \\ U &= \mathrm{H}' \\ T &= Z' \end{aligned}\right\} \qquad (142)$$

when $\xi=0$.

252.] **Ellipsoidals.** Similarly, in dealing with a body whose bounding surface is an ellipsoid

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}=1 \qquad (143)$$

it is convenient to take ξ, η, ζ as the roots of the cubic in p

$$\frac{x^2}{A^2-p}+\frac{y^2}{B^2-p}+\frac{z^2}{C^2-p}=1 \qquad (144)$$

Assuming that A, B, C, and also ζ, η, ξ are in descending order of magnitude, we may shew that

$$A^2>\zeta>B^2>\eta>C^2>\xi>-\infty .$$

The ξ surfaces are the confocal ellipsoids

$$\frac{x^2}{A^2-\xi}+\frac{y^2}{B^2-\xi}+\frac{z^2}{C^2-\xi}=1, \qquad (145)$$

the η surfaces the confocal hyperboloids of one sheet

$$\frac{x^2}{A^2-\eta}+\frac{y^2}{B^2-\eta}-\frac{z^2}{\eta-C^2}=1, \qquad (146)$$

and the ζ surfaces the confocal hyperboloids of two sheets

$$\frac{x^2}{A^2-\zeta}-\frac{y^2}{\zeta-B^2}-\frac{z^2}{\zeta-C^2}=1 \qquad (147)$$

Hence we easily deduce that

$$\left.\begin{aligned} x^2 &= \frac{(A^2-\xi)(A^2-\eta)(A^2-\zeta)}{(A^2-B^2)(A^2-C^2)} \\ y^2 &= \frac{(B^2-\xi)(B^2-\eta)(\zeta-B^2)}{(B^2-C^2)(A^2-B^2)} \\ z^2 &= \frac{(C^2-\xi)(\eta-C^2)(\zeta-C^2)}{(A^2-C^2)(B^2-C^2)} \end{aligned}\right\}\text{.....................}(148)$$

whence

$$\left.\begin{aligned} h_1 &= 2\sqrt{\frac{(A^2-\xi)(B^2-\xi)(C^2-\xi)}{(\eta-\xi)(\zeta-\xi)}} \\ h_2 &= 2\sqrt{\frac{(A^2-\eta)(B^2-\eta)(\eta-C^2)}{(\zeta-\eta)(\eta-\xi)}} \\ h_3 &= 2\sqrt{\frac{(A^2-\zeta)(\zeta-B^2)(\zeta-C^2)}{(\zeta-\xi)(\zeta-\eta)}} \end{aligned}\right\}\text{.................}(149)$$

As in the last Article, the boundary conditions are

$$\left.\begin{aligned} P &= \Xi' \\ U &= H' \\ T &= Z' \end{aligned}\right\}\text{..............................}(150)$$

when $\xi=0$.

EXAMPLES.

1. Show that if ξ_1, η_1; ξ_2, η_2;......be any number of pairs of conjugate functions of x and y, ξ and η will also be conjugate functions, where

$$\left.\begin{aligned} \xi &= A + \Sigma(p_1\xi_1) - \Sigma(q_1\eta_1) \\ \eta &= B + \Sigma(p_1\eta_1) + \Sigma(q_1\xi_1) \end{aligned}\right\}$$

A, B, p_1, p_2, ..., q_1, q_2, ... being any real constants whatever, positive or negative.

2. Show that if ξ and η are conjugate functions of x and y, then x and y (or y and $-x$) are conjugate functions of ξ and η.

3. Show that ξ_1 and η_1, any one of the pairs in Example 1, are conjugate functions of each of the other pairs, or of any pair compounded of them, like ξ and η.

4. Prove that ξ and η, as given by each of the following pairs of equations, are conjugate functions of x and y: and find the value of h for each pair. [r and θ denote the cylindrical polars of § 244.]

(i.) $\begin{cases} \xi = p\log(r/C), \\ \eta = p\,\theta; \end{cases}$

(ii.) $\begin{cases} \xi = (C_1 r^p + C_2 r^{-p}) \cos p\theta, \\ \eta = (C_1 r^p - C_2 r^{-p}) \sin p\theta; \end{cases}$

*(iii.) $\begin{cases} \xi = (C_1 \mathrm{e}^{px} + C_2 \mathrm{e}^{-px}) \cos py, \\ \eta = (C_1 \mathrm{e}^{px} - C_2 \mathrm{e}^{-px}) \sin py; \end{cases}$

(iv.) $\begin{cases} x = (C_1 \cos p\eta + C_2 \sin p\eta) \cosh p\xi, \\ y = (C_1 \sin p\eta - C_2 \cos p\eta) \sinh p\xi; \end{cases}$

(v.) $\begin{cases} \xi = p \log \dfrac{\sqrt{y^2 + (x \pm A)^2}}{C}, \\ \eta = p \tan^{-1} \dfrac{y}{x \pm A}; \end{cases}$

(vi.) $\begin{cases} (x + A \coth \xi)^2 + y^2 = A^2 \operatorname{cosech}^2 \xi, \\ x^2 + (y - A \cot \eta)^2 = A^2 \operatorname{cosec}^2 \eta. \end{cases}$

5. Transform the cylindrical surfaces of the last Example into surfaces of revolution by the method of § 249, and trace them geometrically.

7. If P be on one of the common generators of the conjugate cylinders ξ and η, and if PS_1, PS_2 be normals drawn in the directions in which ξ and η increase, show that their relative position is always such that to make PS_1 coincide with PS_2 we should have to turn it through a right angle in the *positive* direction of rotation about Oz.

7. If any system of orthogonal surfaces be *inverted* as to any centre of inversion, show that the new system thus obtained is also orthogonal.

8. Show that the pure strain defined by

$$\phi = F_1(r) + rF_2(\theta) + r\sin\theta F_3(\omega)$$

has the spherical polars r, θ, ω for its principal coördinates.

9. Show that the pure strain defined by

$$\phi = F_1(r) + rF_2(\theta) + F_3(z)$$

has the cylindrical polars r, θ, z for its principal coördinates.

10. Show that the pure strain defined by

$$\phi = F_1(\xi, \eta) + F_2(z)$$

* Here e denotes—as elsewhere in this work—the base of the Napierian logarithms.

will have the conjugate cylindrics ξ, η, z for its principal coördinates, if F_1 satisfies the differential equation

$$\frac{\partial}{\partial \xi}\left(h^2 \frac{\partial F_1}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(h^2 \frac{\partial F_1}{\partial \xi}\right) = 0.$$

11. Show that in the corresponding case for the conjugate surfaces of revolution of § 249

$$\phi = F_1(\xi, \eta) + \frac{1}{h'} \cdot F_2(\theta),$$

where F_1 satisfies the same differential equation as in the last Example.

12. Show that in the corresponding case for the spheroidals of § 251

$$\phi = F_1(\xi, \eta) + \sqrt{(A^2 - \xi)(A^2 - \eta)} \, . \, F_2(\theta),$$

or

$$\phi = F_1(\xi, \eta) + \sqrt{(A^2 - \xi)(\eta - A^2)} \, . \, F_2(\theta),$$

according as the ξ surfaces are oblate or prolate: F_1 being any root of the differential equation

$$\frac{\partial}{\partial \xi}\left(\frac{1}{\eta - \xi} \frac{\partial F_1}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{\eta - \xi} \frac{\partial F_1}{\partial \xi}\right) = 0.$$

3. Prove that the conditions of § 242 are satisfied by the foll wing forms of irrotational strain—

i.) For spherical polars

$$\phi = F(r).$$

ii.) For cylindrical polars

$$\phi = F_1(r) + F_2(z).$$

(ii.) For conjugate cylindrics

$$\phi = F_1(\xi, \eta) + F_2(z),$$

whe e

$$\left.\begin{aligned} \frac{\partial}{\partial \xi}\left(h^2 \frac{\partial F_1}{\partial \eta}\right) = 0 \\ \frac{\partial}{\partial \eta}\left(h^2 \frac{\partial F_1}{\partial \xi}\right) = 0 \end{aligned}\right\}.$$

(i .) For conjugate surfaces of revolution

$$\phi = F(\xi, \eta),$$

wher F satisfies the same conditions as F_1 in the last example.

14. If the ξ and η surfaces are conjugate surfaces of revolution, as in § 249, show that

$$(\nabla^2\xi)^2 + (\nabla^2\eta)^2 = (hh')^2.$$

15. Show from equations (16) that the η surfaces of § 251 (*ii.*) and of § 252, are of anticlastic curvature.

16. In the case of § 251 (*i.*), what locus is represented by

$$\xi = \eta = A^2\,?$$

17. In the case of § 251 (*ii.*), what locus is represented by

$$\xi = \eta = B^2\,?$$

18. In the case of § 252, what loci are represented by

$$\xi = \eta = C^2,$$

and

$$\eta = \zeta = B^3,$$

respectively ?

19. Deduce from equations (57) the conditions that a Line of Stress may transmit a constant *traction* or *pressure* in the direction of its length, under no Applied Forces.

20. Deduce from equations (56) the conditions that a Tube of Stress may transmit a constant *tension* or *thrust* in the direction of its length, under no Applied Forces.

21. Show from equations (57) that an isotropic medium may be held in equilibrium, under no Applied Forces, by the system of stresses

$$N_2 = N_3 = -N_1 = -\frac{K}{8\pi}\left(\frac{d\Psi}{ds_1}\right)^2,$$

where K is a constant, and Ψ a function of ξ only, satisfying Laplace's equation

$$\nabla^2\Psi = 0.$$

[According to the theory of Faraday and Clerk Maxwell, this represents the condition of a dielectric medium in the neighbourhood of charged conductors. K is the specific inductive capacity of the medium, and Ψ is the electrostatic potential, so that the ξ surfaces are the equipotentials.]

22. Assuming equations (47), (46), (22), (24), (25), (42), deduce (41) and (45) by the method of § 219.

23. Lamé obtains, in his *Coördonnées Curvilignes*, many groups of equations involving h_1, h_2, h_3 and the curvatures of the coördinate surfaces. The following examples may all be deduced from the formulæ of §§ 230-232; each is, of course, the type of a group of similar equations which can easily be deduced from it by the principle of symmetry.

(i.) $$\frac{\partial}{\partial\xi}\left(\frac{\lambda_2}{h_2}\right) - \frac{\partial}{\partial\eta}\left(\frac{\lambda_1}{h_1}\right) = 0.$$

(ii.) $$h_1^2\frac{\partial}{\partial\xi}(\lambda_2 h_2) + h_2^2\frac{\partial}{\partial\eta}(\lambda_1 h_1) = 0.$$

(iii.) $$\begin{cases} \dfrac{\partial\lambda_1}{\partial s_1} = \lambda_2 \cdot {}_\eta\varpi_\xi + \lambda_3 \cdot {}_\zeta\varpi_\xi \\ \dfrac{\partial\lambda_1}{\partial s_2} = -\lambda_2 \cdot {}_\xi\varpi_\eta \\ \dfrac{\partial\lambda_1}{\partial s_3} = -\lambda_3 \cdot {}_\xi\varpi_\zeta \end{cases}$$

(iv.) $$\begin{cases} \lambda_1\dfrac{\partial}{\partial\eta}(\lambda_1 h_1) + \mu_1\dfrac{\partial}{\partial\eta}(\mu_1 h_1) + \nu_1\dfrac{\partial}{\partial\eta}(\nu_1 h_1) = \dfrac{h_1}{h_2} \cdot {}_\eta\varpi_\xi \\ \lambda_2\dfrac{\partial}{\partial\eta}(\lambda_1 h_1) + \mu_2\dfrac{\partial}{\partial\eta}(\mu_1 h_1) + \nu_2\dfrac{\partial}{\partial\eta}(\nu_1 h_1) = -\dfrac{h_1}{h_2} \cdot {}_\xi\varpi_\eta \\ \lambda_3\dfrac{\partial}{\partial\eta}(\lambda_1 h_1) + \mu_3\dfrac{\partial}{\partial\eta}(\mu_1 h_1) + \nu_3\dfrac{\partial}{\partial\eta}(\nu_1 h_1) = 0. \end{cases}$$

(v.) $$\frac{h_2 h_3}{h_1}\frac{\partial^2 h_1}{\partial\eta\partial\zeta} = {}_\eta\varpi_\xi({}_\zeta\varpi_\xi - {}_\zeta\varpi_\eta) + {}_\zeta\varpi_\xi({}_\eta\varpi_\xi - {}_\eta\varpi_\zeta).$$

(vi.) $$\begin{cases} h_2\dfrac{\partial}{\partial s_3}\left(\dfrac{{}_\xi\varpi_\eta}{h_2}\right) + {}_\xi\varpi_\zeta \cdot {}_\zeta\varpi_\eta = 0 \\ h_1\dfrac{\partial}{\partial s_3}\left(\dfrac{{}_\eta\varpi_\xi}{h_1}\right) + {}_\eta\varpi_\zeta \cdot {}_\zeta\varpi_\xi = 0. \end{cases}$$

(vii.) $$\begin{cases} \dfrac{\partial}{\partial s_3}({}_\xi\varpi_\eta) = {}_\zeta\varpi_\eta({}_\xi\varpi_\eta - {}_\xi\varpi_\zeta) \\ \dfrac{\partial}{\partial s_3}({}_\eta\varpi_\xi) = {}_\zeta\varpi_\xi({}_\eta\varpi_\xi - {}_\eta\varpi_\zeta). \end{cases}$$

CHAPTER VI.

GENERAL SOLUTIONS AND EXAMPLES.

The General Problem:—Preliminary Theorems.

253.] **Recapitulation of the General Problem.** Let a homogeneous body of natural density ρ be subjected to a small strain; and let u, v, w be the component displacements, parallel to rectangular axes fixed in space, of that point of the body which in the natural state occupies the position (x, y, z). Then, if e, f, g be the component elongations of the element described about that point, a, b, c the component shears, Δ the cubical dilatation, and θ_1, θ_2, θ_3 the component rotations, we have by §123

$$\left.\begin{aligned}
e=\frac{\partial u}{\partial x},\ f=\frac{\partial v}{\partial y},\ g=\frac{\partial w}{\partial z}\\
a=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z},\ b=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x},\ c=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\\
\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\\
\theta_1=\tfrac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\ \theta_2=\tfrac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right),\ \theta_3=\tfrac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)
\end{aligned}\right\}\ldots\ldots\ldots.(1)$$

Also, if P, Q, R; S, T, U be the normal and tangential components of the stress at the point, we have by §212 for an isotropic body

$$\left.\begin{aligned}
P&=(m+n)e+(m-n)(f+g)\\
Q&=(m+n)f+(m-n)(g+e)\\
R&=(m+n)g+(m-n)(e+f)\\
S&=na\\
T&=nb\\
U&=nc
\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

and by § 214

$$\left.\begin{aligned} e &= P/q - \sigma(Q+R)/q \\ f &= Q/q - \sigma(R+P)/q \\ g &= R/q - \sigma(P+Q)/q \\ a &= S/n \\ b &= T/n \\ c &= U/n \end{aligned}\right\} \qquad (3)$$

the various constants employed—of which two are independent—being connected (§§ 212, 213) by the relations

$$\frac{3kn}{q} = \frac{n}{1-2\sigma} = m = k + \tfrac{1}{3}n \qquad (4)$$

The potential energy V per unit of unstrained volume (§ 212) is given by

$$2V = (m-n)\Delta^2 + 2n(e^2+f^2+g^2) + n(a^2+b^2+c^2) \qquad (5)$$

and this expression can be thrown into various other forms by means of (1), (2), (3), and (4).

If X, Y, Z be the components of the Applied Force per unit mass on the element described about the point (x, y, z), the equations of motion are by § 143

$$\left.\begin{aligned} \frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z} &= \rho(\ddot{u} - X) \\ \frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial S}{\partial z} &= \rho(\ddot{v} - Y) \\ \frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial R}{\partial z} &= \rho(\ddot{w} - Z) \end{aligned}\right\} \qquad (6)$$

which, by virtue of (2), may for an isotropic body be written in either of the forms of § 218,

$$\left.\begin{aligned} m\frac{\partial \Delta}{\partial x} + n\nabla^2 u &= \rho(\ddot{u} - X) \\ m\frac{\partial \Delta}{\partial y} + n\nabla^2 v &= \rho(\ddot{v} - Y) \\ m\frac{\partial \Delta}{\partial z} + n\nabla^2 w &= \rho(\ddot{w} - Z) \end{aligned}\right\} \qquad (7)$$

or

$$\left.\begin{aligned} (m+n)\frac{\partial \Delta}{\partial x} - 2n\left(\frac{\partial \theta_3}{\partial y} - \frac{\partial \theta_2}{\partial z}\right) &= \rho(\ddot{u} - X) \\ (m+n)\frac{\partial \Delta}{\partial y} - 2n\left(\frac{\partial \theta_1}{\partial z} - \frac{\partial \theta_3}{\partial x}\right) &= \rho(\ddot{v} - Y) \\ (m+n)\frac{\partial \Delta}{\partial z} - 2n\left(\frac{\partial \theta_2}{\partial x} - \frac{\partial \theta_1}{\partial y}\right) &= \rho(\ddot{w} - Z) \end{aligned}\right\} \qquad (8)$$

The equations of equilibrium are at once derived from these by making

$$\ddot{u} = \ddot{v} = \ddot{w} = 0;$$

while, if the body be free from Applied Force, we have only to make

$$X = Y = Z = 0.$$

If F, G, H be the components of the Surface Traction per unit area on the element of the bounding surface of the body described about the point (x, y, z), and if λ, μ, ν be the direction-cosines of the outward normal to the surface at that point, the conditions to be satisfied at every point of the surface are by §§ 144, 145,

$$\left.\begin{aligned} \lambda P + \mu U + \nu T &= F \\ \lambda U + \mu Q + \nu S &= G \\ \lambda T + \mu S + \nu R &= H \end{aligned}\right\} \qquad (9)$$

which can be expressed in terms of the strain components or of the displacements, for an isotropic body, by means of (1) and (2).

If the component displacements of any point (x, y, z) on the bounding surface be u_0, v_0, w_0, then u, v, w must be such functions of (x, y, z) as will satisfy the equations

$$\left.\begin{aligned} u &= u_0 \\ v &= v_0 \\ w &= w_0 \end{aligned}\right\} \qquad (10)$$

at every point of the surface.

The General Problem for an isotropic body is to determine values of u, v, w as functions of x, y, z which will satisfy (6), (7), and (8)—or the corresponding equations for the case of equilibrium—under a given system of applied forces, at every point in the interior of the body, and which will at the same time satisfy the boundary conditions (9) or (10)—according as the Surface Tractions or surface displacements are given—at every point of the bounding surface.

Of this problem no *universal* solution can be obtained—that is to say, no solution for u, v, w as functions of X, Y, Z, F, G, H, without reference to the forms of these quantities themselves as functions of x, y, z—but several very general solutions have been worked out, each applicable to a large class of cases.

Before proceeding to the consideration of these solutions, we shall state and prove five general theorems concerning small strains which will very much simplify our task. The only general principle which these theorems involve is that of the superposition of small strains and stresses, which has already been sufficiently established (§§ 87, 88, 153-155), and which finds

its mathematical expression in the perfectly *linear form* of all the fundamental differential equations of our theory. For the sake of the greater simplicity of the formulæ involved, they are proved for an isotropic body, but the method of proof is perfectly general, and they are equally true of *all perfectly elastic solids, under strains which are sufficiently small to admit of the application of the principle of superposition.*

Preliminary Theorems.

254.] **THEOREM I.** *No distribution of Strain, of whatever kind, throughout a perfectly elastic body, can be maintained unaltered except by the continued exertion either of Applied Forces, or of Surface Tractions, or of the two systems combined. Consequently, any distribution of Displacement which can be so maintained must be such as can be suffered by a perfectly rigid body: that is, compounded of a translation and a rotation of the body as a whole.*

It will be observed that this Theorem is simply a statement of one of the fundamental properties of Perfectly Elastic Solids § 18 (*iii*)] on which we have based our analytical theory (Chapter IV). The following proof is therefore merely a return to first principles, and is, so far as it goes, a test of the accuracy of our deductions.

If it be possible, let the body be maintained in the state of strain $\{e, f, g, a, b, c, \theta_1, \theta_2, \theta_3\}$ in the absence of any Applied Forces or Surface Tractions.

By § 194, the total potential energy W of the strained body is equal to the work done by the system of Applied Forces and Surface Tractions which maintain the strain, in bringing it from its natural state to its given condition. And since the work done by a zero system of forces must necessarily be zero, it follows that in the supposed state of strain we must have

$$W = 0.$$

But, if V be the potential energy per unit volume,

$$W = \iiint V dxdydz;$$

and the form (5) of V shews that it is an essential positive quantity. Hence the integral W is the sum of a number of essentially positive quantities, and this sum cannot possibly vanish unless each of its terms vanishes separately. Thus for every element of the body we must have .

$$V = 0.$$

But by (5) V is itself the sum of a number of essentially positive

quantities, and if V is zero we must have, everywhere throughout the body,

$$e = f = g = a = b = c = 0:$$

that is to say, the component elongations and shears vanish at every point, and the supposed strain, if it exists at all, must consist simply of *varying* rotations (§ 48).

Now, by (1) we have

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0 \\ &\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \end{aligned}\right\} \quad \text{.....................(11)}$$

and therefore

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial z \partial x} = 0 \\ \frac{\partial^2 v}{\partial x \partial y} &= \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial z} = 0 \\ \frac{\partial^2 w}{\partial z \partial x} &= \frac{\partial^2 w}{\partial y \partial z} = \frac{\partial^2 w}{\partial z^2} = 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= -\frac{\partial^2 v}{\partial x \partial y} = 0, & \frac{\partial^2 u}{\partial z^2} &= -\frac{\partial^2 w}{\partial z \partial x} = 0 \\ \frac{\partial^2 v}{\partial z^2} &= -\frac{\partial^2 w}{\partial y \partial z} = 0, & \frac{\partial^2 v}{\partial x^2} &= -\frac{\partial^2 u}{\partial x \partial y} = 0 \\ \frac{\partial^2 w}{\partial x^2} &= -\frac{\partial^2 u}{\partial z \partial x} = 0, & \frac{\partial^2 w}{\partial y^2} &= -\frac{\partial^2 v}{\partial y \partial z} = 0 \end{aligned}\right\}$$

Thus all the second derivatives of u, v, w vanish, except

$$\frac{\partial^2 u}{\partial y \partial z}, \frac{\partial^2 v}{\partial z \partial x}, \frac{\partial^2 w}{\partial x \partial y};$$

and u, v, w must be of the form

$$\left.\begin{aligned} u &= A_1 + B_1 y + C_1 z + D_1 yz \\ v &= A_2 + B_2 z + C_2 x + D_2 zx \\ w &= A_3 + B_3 x + C_3 y + D_3 xy \end{aligned}\right\} \quad \text{.......................(12)}$$

where the coefficients are absolute constants.

Substituting from (12) in (11) we get the three relations

$$\left.\begin{aligned} B_2 + C_3 + (D_2 + D_3)x = 0 \\ B_3 + C_1 + (D_3 + D_1)y = 0 \\ B_1 + C_2 + (D_1 + D_2)z = 0 \end{aligned}\right\},$$

which, since B, C, D are constants, are really equivalent to the six

$$\begin{aligned} B_2 + C_3 = B_3 + C_1 = B_1 + C_2 = D_2 + D_3 \\ = D_3 + D_1 = D_1 + D_2 = 0. \end{aligned}$$

Thus
$$D_1 = D_2 = D_3 = 0,$$

and
$$\left.\begin{aligned} u &= A_1 - C_2 y + C_1 z \\ v &= A_2 - C_3 z + C_2 x \\ w &= A_3 - C_1 x + C_3 y \end{aligned}\right\} \qquad (13)$$

The only distribution of displacement which can be maintained without Applied Forces or Surface Tractions is therefore compounded of the *bodily* translation

$$\left.\begin{aligned} u &= A_1 \\ v &= A_2 \\ w &= A_3 \end{aligned}\right\},$$

and the *bodily* rotation (§§ 85, 86)

$$\left.\begin{aligned} \theta_1 &= C_3 \\ \theta_2 &= C_1 \\ \theta_3 &= C_2 \end{aligned}\right\},$$

which is simply such as can be suffered by any perfectly rigid body (§ 48), and does not constitute a strain at all.

255.] **THEOREM II.** *The distribution of Strain throughout a perfectly elastic solid in equilibrium under any given system of Applied Forces and Surface Tractions is perfectly determinate. Consequently, the distribution of Displacement under the same conditions is also determinate, with the exception of an arbitrary displacement of the kind that can be suffered by a perfectly rigid body.*

For if X, Y, Z be the components of the given system of Applied Forces, and F, G, H of the given system of Surface Tractions: and if $\{e, f, g, a, b, c\}$ be a distribution of strain consistent with the given conditions, it must satisfy the equations

$$\left.\begin{aligned} (m+n)\frac{\partial e}{\partial x} + (m-n)\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) + n\left(\frac{\partial c}{\partial y} + \frac{\partial b}{\partial z}\right) + \rho X = 0 \\ (m+n)\frac{\partial f}{\partial y} + (m-n)\left(\frac{\partial g}{\partial y} + \frac{\partial e}{\partial y}\right) + n\left(\frac{\partial a}{\partial z} + \frac{\partial c}{\partial x}\right) + \rho Y = 0 \\ (m+n)\frac{\partial g}{\partial z} + (m-n)\left(\frac{\partial e}{\partial z} + \frac{\partial f}{\partial z}\right) + n\left(\frac{\partial b}{\partial x} + \frac{\partial a}{\partial y}\right) + \rho Z = 0 \end{aligned}\right\} \quad \ldots\ldots (14)$$

throughout the body, and

$$\left.\begin{aligned} \lambda[(m+n)e + (m-n)(f+g)] + \mu nc + \nu nb &= F \\ \lambda nc + \mu[(m+n)f + (m-n)(g+e)] + \nu na &= G \\ \lambda nb + \mu na + \nu[(m+n)g + (m-n)(e+f)] &= H \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (14$$

at the bounding surface. Similarly, if $\{e', f', g', a', b', c'\}$ by any other distribution of strain consistent with the given conditions, we must have

$$\left.\begin{aligned}(m+n)\frac{\partial e'}{\partial x}+(m-n)\left(\frac{\partial f'}{\partial x}+\frac{\partial g'}{\partial x}\right)+n\left(\frac{\partial c'}{\partial y}+\frac{\partial b'}{\partial z}\right)+\rho X=0\\(m+n)\frac{\partial f'}{\partial y}+(m-n)\left(\frac{\partial g'}{\partial y}+\frac{\partial e'}{\partial y}\right)+n\left(\frac{\partial a'}{\partial z}+\frac{\partial c'}{\partial x}\right)+\rho Y=0\\(m+n)\frac{\partial g'}{\partial z}+(m-n)\left(\frac{\partial e'}{\partial z}+\frac{\partial'}{\partial z}\right)+n\left(\frac{\partial b'}{\partial x}+\frac{\partial a'}{\partial y}\right)+\rho Z=0\end{aligned}\right\}$$

and
$$\left.\begin{aligned}\lambda[(m+n)e'+(m-n)(f'+g')]+\mu nc'+\nu nb'=F\\\lambda nc'+\mu[(m+n)f'+(m-n)(g'+e')]+\nu na'=G\\\lambda nb'+\mu na'+\nu[(m+n)g'+(m-n)(e'+f')]=H\end{aligned}\right\}.$$

Let $e'=e+e''$, $f'=f+f''$, $g'=g+g''$, $a'=a+a''$, $b'=b+b''$, $c'=c+c''$. Then by subtraction of the two systems of linear equations we find that

$$\left.\begin{aligned}(m+n)\frac{\partial e''}{\partial x}+(m-n)\left(\frac{\partial f''}{\partial x}+\frac{\partial g''}{\partial x}\right)+n\left(\frac{\partial c''}{\partial y}+\frac{\partial b''}{\partial z}\right)=0\\(m+n)\frac{\partial f''}{\partial y}+(m-n)\left(\frac{\partial g''}{\partial y}+\frac{\partial e''}{\partial y}\right)+n\left(\frac{\partial a''}{\partial z}+\frac{\partial c''}{\partial x}\right)=0\\(m+n)\frac{\partial g''}{\partial z}+(m-n)\left(\frac{\partial e''}{\partial z}+\frac{\partial f''}{\partial z}\right)+n\left(\frac{\partial b''}{\partial x}+\frac{\partial a''}{\partial y}\right)=0\end{aligned}\right\}\dots\dots\dots(16)$$

throughout the body, and that

$$\left.\begin{aligned}\lambda[(m+n)e''+(m-n)(f''+g'')]+\mu nc''+\nu nb''=0\\\lambda nc''+\mu[(m+n)f''+(m-n)(g''+e'')]+\nu na''=0\\\lambda nb''+\mu na''+\nu[(m+n)g''+(m-n)(e''+f'')]=0\end{aligned}\right\}\dots\dots\dots(17).$$

at the boundary surface.

Comparing (16) and (17) with the standard forms of the equations [(47) and (49) of § 217], we see at once that $\{e'', f'', g'', a'', b'', c''\}$ is the specification of a strain such as could be maintained unaltered without Applied Forces or Surface Tractions.

Thus, by Theorem I,

$$e''=f''=g''=a''=b''=c''=0;$$

and consequently

$$e'=e,\ f'=f,\ g'=g,\ a'=a,\ b'=b,\ c'=c.$$

Thus *only one* distribution of Strain can satisfy the given conditions, and the solution is completely determinate as to the strain.

Consequently, the distribution of displacement is also determinate, *in so far as it constitutes a strain:* that is to say, with

the sole exception of an arbitrary translation and rotation of the body as a whole. Of course, as we expressly excluded such displacements from consideration (§§ 48-50), we cannot expect our equations to give us any information on the subject.

It should be observed that when the surface displacements (10) are given, the distribution of displacement is *absolutely* determinate.

256.] **THEOREM III.** *The most general distribution of motion possible in a perfectly elastic body, free from Applied Forces and Surface Tractions, consists of a series of superposed small harmonic vibrations of the points of the body about their natural positions; translations and rotations of the body as a whole being excluded.*

The equations of motion (7) become, when the Applied Forces are zero,

$$\left.\begin{aligned} m\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+n\nabla^2 u&=\rho\frac{\partial^2 u}{\partial t^2}\\ m\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+n\nabla^2 v&=\rho\frac{\partial^2 v}{\partial t^2}\\ m\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+n\nabla^2 w&=\rho\frac{\partial^2 w}{\partial t^2}\end{aligned}\right\}\quad\text{..................(18)}$$

These equations are linear, and consequently the most general solutions for u, v, w, as functions of x, y, z, and t, may be considered as built up by adding together simpler values of u, v, w, which will simultaneously satisfy (18). But any function of x, y, z, t may be expanded in a series of terms, each of which is of one of the three following forms:—

(*i.*) Function of t only.
(*ii.*) Function of (x, y, z) only.
(*iii.*) Product of a function of t into a function of (x, y, z).

Now any solution which gives u, v, w, or any of them in the form (*i.*)—that is, independent of x, y, z—represents a translation of the body as a whole. Also any solution which gives u, v, w in form (*ii.*)—that is, independent of t—must also be a solution of equations (16) and (17), and therefore represents a superposed translation and rotation of the body as a whole (Theorem I).

Both these solutions have been expressly excluded, and we must, therefore, assume that the most general solution for *straining motion* is of the form

$$\left.\begin{aligned} u&=u_1\tau_1+u_2\tau_2+\ldots\ldots u_i\tau_i+\ldots\ldots\\ v&=v_1\tau_1'+v_2\tau_2'+\ldots\ldots v_i\tau_i'+\ldots\ldots\\ w&=w_1\tau_1''+w_2\tau_2''+\ldots\ldots w_i\tau_i''+\ldots\ldots\end{aligned}\right\}\quad\text{..................(19)}$$

where u_i, v_i, w_i are functions of x, y, z only, and τ_i, τ_i', τ_i'' of t only; and

$$\left.\begin{aligned} u &= u_i\tau_i \\ v &= v_i\tau_i' \\ w &= w_i\tau_i'' \end{aligned}\right\}$$

are simultaneous simple solutions of equations (18).

Substituting in these equations, we get

$$m\left\{\tau_i\frac{\partial^2 u_i}{\partial x^2} + \tau_i'\frac{\partial^2 v_i}{\partial x\partial y} + \tau_i''\frac{\partial^2 w_i}{\partial z\partial x}\right\} + n\tau_i\nabla^2 u_i = \rho u_i\frac{d^2\tau_i}{dt^2},$$

and two similar equations.

Now it is obviously impossible, in general, that these equations can be satisfied by values of u_i, v_i, w_i which are independent of t, unless all functions of t can be cleared from the left-hand side. The necessary and sufficient conditions for this are

$$\tau_i = \tau_i' = \tau_i'',$$

for all values of i.

Making this assumption, the equations may now be written

$$\left.\begin{aligned} \frac{m}{u_i}\cdot\frac{\partial}{\partial x}\left\{\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}+\frac{\partial w_i}{\partial z}\right\} + \frac{n}{u_i}\nabla^2 u_i &= \frac{\rho}{\tau_i}\cdot\frac{d^2\tau_i}{dt^2} \\ \frac{m}{v_i}\cdot\frac{\partial}{\partial y}\left\{\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}+\frac{\partial w_i}{\partial z}\right\} + \frac{n}{v_i}\nabla^2 v_i &= \frac{\rho}{\tau_i}\cdot\frac{d^2\tau_i}{dt^2} \\ \frac{m}{w_i}\cdot\frac{\partial}{\partial z}\left\{\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}+\frac{\partial w_i}{\partial z}\right\} + \frac{n}{w_i}\nabla^2 w_i &= \frac{\rho}{\tau_i}\cdot\frac{d^2\tau_i}{dt^2} \end{aligned}\right\}.$$

Here we have three expressions which are known to be independent of t equated to an expression which is known to be independent of x, y, z. In order that this may be possible, each of the four expressions must be equal to an absolute constant.

Let this constant be denoted by $\mathfrak{i}$; then we shall have

$$\frac{d^2\tau_i}{dt^2} + \mathfrak{i}\tau_i = 0 \qquad (20)$$

and

$$\left.\begin{aligned} m\frac{\partial}{\partial x}\left\{\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}+\frac{\partial w_i}{\partial z}\right\} + n\nabla^2 u_i + \rho\mathfrak{i}u_i &= 0 \\ m\frac{\partial}{\partial y}\left\{\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}+\frac{\partial w_i}{\partial z}\right\} + n\nabla^2 v_i + \rho\mathfrak{i}v_i &= 0 \\ m\frac{\partial}{\partial z}\left\{\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}+\frac{\partial w_i}{\partial z}\right\} + n\nabla^2 w_i + \rho\mathfrak{i}w_i &= 0 \end{aligned}\right\} \qquad (21)$$

and the general solution will be of the form

$$\left.\begin{aligned} u &= \Sigma(u_i \tau_i) \\ v &= \Sigma(v_i \tau_i) \\ w &= \Sigma(w_i \tau_i) \end{aligned}\right\} \quad \text{.................................(22)}$$

The solution of (20) depends upon the form of $\mathfrak{i}$. If $\mathfrak{i}$ be real and positive

$$\tau_i = A \sin \sqrt{\mathfrak{i}}\,.\,t + B \cos \sqrt{\mathfrak{i}}\,.\,t\,;$$

if $\mathfrak{i}$ be real and negative

$$\tau_i = A e^{\sqrt{-\mathfrak{i}}\,.\,t} + B e^{-\sqrt{-\mathfrak{i}}\,.\,t},$$

where e denotes the base of Napier's logarithms: and lastly, if $\mathfrak{i}$ be partly real and partly imaginary, the solution is of a mixed form.

We now proceed to shew that *every possible value of* $\mathfrak{i}$ *which permits of solutions of* (21) *consistent with the boundary conditions*

$$F = 0,\ G = 0,\ H = 0,$$

is both real and positive.

Let

$$e_i\tau_i,\ f_i\tau_i,\ \ldots\ldots c_i\tau_i,\ \Delta_i\tau_i$$

be the components of the strain corresponding to the partial solution

$$\left.\begin{aligned} u &= u_i\tau_i \\ v &= v_i\tau_i \\ w &= w_i\tau_i \end{aligned}\right\} \quad \text{.................................(23)}$$

and let

$$P_i\tau_i,\ Q_i\tau_i,\ \ldots\ldots U_i\tau_i$$

be the components of the corresponding stress.

Then by (1)

$$\left.\begin{aligned} e_i &= \frac{\partial u_i}{\partial x}, \ldots\ldots\ldots\ldots \\ a_i &= \frac{\partial w_i}{\partial y} + \frac{\partial v_i}{\partial z}, \ldots\ldots \end{aligned}\right\},$$

and by (2)

$$\left.\begin{aligned} P_i &= m\Delta_i + 2ne_i, \ldots\ldots \\ S_i &= na_i\ \ldots\ldots\ldots\ldots\ldots \end{aligned}\right\}.$$

Thus (21) may be written

$$\left.\begin{aligned}\frac{\partial P_i}{\partial x}+\frac{\partial U_i}{\partial y}+\frac{\partial T_i}{\partial z}+\mathfrak{i}\rho u_i=0\\ \frac{\partial U_i}{\partial x}+\frac{\partial Q_i}{\partial y}+\frac{\partial S_i}{\partial z}+\mathfrak{i}\rho v_i=0\\ \frac{\partial T_i}{\partial x}+\frac{\partial S_i}{\partial y}+\frac{\partial R_i}{\partial z}+\mathfrak{i}\rho w_i=0\end{aligned}\right\}\quad\text{(24)}$$

while the conditions to be satisfied at the bounding surface are

$$\left.\begin{aligned}\lambda P_i+\mu U_i+\nu T_i=0\\ \lambda U_i+\mu Q_i+\nu S_i=0\\ \lambda T_i+\mu S_i+\nu R_i=0\end{aligned}\right\}\quad\text{(25)}$$

Let any other particular solution of equations (18) be

$$\left.\begin{aligned}u=u_j\tau_j\\ v=v_j\tau_j\\ w=w_j\tau_j\end{aligned}\right\}\quad\text{(26)}$$

and let a similar notation be adopted with regard to j and the functions depending upon it.

Consider the integral

$$\mathrm{I}_{ij}=\iiint[u_iu_j+v_iv_j+w_iw_j]dxdydz,$$

taken throughout the entire volume of the body. Substituting for u_i, v_i, w_i from (24)

$$\begin{aligned}-\mathfrak{i}\rho\mathrm{I}_{ij}=\iiint\Big\{&u_j\left[\frac{\partial P_i}{\partial x}+\frac{\partial U_i}{\partial y}+\frac{\partial T_i}{\partial z}\right]\\ &+v_j\left[\frac{\partial U_i}{\partial x}+\frac{\partial Q_i}{\partial y}+\frac{\partial S_i}{\partial z}\right]\\ &+w_j\left[\frac{\partial T_i}{\partial x}+\frac{\partial S_i}{\partial y}+\frac{\partial R_i}{\partial z}\right]\Big\}dxdydz.\end{aligned}$$

Integrating by parts, as in §§ 146, 194, 219, we have

$$\begin{aligned}-\mathfrak{i}\rho\mathrm{I}_{ij}=\iint\{&u_j[\lambda P_i+\mu U_i+\nu T_i]\\ &+v_j[\lambda U_i+\mu Q_i+\nu S_i]\\ &+w_j[\lambda T_i+\mu S_i+\nu R_i]\}dS\\ -\iiint\Big\{&P_i\frac{\partial u_j}{\partial x}+Q_i\frac{\partial v_j}{\partial y}+R_i\frac{\partial w_j}{\partial z}\\ &+S_i\left(\frac{\partial w_j}{\partial y}+\frac{\partial v_j}{\partial z}\right)+T_i\left(\frac{\partial u_j}{\partial z}+\frac{\partial w_j}{\partial x}\right)\\ &+U_i\left(\frac{\partial v_j}{\partial x}+\frac{\partial u_j}{\partial y}\right)\Big\}dxdydz.\end{aligned}$$

By (25) the surface integral vanishes, and thus

$$i\rho \mathbf{I}_{ij} = \iiint \{e_j P_i + f_j Q_i + g_j R_i + a_j S_i + b_j T_i + c_j U_i\} dxdydz.$$

Now, looking back to the original form of $\mathbf{I}_{ij}$, it is obvious that it is symmetrical with regard to i and j. That is

$$\mathbf{I}_{ij} = \mathbf{I}_{ji},$$

and, interchanging i and j and i and j in the formula just obtained,

$$j\rho \mathbf{I}_{ji} = \iiint \{e_i P_j + f_i Q_j + g_i R_j + a_i S_j + b_i T_j + c_i U_j\} dxdydz.$$

But the stress components are linear functions of the strain components, and by direct substitution in these two integrals we obtain

$$\begin{aligned} i\rho \mathbf{I}_{ij} = j\rho \mathbf{I}_{ji} = \iiint \{ &(m+n)(e_i e_j + f_i f_j + g_i g_j) \\ &+ (m-n)(f_i g_j + f_j g_i + g_i e_j + g_j e_i + e_i f_j + e_j f_i) \\ &+ (a_i a_j + b_i b_j + c_i c_j)\} dxdydz. \end{aligned}$$

Thus

$$(\mathrm{i} - \mathrm{j})\mathbf{I}_{ij} = 0,$$

and if i and j be *any two different* quantities which admit of solutions of (21) or (24) consistent with the boundary conditions (25), the integral

$$\mathbf{I} \equiv \iiint (u_i u_j + v_i v_j + w_i w_j) dxdydz$$

must vanish identically.

Now we are only concerned with real displacements, and therefore we have only to deal with real values of u, v, w. Thus, by a well-known principle, if in the series (22) there occurs any imaginary value of i, of the form

$$\alpha + \beta\sqrt{-1},$$

there must also occur the conjugate form

$$\alpha - \beta\sqrt{-1}.$$

Let us then assume

$$\left.\begin{aligned} \mathrm{i} &= \alpha + \beta\sqrt{-1} \\ \mathrm{j} &= \alpha - \beta\sqrt{-1} \end{aligned}\right\}.$$

so that the corresponding displacements are of the forms

$$\left.\begin{aligned} u_i &= \mathrm{u} + \beta\mathrm{u}'\sqrt{-1} \\ v_i &= \mathrm{v} + \beta\mathrm{v}'\sqrt{-1} \\ w_i &= \mathrm{w} + \beta\mathrm{w}'\sqrt{-1} \end{aligned}\right\} \qquad \left.\begin{aligned} u_j &= \mathrm{u} - \beta\mathrm{u}'\sqrt{-1} \\ v_j &= \mathrm{v} - \beta\mathrm{v}'\sqrt{-1} \\ w_j &= \mathrm{w} - \beta\mathrm{w}'\sqrt{-1} \end{aligned}\right\}$$

where u, v, w, u′, v′, w′ are all real.

Thus

$$\mathbf{I}_{ij} = \iiint \{\mathrm{u}^2 + \mathrm{v}^2 + \mathrm{w}^2 + \beta^2(\mathrm{u}'^2 + \mathrm{v}'^2 + \mathrm{w}'^2)\} dxdydz,$$

which is the sum of six essentially positive quantities; and since $\mathbf{I}_{ij}$ is identically zero, each of these six quantities must vanish separately. Thus at every point of the body

$$\left.\begin{array}{r} \mathrm{u} = \mathrm{v} = \mathrm{w} = 0 \\ \beta\mathrm{u}' = \beta\mathrm{v}' = \beta\mathrm{w}' = 0 \end{array}\right\},$$

and the solution is therefore null.

Thus it is conclusively shown that any value of $\mathfrak{i}$ which admits of solutions of (21) or (24), consistent with the boundary conditions (25), must be *real*.

Again, consider the second integral

$$\mathbf{I}_i = \iiint (u_i^2 + v_i^2 + w_i^2) dxdydz.$$

Since $\mathfrak{i}$ is necessarily real, $\mathbf{I}_i$ is necessarily positive: but by (24)

$$\begin{aligned} -\mathfrak{i}\rho\mathbf{I}_i = \iiint \Big\{ & u_i\left(\frac{\partial P_i}{\partial x} + \frac{\partial U_i}{\partial y} + \frac{\partial T_i}{\partial z}\right) \\ & + v_i\left(\frac{\partial U_i}{\partial x} + \frac{\partial Q_i}{\partial y} + \frac{\partial S_i}{\partial z}\right) \\ & + w_i\left(\frac{\partial T_i}{\partial x} + \frac{\partial S_i}{\partial y} + \frac{\partial R_i}{\partial z}\right) \Big\} dxdydz. \end{aligned}$$

Integrating by parts, as before,

$$\begin{aligned} \mathfrak{i}\rho\mathbf{I}_i &= \iiint \{e_iP_i + f_iQ_i + g_iR_i + a_iS_i + b_iT_i + c_iU_i\} dxdydz \\ &= 2\iiint V_i dxdydz \\ &= 2W_i \end{aligned}$$

by (19) and (20) of § 199; $V_{i\mathcal{T}i}$ being the potential energy per unit volume, and $W_{i\mathcal{T}i}$ the total potential energy of the body, due to the partial solution (23) above. Or, which amounts to the same thing, W_i is the potential energy due to the strain

$$\{e_i, f_i, g_i, a_i, b_i, c_i\}.$$

Thus $\mathfrak{i}\mathbf{I}_i$ is essentially positive, as well as $\mathbf{I}_i$; and consequently $\mathfrak{i}$ is also essentially positive.

Finally then we see that *every value of* $\mathfrak{i}$ *which admits of a solution of* (21) *consistent with the boundary conditions* (25) *is essentially real and positive.*

Thus we may obviously write

$$\mathfrak{i} = i^2;$$

and we shall then have, for the most general equations of straining motion, under no Applied Forces or Surface Tractions,

$$\left.\begin{aligned} u &= \Sigma(u_i \sin it + u_i' \cos it) \\ v &= \Sigma(v_i \sin it + v_i' \cos it) \\ w &= \Sigma(w_i \sin it + w_i' \cos it) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

where u_i, v_i, w_i and u_i', v_i', w_i' are any two sets of values of u, v, w which satisfy

$$\left.\begin{aligned} m\frac{\partial}{\partial x}\left(\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}+\frac{\partial w_i}{\partial z}\right)+n\nabla^2 u_i+i^2\rho u_i &= 0 \\ m\frac{\partial}{\partial y}\left(\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}+\frac{\partial w_i}{\partial z}\right)+n\nabla^2 v_i+i^2\rho v_i &= 0 \\ m\frac{\partial}{\partial z}\left(\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}+\frac{\partial w_i}{\partial z}\right)+n\nabla^2 w_i+i^2\rho w_i &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(28)$$

throughout the body, and the conditions (25) over the bounding surface.

The most general form of small straining motion, under no Applied Forces or Surface Tractions, is therefore to be obtained by superposing all such possible systems of small simple harmonic vibrations of points in the body about their natural positions.

Equations (27), (28), and (25) thus present to us the analytical statement of the *Problem of Free Vibrations*.

In general there are an infinite number of partial solutions of (28) for each value of i. In any particular problem, the boundary conditions, combined with considerations of symmetry, restrictions on the mode of propagation, etc., will enable us to select appropriate solutions.

257.] **THEOREM IV.** *The most general distribution of motion possible in a perfectly elastic body, under any system whatever of Applied Forces and Surface Tractions capable of maintaining equilibrium, consists of a series of superposed small harmonic vibrations—governed by the same laws as in the preceding case, and consequently dependent only on the properties of the body itself—of points in the body about mean positions which are identical with those to which the same points are displaced, when the body is in equilibrium under the same system of Applied Forces and Surface Tractions.*

For if u, v, w be the component displacements at time t of the point which in the natural state occupies the position (x, y, z), we have

$$m\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+n\nabla^2 u+\rho(X-\ddot{u})=0,$$

etc.,

throughout the body, and

$$\lambda\left[(m+n)\frac{\partial u}{\partial x}+(m-n)\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]+\mu n\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\nu n\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=F,$$

etc.,

at every point of its surface.

But if u', v', w' be the displacements which the same point would experience, if the body were *in equilibrium* under the same system of Applied Forces and Surface Tractions, then

$$m\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)+n\nabla^2 u'+\rho X=0,$$

etc.,

throughout the body, and

$$\lambda\left[(m+n)\frac{\partial u'}{\partial x}+(m-n)\left(\frac{\partial v'}{\partial x}+\frac{\partial w'}{\partial y}\right)\right]+\mu n\left(\frac{\partial v'}{\partial x}+\frac{\partial u'}{\partial y}\right)$$
$$+\nu n\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right)=F,$$

etc.,

over the surface.

Thus, if we assume

$$u=u'+u'',\quad v=v'+v'',\quad w=w'+w'',$$

the displacements u'', v'', w'' satisfy

$$\left.\begin{aligned}
m\frac{\partial}{\partial x}\left(\frac{\partial u''}{\partial x}+\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}\right)+n\nabla^2 u''&=\rho\ddot{u}''\\
m\frac{\partial}{\partial y}\left(\frac{\partial u''}{\partial x}+\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}\right)+n\nabla^2 v''&=\rho\ddot{v}''\\
m\frac{\partial}{\partial z}\left(\frac{\partial u''}{\partial x}+\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}\right)+n\nabla^2 w''&=\rho\ddot{w}''
\end{aligned}\right\}$$

throughout the body, and

$$\lambda\left[(m+n)\frac{\partial u''}{\partial x}+(m-n)\left(\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}\right)\right]+\mu n\left(\frac{\partial v''}{\partial x}+\frac{\partial u''}{\partial y}\right)$$
$$+\nu n\left(\frac{\partial u''}{\partial z}+\frac{\partial w''}{\partial x}\right)=0,$$

etc.,

over the bounding surface.

The distribution of motion represented by u'', v'', w'' is therefore such as might take place if the body were in motion under no Applied Forces or Surface Tractions, and by the last Theorem we know that this consists of a series of small superposed harmonic vibrations about the positions given by

$$u''=v''=w''=0,$$

or by

$$\left.\begin{aligned} u &= u' \\ v &= v' \\ w &= w' \end{aligned}\right\};$$

and this system of vibrations is of course absolutely independent of the external forces.

The assumption in the enunciation of this Theorem that the Applied Forces and Surface Tractions are such as are *capable of maintaining a state of equilibrium*, expressly excludes all such systems as *vary with the time.* These latter affect the mode of vibration, but not the mean configuration: the most important case is that in which all the external forces are harmonic functions of the time, of the same period throughout. The problem then becomes that of *Forced Vibrations*, to be considered in the next Theorem.

258.] **THEOREM V.** *A system of Applied Forces and Surface Tractions which varies as any simple harmonic function of the time, gives rise to a distribution of Forced Vibrations, of the same period as themselves, about the natural configuration; and the most general form of straining motion possible under such a system consists of this (perfectly definite) mode of forced vibration, with any (perfectly arbitrary) modes of free vibration superposed on it.*

For if the Applied Forces and Surface Tractions be given by

$$\left.\begin{aligned} X &= \mathbf{X}\sin it \\ Y &= \mathbf{Y}\sin it \\ Z &= \mathbf{Z}\sin it \end{aligned}\right\} \qquad \left.\begin{aligned} F &= \mathbf{F}\sin it \\ G &= \mathbf{G}\sin it \\ H &= \mathbf{H}\sin it \end{aligned}\right\},$$

the general equations of motion will be

$$\left.\begin{aligned} m\frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right\} + n\nabla^2 u - \rho\ddot{u} + \rho\mathbf{X}\sin it = 0 \\ m\frac{\partial}{\partial y}\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right\} + n\nabla^2 v - \rho\ddot{v} + \rho\mathbf{Y}\sin it = 0 \\ m\frac{\partial}{\partial z}\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right\} + n\nabla^2 w - \rho\ddot{w} + \rho\mathbf{Z}\sin it = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots(29)$$

and the boundary conditions

$$\left.\begin{aligned} \lambda\left[(m+n)\frac{\partial u}{\partial x} + (m-n)\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right] + \mu n\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \\ + \nu n\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) = \mathbf{F}\sin it, \\ \text{etc.,} \end{aligned}\right\} \quad \ldots\ldots(30)$$

In the first place it is obvious that if (u, v, w) represent *any distribution whatever* of displacement, satisfying the general equations (18) of free vibration, and the corresponding boundary conditions, viz., $F=G=H=0$, the distribution of displacement $(u+\mathrm{u}, v+\mathrm{v}, w+\mathrm{w})$ formed by superposing this system on the particular solution of (29) and (30) which depend upon X, Y, Z, F, G, H, will also satisfy (29) and (30). That is to say, the most general solution of (29) and (30) consists of the particular solution, with any arbitrary modes of free vibration superposed upon it.

From the form of (29) and (30) it is obvious that the particular solution must be of the form

$$\left.\begin{aligned} u &= \mathbf{u}\sin it \\ v &= \mathbf{v}\sin it \\ w &= \mathbf{w}\sin it \end{aligned}\right\} \qquad (31)$$

where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are determined by the general equations

$$\left.\begin{aligned} m\frac{\partial}{\partial x}\left\{\frac{\partial \mathbf{u}}{\partial x}+\frac{\partial \mathbf{v}}{\partial y}+\frac{\partial \mathbf{w}}{\partial z}\right\}+n\nabla^2\mathbf{u}+\rho(i^2\mathbf{u}+\mathbf{X})=0 \\ m\frac{\partial}{\partial y}\left\{\frac{\partial \mathbf{u}}{\partial x}+\frac{\partial \mathbf{v}}{\partial y}+\frac{\partial \mathbf{w}}{\partial z}\right\}+n\nabla^2\mathbf{v}+\rho(i^2\mathbf{v}+\mathbf{Y})=0 \\ m\frac{\partial}{\partial z}\left\{\frac{\partial \mathbf{u}}{\partial x}+\frac{\partial \mathbf{v}}{\partial y}+\frac{\partial \mathbf{w}}{\partial z}\right\}+n\nabla^2\mathbf{w}+\rho(i^2\mathbf{w}+\mathbf{Z})=0 \end{aligned}\right\} \qquad (29a)$$

and the boundary conditions

$$\left.\begin{aligned} \lambda\left[(m+n)\frac{\partial \mathbf{u}}{\partial x}+(m-n)\left(\frac{\partial \mathbf{v}}{\partial y}+\frac{\partial \mathbf{w}}{\partial z}\right)\right]+\mu n\left(\frac{\partial \mathbf{v}}{\partial x}+\frac{\partial \mathbf{u}}{\partial y}\right) \\ +\nu n\left(\frac{\partial \mathbf{u}}{\partial z}+\frac{\partial \mathbf{w}}{\partial x}\right)=\mathbf{F}, \\ \text{etc.,} \end{aligned}\right\} \qquad (30a)$$

It therefore consists of a system of simple harmonic vibrations about the natural configuration, having the same period as the external forces, while the *mode* of vibration—or distribution of amplitudes as functions of x, y, z—depends on the form of these forces.

259.] **Subdivision of the General Problem.** Availing ourselves of these Theorems, we may now greatly simplify the General Problem by subdividing it into the five following:—

(*i.*) The problem of **Free Vibrations,** under *no* Applied Forces or Surface Tractions.

(*ii.*) The problem of **Forced Vibrations** under any given *periodic* system of Surface Tractions only.

(*iii.*) The problem of **Forced Vibrations** under any given *periodic* system of Applied Forces and Surface Tractions.

(*iv.*) The problem of **Equilibrium** under no Applied Forces, with a given distribution of *equilibrating* Surface Tractions, or of Surface Displacements.

(*v.*) The problem of **Equilibrium** under any given *equilibrating* systems of Applied Forces and Surface Tractions.

The Problem of Free Vibrations.

260.] **General Statement of the Problem.** The general equations to be satisfied throughout the body are

$$\left.\begin{aligned} m\frac{\partial\Delta}{\partial x} + n\nabla^2 u - \rho\frac{\partial^2 u}{\partial t^2} &= 0\\ m\frac{\partial\Delta}{\partial y} + n\nabla^2 v - \rho\frac{\partial^2 v}{\partial t^2} &= 0\\ m\frac{\partial\Delta}{\partial z} + n\nabla^2 w - \rho\frac{\partial^2 w}{\partial t^2} &= 0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(32)$$

and the boundary conditions are

$$\left.\begin{aligned} \lambda P + \mu U + \nu T &= 0\\ \lambda U + \mu Q + \nu S &= 0\\ \lambda T + \mu S + \nu R &= 0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(33)$$

By § 256, the general solution is of the form

$$\left.\begin{aligned} u &= \Sigma(u_i \sin it + u_i' \cos it)\\ v &= \Sigma(v_i \sin it + v_i' \cos it)\\ w &= \Sigma(w_i \sin it + w_i' \cos it)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(34)$$

the strain components being given by

$$\left.\begin{aligned} e &= \Sigma(e_i \sin it + e_i' \cos it)\\ &\text{etc.,}\\ a &= \Sigma(a_i \sin it + a_i' \cos it)\\ &\text{etc.,}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(35)$$

where

$$\left.\begin{aligned} e_i &= \frac{\partial u_i}{\partial x}, \text{ etc.,}\\ a_i &= \frac{\partial w_i}{\partial y} + \frac{\partial v_i}{\partial z}, \text{ etc.}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(36)$$

and the stress components by

$$\left.\begin{aligned} P &= \Sigma(P_i \sin it + P_i' \cos it)\\ &\text{etc.}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(37)$$

where

$$\left.\begin{aligned} P_i &= (m+n)e_i + (m-n)(f_i + g_i) \\ &\text{etc.,} \\ S_i &= na_i \\ &\text{etc.} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(38)$$

Similarly, if the motion be irrotational, the displacement potential ϕ must be of the form

$$\phi = \Sigma(\phi_i \sin it + \phi_i' \cos it) \quad\ldots\ldots\ldots\ldots(39)$$

where

$$\left.\begin{aligned} u_i &= \frac{\partial\phi_i}{\partial x} \\ v_i &= \frac{\partial\phi_i}{\partial y} \\ w_i &= \frac{\partial\phi_i}{\partial z} \end{aligned}\right\} \qquad \left.\begin{aligned} u_i' &= \frac{\partial\phi_i'}{\partial x} \\ v_i' &= \frac{\partial\phi_i'}{\partial y} \\ w_i' &= \frac{\partial\phi_i'}{\partial z} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(40)$$

Selecting the partial solution of order i from (34) and substituting in (32) and (33), we see that each of the systems of displacement (u_i, v_i, w_i) and (u'_i, v'_i, w'_i) satisfies the general equations

$$\left.\begin{aligned} m\frac{\partial\Delta_i}{\partial x} + n\nabla^2 u_i + \rho i^2 u_i &= 0 \\ m\frac{\partial\Delta_i}{\partial y} + n\nabla^2 v_i + \rho i^2 v_i &= 0 \\ m\frac{\partial\Delta_i}{\partial z} + n\nabla^2 w_i + \rho i^2 w_i &= 0 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(41)$$

and the boundary conditions

$$\left.\begin{aligned} \lambda P_i + \mu U_i + \nu T_i &= 0 \\ \lambda U_i + \mu Q_i + \nu S_i &= 0 \\ \lambda T_i + \mu S_i + \nu R_i &= 0 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(42)$$

261.] **How does i enter into the solutions for u_i, v_i, w_i?** Writing in (41) and (42)

$$ix = \mathrm{x}_i, \quad iy = \mathrm{y}_i, \quad iz = \mathrm{z}_i,$$

they become

$$m\frac{\partial}{\partial \mathrm{x}_i}\left(\frac{\partial u_i}{\partial \mathrm{x}_i} + \frac{\partial v_i}{\partial \mathrm{y}_i} + \frac{\partial w_i}{\partial \mathrm{z}_i}\right) + n\left(\frac{\partial^2}{\partial \mathrm{x}_i^2} + \frac{\partial^2}{\partial \mathrm{y}_i^2} + \frac{\partial^2}{\partial \mathrm{z}_i^2}\right)u_i + \rho u_i = 0, \text{ etc.}$$

$$\lambda\left[(m+n)\frac{\partial u_i}{\partial \mathrm{x}_i} + (m-n)\left(\frac{\partial v_i}{\partial \mathrm{y}_i} + \frac{\partial w_i}{\partial \mathrm{z}_i}\right)\right] + \mu n\left(\frac{\partial v_i}{\partial \mathrm{x}_i} + \frac{\partial u_i}{\partial \mathrm{y}_i}\right) + \nu n\left(\frac{\partial u_i}{\partial \mathrm{z}_i} + \frac{\partial w_i}{\partial \mathrm{x}_i}\right) = 0$$

etc.

and it is obvious that these equations retain precisely the same form when any other suffix is substituted for i.

Thus we must have

$$\left.\begin{aligned} u_i &= \mathrm{F}_1(ix,\ iy,\ iz) \\ v_i &= \mathrm{F}_2(ix,\ iy,\ iz) \\ w_i &= \mathrm{F}_3(ix,\ iy,\ iz) \end{aligned}\right\}$$

where the forms of the functions F_1, F_2, F_3 are independent of i.

262.] **Distribution of Kinetic and Potential Energy among the partial components.** The potential energy due to any distribution of strain may [§ 199 (20)] be put into the form

$$W = \tfrac{1}{2}\iiint [Pe + Qf + Rg + Sa + Tb + Uc]\,dxdydz.$$

Substituting from (1) for $e, \ldots c$, and integrating by parts,

$$W = \tfrac{1}{2}\iint [u(\lambda P + \mu U + \nu T) + v(\lambda U + \mu Q + \nu S) + w(\lambda T + \mu S + \nu R)]\,dS.$$

$$-\tfrac{1}{2}\iiint\left[u\left(\frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z}\right) + v\left(\frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial S}{\partial z}\right)\right.$$
$$\left. + w\left(\frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial R}{\partial z}\right)\right]dxdydz.$$

Thus in a state of free vibration [putting $X = Y = Z = 0$ in (6), and $F = G = H = 0$ in (9)] we have

$$W = -\frac{\rho}{2}\iiint [u\ddot{u} + v\ddot{v} + w\ddot{w}]\,dxdydz \ldots\ldots\ldots\ldots\ldots\ldots(43)$$

Also, if $\mathfrak{T}$ be the kinetic energy,

$$\mathfrak{T} = \frac{\rho}{2}\iiint [\dot{u}^2 + \dot{v}^2 + \dot{w}^2]\,dxdydz \ldots\ldots\ldots\ldots\ldots\ldots(44)$$

Thus, if W_i and $\mathfrak{T}_i$ be the potential and kinetic energies due to the partial component of order i,

$$\begin{aligned} W_i = \frac{i^2\rho}{2}\Big\{&\sin^2 it\iiint [u_i^2 + v_i^2 + w_i^2]\,dxdydz \\ &+ \cos^2 it\iiint [u_i'^2 + v_i'^2 + w_i'^2]\,dxdydz \\ &+ 2\sin it\cos it\iiint [u_iu_i' + v_iv_i' + w_iw_i']\,dxdydz\Big\} \ldots\ldots\ldots(45) \end{aligned}$$

$$\begin{aligned} \mathfrak{T}_i = \frac{i^2\rho}{2}\Big\{&\cos^2 it\iiint [u_i^2 + v_i^2 + w_i^2]\,dxdydz \\ &+ \sin^2 it\iiint [u_i'^2 + v_i'^2 + w_i'^2]\,dxdydz \\ &- 2\sin it\cos it\iiint [u_iu_i' + v_iv_i' + w_iw_i']\,dxdydz\Big\} \ldots\ldots\ldots(46) \end{aligned}$$

and the whole of the energy due to the partial component is given by

$$\mathfrak{E}_i = W_i + \mathfrak{T}_i = \frac{i^2\rho}{2}\iiint [u_i^2 + v_i^2 + w_i^2 + u_i'^2 + v_i'^2 + w_i'^2]dxdydz \dots\dots(47)$$

which is independent of the time, as of course it ought to be—no work being done on the body.

Now, substituting from (34) in (43), we obtain for the resultant potential energy

$$\begin{aligned} W = \frac{\rho}{2}\iiint \{&\Sigma i^2(u_i \sin it + u_i' \cos it) \,.\, \Sigma(u_j \sin jt + u_j' \cos jt) \\ &+ \Sigma i^2(v_i \sin it + v_i' \cos it) \,.\, \Sigma(v_j \sin jt + v_j' \cos jt) \\ &+ \Sigma i^2(w_i \sin it + w_i' \cos it) \,.\, \Sigma(w_j \sin jt + w_j' \cos jt)\}dxdydz\,; \end{aligned}$$

where both i and j are to receive in succession all the values included in the series (34).

This again may be written

$$\begin{aligned} W = \frac{\rho}{2}\{&\Sigma i^2 \sin^2 it \iiint [u_i^2 + v_i^2 + w_i^2]dxdydz \\ &+ \Sigma i^2 \cos^2 it \iiint [u_i'^2 + v_i'^2 + w_i'^2]dxdydz \\ &+ 2\Sigma i^2 \sin it \cos it \iiint [u_i u_i' + v_i v_i' + w_i w_i']dxdydz \\ &+ \Sigma\Sigma i^2 \sin it \sin jt \iiint [u_i u_j + v_i v_j + w_i w_j]dxdydz \\ &+ \Sigma\Sigma i^2 \cos it \cos jt \iiint [u_i' u_j' + v_i' v_j' + w_i' w_j']dxdydz \\ &+ \Sigma\Sigma i^2 \sin it \cos jt \iiint [u_i u_j' + v_i v_j' + w_i w_j']dxdydz\}\,; \end{aligned}$$

where the single summations are to be taken for all the values of i included in the series (34), and the double summations for all *different* values of i and j.

But it is obvious that each term included in either of the double summations is of the same form as the integral I_{ij} of § 256, and is therefore identically zero.

Thus we finally have

$$\begin{aligned} W = \frac{\rho}{2}\Sigma i^2\{&\sin^2 it \iiint [u_i^2 + v_i^2 + w_i^2]dxdydz \\ &+ \cos^2 it \iiint [u_i'^2 + v_i'^2 + w_i'^2]dxdydz \\ &+ 2 \sin it \cos it \iiint [u_i u_i' + v_i v_i' + w_i w_i']dxdydz\}, \end{aligned}$$

and, comparing this with (45), we see that

$$W = \Sigma(W_i) \dots\dots\dots\dots\dots\dots\dots\dots(48)$$

In a precisely similar manner we may show that

$$\mathfrak{T} = \Sigma(\mathfrak{T}_i) \dots\dots\dots\dots\dots\dots\dots\dots(49)$$

This is a very remarkable result, the significance of which must not be overlooked. It extends the principle of superposition —*in the case of free vibrations only*—from the components of strain and stress, which are *linear* functions of u, v, w, to W and $\mathfrak{T}$ which are *volume-integrals** of *homogeneous quadratic functions of the same quantities.* The theorem is originally due to Barré de St. Venant (*Comptes Rendus*; 1865, p. 3 and 1866, p. 195).

If $\mathfrak{E}$ be the total energy of the body, due to the resultant strain, we have

$$\left.\begin{aligned}\mathfrak{E} &= W + \mathfrak{T} = \Sigma(W_i + \mathfrak{T}_i) = \Sigma(\mathfrak{E}_i)\\ &= \frac{\rho}{2}\Sigma i^2 \iiint [u_i^2 + v_i^2 + w_i^2 + u_i'^2 + v_i'^2 + w_i'^2]\,dx\,dy\,dz\end{aligned}\right\} \quad\ldots\ldots\ldots (50)$$

Let $\overline{W}_i$ and $\overline{\mathfrak{T}}_i$ denote the *mean values* of W_i and $\mathfrak{T}_i$, throughout any interval of time which is an exact multiple of the period $(2\pi/i)$ of the corresponding displacements. Then since

$$\left.\begin{aligned}\frac{i}{2p\pi}\int_0^{\frac{2\pi p}{i}} \sin^2 it\,dt &= \frac{i}{2p\pi}\int_0^{\frac{2\pi p}{i}} \cos^2 it\,dt = \tfrac{1}{2}\\ \text{and}\qquad \int_0^{\frac{2\pi p}{i}} \sin it \cos it\,dt &= 0\end{aligned}\right\}$$

for all integral values of p, it is easy to show from (45), (46) and (47) that

$$\overline{W}_i = \overline{\mathfrak{T}}_i = \tfrac{1}{2}\mathfrak{E}_i\,; \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (51)$$

and consequently, if

$$\overline{W} = \Sigma(\overline{W}_i),\quad \overline{\mathfrak{T}} = \Sigma(\overline{\mathfrak{T}}_i),$$

we have from (50) and (51)

$$\overline{W} = \overline{\mathfrak{T}} = \tfrac{1}{2}\mathfrak{E} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52)$$

Equations (51) and (52) are the analytical expressions of a statement which the student is very likely to meet with:—that, in any state of free vibratory motion, the energy is *on the whole* equally distributed into kinetic and potential energies.

* The student must be careful to avoid the error of applying this principle to the quadratic functions themselves—viz., the potential and kinetic energies per unit volume. It is only *on integration throughout the whole volume of the body* that the terms involving products of i-functions and j-functions disappear. Thus, although it is true that

$$\iiint V\,dx\,dy\,dz = \Sigma \iiint V\,dx\,dy\,dz,$$

it is *not* true that

$$V = \Sigma(V),$$

as will be at once obvious on substituting from (5).

263.] **Future investigation may be confined to one partial solution.** It is sufficiently obvious from the last three Articles that, when the properties of the partial solution of order i are completely known, those of the most general solution can be at once deduced by simple summation as to i.

We shall therefore confine ourselves to the discussion of Equations (41) and (42). In the former, we may conveniently drop the suffix, of which the presence of i^2 will be a sufficient reminder, and write

$$\left.\begin{aligned} m\frac{\partial\Delta}{\partial x} + n\nabla^2 u + \rho i^2 u &= 0 \\ m\frac{\partial\Delta}{\partial y} + n\nabla^2 v + \rho i^2 v &= 0 \\ m\frac{\partial\Delta}{\partial z} + n\nabla^2 w + \rho i^2 w &= 0 \end{aligned}\right\} \quad\text{.......................(53)}$$

or

$$\left.\begin{aligned} (m+n)\frac{\partial\Delta}{\partial x} - 2n\left(\frac{\partial\theta_3}{\partial y} - \frac{\partial\theta_2}{\partial z}\right) + \rho i^2 u &= 0 \\ (m+n)\frac{\partial\Delta}{\partial y} - 2n\left(\frac{\partial\theta_1}{\partial z} - \frac{\partial\theta_3}{\partial x}\right) + \rho i^2 v &= 0 \\ (m+n)\frac{\partial\Delta}{\partial z} - 2n\left(\frac{\partial\theta_2}{\partial x} - \frac{\partial\theta_1}{\partial y}\right) + \rho i^2 w &= 0 \end{aligned}\right\} \quad\text{..............(54)}$$

The boundary conditions, however, it will be better to retain in the form (42).

The Problem of Forced Vibrations, under Periodic Surface Tractions only.

264.] **General Equations.** Let the body be free from all Applied Forces, as before, but subject to any distribution of Surface Tractions that is strictly periodic as to the time. The traction components will then be of the general form

$$\left.\begin{aligned} F &= \Sigma(F_p \sin pt + F_p' \cos pt) \\ G &= \Sigma(G_p \sin pt + G_p' \cos pt) \\ H &= \Sigma(H_p \sin pt + H_p' \cos pt) \end{aligned}\right\} \quad\text{.......................(55)}$$

Equations (32) will still represent the conditions to be satisfied throughout the body, and these may be decomposed, as before into systems of the form (41) for all values of i.

The boundary conditions (9) may be written

$$\Sigma[(\lambda P_i + \mu U_i + \nu T_i)\sin it] + \Sigma[(\lambda P_i' + \mu U_i' + \nu T_i')\cos it] = \Sigma[F_p \sin pt] + \Sigma[F_p' \cos pt]$$

and so on.

Thus to every value of i in the series (34) which coincides with a value of p occurring in the series (55), corresponds a solution of (41) satisfying the boundary conditions

$$\left.\begin{aligned} \lambda P_p + \mu U_p + \nu T_p &= F_p \\ \lambda U_p + \mu Q_p + \nu S_p &= G_p \\ \lambda T_p + \mu S_p + \nu R_p &= H_p \end{aligned}\right\},$$

while the solutions corresponding to all values of i which are absent from the series (55) satisfy the boundary conditions (42), as before.

In other words, the general solution consists of a system of forced vibrations of the same periods as, and depending upon the form of the Surface Tractions, with any arbitrarily chosen system of free vibrations, satisfying (41) and (42), superposed upon it.

It is thus convenient to consider the two problems together, taking (53) or (54) for the general equations of vibration, and

$$\left.\begin{aligned} \lambda P_i + \mu U_i + \nu T_i &= F_i \\ \lambda U_i + \mu Q_i + \nu S_i &= G_i \\ \lambda T_i + \mu S_i + \nu R_i &= H_i \end{aligned}\right\} \qquad (56)$$

for the general boundary conditions; F_i, G_i, H being each zero, except when i represents one of the values of p in the series (55).

The Two Problems Combined.

Sir William Thomson's Method of Solution.

265.] **Resolution of the Strain.** By the principle of superposition and its converse (§ 88), any system of small displacements, or the system of small strains produced by it, may be resolved in an arbitrary manner into any number of systems, subject only to the condition that the algebraic sums of the components of the latter shall be identically equal to the corresponding components of the original system.

Since the three component displacements at each point of the body must in general be supposed independent functions of position, the number of *independent* functions involved in any resolution of the strain must be *exactly three* in the most general case.

Now the most general form of strain consists of dilatation, distortion and rotation; and it is characteristic of vibrations under no applied forces that the strain must involve *either* dilatation *or* rotation at every point. [This is at once obvious from equations (58) below.] This strain then may be resolved into two, of which the first may be supposed to give rise to cubical

dilatation, but to be irrotational, and therefore to involve only one independent function of position—its displacement potential. Thus, if we suppose the second (rotational) strain to be independent of the first, its component displacements must be connected with one another by some one arbitrary relation:—such, for instance, as that they shall contribute nothing to the cubical dilatation at any point of the body.

We shall then have resolved the most general form of small strain into two independent small strains, one of which contributes dilatation and distortion without rotation, and the other distortion and rotation without dilatation.

266. **Decomposition of the General Equations.** If we write in equations (54)

$$\Omega^2 = (m+n)/\rho\,;\ \Omega'^2 = n/\rho \qquad\qquad (57)$$

they become

$$\left.\begin{aligned}\Omega^2\frac{\partial\Delta}{\partial x} - 2\Omega'^2\left(\frac{\partial\theta_3}{\partial y} - \frac{\partial\theta_2}{\partial z}\right) + i^2u &= 0\\ \Omega^2\frac{\partial\Delta}{\partial y} - 2\Omega'^2\left(\frac{\partial\theta_1}{\partial z} - \frac{\partial\theta_3}{\partial x}\right) + i^2v &= 0\\ \Omega^2\frac{\partial\Delta}{\partial z} - 2\Omega'^2\left(\frac{\partial\theta_2}{\partial x} - \frac{\partial\theta_1}{\partial y}\right) + i^2w &= 0\end{aligned}\right\} \qquad\qquad (58)$$

(*i.*) Let us suppose the mode of vibration to be irrotational, with a displacement potential ϕ. We have then

$$\left.\begin{aligned}\Omega^2\frac{\partial\Delta}{\partial x} + i^2u &= 0\\ \Omega^2\frac{\partial\Delta}{\partial y} + i^2v &= 0\\ \Omega^2\frac{\partial\Delta}{\partial z} + i^2w &= 0\end{aligned}\right\} \qquad\qquad (59)$$

or by (59) of § 123

$$\left.\begin{aligned}\frac{\partial}{\partial x}(\Omega^2\nabla^2\phi + i^2\phi) &= 0\\ \frac{\partial}{\partial y}(\Omega^2\nabla^2\phi + i^2\phi) &= 0\\ \frac{\partial}{\partial z}(\Omega^2\nabla^2\phi + i^2\phi) &= 0\end{aligned}\right\} \qquad\qquad (60)$$

Also, by differentiating (59) as to x, y, z respectively, and adding the results

$$\Omega^2\nabla^2\Delta + i^2\Delta = 0 \qquad\qquad (61)$$

where

$$\Delta = \nabla^2\phi \quad \text{...(62)}$$

By the theory of the ordinary potential, we know that the solution of (62) is

$$\phi = -\frac{1}{4\pi}\iiint \frac{\Delta'}{r}\,dx'dy'dz',$$

where Δ' is the value of Δ at (x', y', z'), and

$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\,;$$

the integral being taken throughout all those portions of space where Δ' differs from zero.

Hence we deduce

$$\nabla^2\phi = -\frac{1}{4\pi}\iiint \frac{\nabla'^2\Delta'}{r}\,dx'dy'dz',$$

where the symbol ∇'^2 denotes the operator

$$\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}.$$

Thus

$$\Omega^2\nabla^2\phi + i^2\phi = -\frac{1}{4\pi}\iiint \frac{(\Omega^2\nabla'^2\Delta' + i^2\Delta')}{r}\,dx'dy'dz'.$$

But by (61)

$$\Omega^2\nabla'^2\Delta' + i^2\Delta' = 0\,;$$

and therefore also

$$\Omega^2\nabla^2\phi + i^2\phi = 0 \quad \text{.........................(63)}$$

This is the general equation to be satisfied by ϕ. Any solution of it will obviously satisfy (60), and therefore also (59).

(*ii.*) Let us suppose the strain to be rotational, and such that the cubical dilatation is everywhere zero.

If u, v, w be the component displacements in this case, the requisite condition is [by (1)]

$$\frac{\partial \mathrm{u}}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} + \frac{\partial \mathrm{w}}{\partial z} = 0, \quad \text{..........................(64)}$$

while equations (58) now reduce to

$$\left.\begin{aligned} 2\Omega'^2\left(\frac{\partial\theta_2}{\partial z} - \frac{\partial\theta_3}{\partial y}\right) + i^2\mathrm{u} &= 0 \\ 2\Omega'^2\left(\frac{\partial\theta_3}{\partial x} - \frac{\partial\theta_1}{\partial z}\right) + i^2\mathrm{v} &= 0 \\ 2\Omega'^2\left(\frac{\partial\theta_1}{\partial y} - \frac{\partial\theta_2}{\partial x}\right) + i^2\mathrm{w} &= 0 \end{aligned}\right\}.$$

or

$$\left.\begin{aligned}\Omega'^2\nabla^2\mathrm{u} + i^2\mathrm{u} = 0\\ \Omega'^2\nabla^2\mathrm{v} + i^2\mathrm{v} = 0\\ \Omega'^2\nabla^2\mathrm{w} + i^2\mathrm{w} = 0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(65)$$

Equations (65) and (64) are therefore the general equations to be satisfied by u, v, w in this case. In virtue of (64) only two of these quantities are independent.

(*iii.*) If we write in equations (58)

$$\left.\begin{aligned}u &= \frac{\partial\phi}{\partial x} + \mathrm{u}\\ v &= \frac{\partial\phi}{\partial y} + \mathrm{v}\\ w &= \frac{\partial\phi}{\partial x} + \mathrm{w}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(66)$$

they become

$$\left.\begin{aligned}\frac{\partial}{\partial x}(\Omega^2\nabla^2\phi + i^2\phi) + \Omega'^2\nabla^2\mathrm{u} + i^2\mathrm{u} + \Omega^2\frac{\partial}{\partial x}\left(\frac{\partial\mathrm{u}}{\partial x} + \frac{\partial\mathrm{v}}{\partial y} + \frac{\partial\mathrm{w}}{\partial z}\right) = 0\\ \frac{\partial}{\partial y}(\Omega^2\nabla^2\phi + i^2\phi) + \Omega'^2\nabla^2\mathrm{v} + i^2\mathrm{v} + \Omega^2\frac{\partial}{\partial y}\left(\frac{\partial\mathrm{u}}{\partial x} + \frac{\partial\mathrm{v}}{\partial y} + \frac{\partial\mathrm{w}}{\partial z}\right) = 0\\ \frac{\partial}{\partial z}(\Omega^2\nabla^2\phi + i^2\phi) + \Omega'^2\nabla^2\mathrm{w} + i^2\mathrm{w} + \Omega^2\frac{\partial}{\partial z}\left(\frac{\partial\mathrm{u}}{\partial x} + \frac{\partial\mathrm{v}}{\partial y} + \frac{\partial\mathrm{w}}{\partial z}\right) = 0\end{aligned}\right\}.$$

These equations are obviously satisfied identically, if we take in (66), for ϕ *any* solution of (63), and for u, v, w *any* solutions of (65) which satisfy (64). Thus, so far as the general equations go, u, v, w may be supposed perfectly independent of ϕ, and the system of displacements represented by (63), (64), (65) and (66) will fulfil the conditions of § 265, and at the same time satisfy the general equations (58).

The boundary conditions (56) will of course impose restrictions upon the generality of this solution, which can easily be deduced by substituting from (66), and supplying the suffix [see (67) and (86) below].

The irrotational or ϕ solution.

267.] **General Equations.** If the mode of vibration be wholly irrotational, the potential ϕ satisfies the equation

$$\Omega^2\nabla^2\phi + i^2\phi = 0\ldots\ldots\ldots\ldots(63)$$

throughout the body; and by substituting the formulæ (61) of

§ 124 in equations (56) above, we find for the boundary conditions

$$\left.\begin{aligned}(m-n)\lambda\nabla^2\phi_i + 2n\left[\lambda\frac{\partial^2\phi_i}{\partial x^2} + \mu\frac{\partial^2\phi_i}{\partial x\partial y} + \nu\frac{\partial^2\phi_i}{\partial z\partial x}\right] &= F_i\\ (m-n)\mu\nabla^2\phi_i + 2n\left[\lambda\frac{\partial^2\phi_i}{\partial x\partial y} + \mu\frac{\partial^2\phi_i}{\partial y^2} + \nu\frac{\partial^2\phi_i}{\partial y\partial z}\right] &= G_i\\ (m-n)\nu\nabla^2\phi_i + 2n\left[\lambda\frac{\partial^2\phi_i}{\partial z\partial x} + \mu\frac{\partial^2\phi_i}{\partial y\partial z} + \nu\frac{\partial^2\phi_i}{\partial z^2}\right] &= H_i\end{aligned}\right\}\ldots\ldots\ldots(67)$$

268.] **Plane Waves of Sound.** Let us first suppose the vibrations to be everywhere parallel to Ox. Then

$$u=\frac{d\phi}{dx},\ v=0,\ w=0.$$

Thus ϕ is a function of x only, and (63) reduces to

$$\frac{d^2\phi}{dx^2}+\frac{i^2}{\Omega^2}\phi=0\,;$$

every solution of which is of the form

$$\phi_i = A_i\sin\frac{ix}{\Omega}+B_i\cos\frac{ix}{\Omega}.$$

The corresponding partial solution for ϕ [see (39) above] is therefore

$$\left(A_i\sin\frac{ix}{\Omega}+B_i\cos\frac{ix}{\Omega}\right)\sin it+\left(A_i'\sin\frac{ix}{\Omega}+B_i'\cos\frac{ix}{\Omega}\right)\cos it,\ldots\ldots(68)$$

which may also be written in the form

$$C_i\sin\frac{i}{\Omega}(x-\Omega t-\beta_i)+C_i'\sin\frac{i}{\Omega}(x+\Omega t-\beta_i')\ldots\ldots\ldots\ldots(69)$$

where C_i, C_i', β_i, β_i' are arbitrary constants.

This partial solution represents two systems of plane waves, of period $2\pi/i$ and wave length $2\pi\Omega/i$, but of arbitrary amplitudes and phases, propagated with the same velocity Ω in the positive and negative directions of x, respectively.

The wave surfaces (or equipotentials) are planes perpendicular to Ox. The vibrations are therefore everywhere normal to the wave surfaces and parallel to the direction of propagation, and disturbances of this character are in consequence known as waves of *normal* or *longitudinal* vibration.

When their periods are within certain limits, they are capable of giving rise to the sensation of sound, if transmitted to the auditory nerves, and waves of longitudinal vibration are therefore often distinguished as *sound waves*.

The velocity of propagation Ω is independent of i, and therefore of the period of vibration: or, in other words, the *Velocity*

of Sound in an isotropic elastic solid is the same for notes of every pitch, and depends solely on the elastic constitution of the material, being given by

$$\Omega = \sqrt{(m+n)/\rho} = \sqrt{(k+\tfrac{4}{3}n)/\rho} \qquad \text{.....................(57)}$$

In calculating the values of Ω numerically, by means of Table (C), page 201, we must remember that the moduli enter into the equations of motion as absolute accelerating forces per unit area, and must consequently be expressed in absolute—not in gravitational—units.

Thus if k, n be the gravitational measures of the moduli in grammes weight per square centimetre, and ρ the density in grammes per cubic centimetre, as given in Table (C), equation (57) may be written

$$\Omega = \sqrt{(\mathrm{k}+\tfrac{4}{3}\mathrm{n})\mathbf{g}/\rho},$$

where $\mathbf{g}$ denotes the acceleration due to gravity, in centimetres per second per second; giving Ω in centimetres per second.

Now by §§ 221, 222

$$\mathrm{k} = \rho\mathfrak{k}, \quad \mathrm{n} = \rho\mathfrak{n},$$

where $\mathfrak{k}$ and $\mathfrak{n}$ are the lengths in centimetres of the moduli of compression and rigidity. Hence

$$\Omega = \sqrt{\mathbf{g}(\mathfrak{k}+\tfrac{4}{3}\mathfrak{n})}, \qquad \text{.............................(70)}$$

which is equal to the velocity that would be acquired by the body in falling under gravity through a height $(\tfrac{1}{2}\mathfrak{k}+\tfrac{2}{3}\mathfrak{n})$.

Taking the value of $\mathbf{g}$ at 981·4, we obtain the following values of Ω:

Velocity of Sound in	Metres per Second.
Steel, - - - - - - -	191,550
Wrought Iron, - - - -	176,790
Flint Glass, - - - - -	157,840
Cast Iron, - - - - - -	148,750
Copper, - - - - - -	138,170
Water at 8° C., - - - - - -	1,435
Air at 10° C., - - - - - -	337

The velocities in water at 8° C., and in air at 10° C., determined by experiment, are added for the sake of comparison. We have seen in Appendix IV., Section A, that a "perfect liquid" may be regarded as analogous to a perfectly

elastic solid totally devoid of rigidity. Thus by analogy the velocity of sound in a perfect liquid should be given by

$$\Omega = \sqrt{k/\rho},$$

where k is the modulus of compression in *dynes* per square centimetre.

Now, according to Table (B), page 200, the values of k for water are 20,300 at 4°·1 Cent., and 21,100 at 10°·8 C.; the corresponding values of ρ being ·999999 and ·999668. Thus the velocity of sound in water, on the supposition that it is a perfect fluid, should be

at 4°·1,	-	1,425 metres per second,
and at 10°·8,	-	1,453 „ „

giving by interpolation about 1,450 metres per second at 10° Cent. Thus the experimental and theoretical results agree quite as closely as we have any right to expect, when the existence of viscosity is taken into account.

269.] Summing (69) as to i, we have for the general equation of the propagation of plane sound waves, in a medium of indefinite extent

$$\phi = \Sigma C_i \sin\frac{i}{\Omega}(x - \Omega t - \beta_i) + \Sigma C_i' \sin\frac{i}{\Omega}(x + \Omega t - \beta_i') \dots\dots\dots (71)$$

which includes waves of all periods and wave-lengths.

In the case of a finite body, we find by substituting in (67) that the maintenance of this state of vibration requires the continual exertion of the system of periodic surface tractions

$$\left.\begin{aligned} F &= -\lambda\rho\sum\Big[i^2 C_i \sin\frac{i}{\Omega}(x - \Omega t - \beta_i) \\ &\qquad + i^2 C_i' \sin\frac{i}{\Omega}(x + \Omega t - \beta_i')\Big] \\ G &= -\mu\rho\frac{m-n}{m+n}\sum\Big[i^2 C_i \sin\frac{i}{\Omega}(x - \Omega t - \beta_i) \\ &\qquad + i^2 C_i' \sin\frac{i}{\Omega}(x + \Omega t - \beta_i')\Big] \\ H &= -\nu\rho\frac{m-n}{m+n}\sum\Big[i^2 C_i \sin\frac{i}{\Omega}(x - \Omega t - \beta_i) \\ &\qquad + i^2 C_i' \sin\frac{i}{\Omega}(x + \Omega t - \beta_i')\Big] \end{aligned}\right\} \dots\dots\dots\dots (72)$$

t is therefore impossible for it to exist alone as a state of free ibration in a body which is bounded on all sides.

270.] **Transmission of free sound vibrations through an infinite plate of any thickness.** If however we suppose the body to be indefinitely extended in all directions perpendicular to Ox, and bounded only by two planes perpendicular to that axis,

we shall only have to deal with the surface conditions over these two faces, the direction-cosines at every point of which are

$$\lambda = \pm 1,\ \mu = 0,\ \nu = 0.$$

Thus if the faces are given by

$$x = d,\ x = -d',$$

the only boundary conditions to be satisfied are

$$\left.\begin{aligned} \Sigma i^2\left[C_i \sin\frac{i}{\Omega}(d' - \Omega t - \beta_i') + C_i' \sin\frac{i}{\Omega}(d + \Omega t - \beta_i')\right] = 0 \\ \Sigma i^2\left[C_i \sin\frac{i}{\Omega}(d' + \Omega t + \beta_i) + C_i' \sin\frac{i}{\Omega}(d' - \Omega t + \beta_i')\right] = 0 \end{aligned}\right\}$$

for all values of t; thus the coefficients of $\sin it$ and $\cos it$ in these series must vanish for all values of i. These conditions, however, are much more easily interpreted if we retain the original form (68) for ϕ. We then have

$$\left.\begin{aligned} A_i \sin\frac{id}{\Omega} + B_i \cos\frac{id}{\Omega} &= 0 \\ A_i' \sin\frac{id}{\Omega} + B_i' \cos\frac{id}{\Omega} &= 0 \\ A_i' \sin\frac{id'}{\Omega} - B_i \cos\frac{id'}{\Omega} &= 0 \\ A_i' \sin\frac{id'}{\Omega} - B_i' \cos\frac{id'}{\Omega} &= 0 \end{aligned}\right\} \quad \text{.........................(73)}$$

This system of equations admits of three solutions.

(*i.*) Let $l = d + d'$ be the thickness of the plate. Then equations (73) are satisfied by

$$\left.\begin{aligned} i &= \frac{i\pi\Omega}{l} \\ \frac{B_i}{A_i} = \frac{B_i'}{A_i'} &= -\tan\frac{i\pi d}{l} \end{aligned}\right\},$$

where i is any integer. This gives for the general solution

$$\phi = \sum_{i=0}^{i=\infty} \mathfrak{A}_i \sin\frac{i\pi(d-x)}{l} \sin\frac{i\pi(\Omega t - a_i)}{l} \quad \text{..................(74)}$$

where the summation includes all positive integral values of i, and $\mathfrak{A}_i$ and a_i are arbitrary constants.

(*ii.*) Again, if the ratio $d : d'$ be reduced to its lowest terms, and then take the form $r : s$, so that r and s are integers of which one at least must be odd, a second solution of (73) is

$$\left.\begin{aligned} i &= \frac{i\pi(r+s)\Omega}{l} \\ B_i &= B_i' = 0 \end{aligned}\right\}.$$

This gives for the general solution

$$\phi = \sum_{i=0}^{i=\infty} \mathfrak{B}_i \sin \frac{i(r+s)\pi x}{l} \sin \frac{i(r+s)\pi(\Omega t - \beta_i)}{l} \qquad \text{.........(75)}$$

to be summed for all positive integral values of i, as before.

(*iii.*) Finally, a third solution of (73) is given by

$$\left.\begin{aligned} i &= \frac{(2i+1)\pi(r+s)\Omega}{2^{p+1}\,.\,l} \\ A_i &= A_i' = 0 \end{aligned}\right\},$$

where 2^p is the highest power of 2 in the product rs. This gives for the general solution

$$\phi = \sum_{i=0}^{i=\infty} \mathfrak{C}_i \cos \frac{(2i+1)(r+s)\pi x}{2^{p+1}\,.\,l} \sin \frac{(2i+1)(r+s)\pi(\Omega t - \gamma_i)}{2^{p+1}\,.\,l} \qquad \text{......(76)}$$

The sum of the three series (74), (75) and (76) represents all the plane waves of normal vibration that can maintain themselves unchanged in such a plate, without the application of periodic tractions over its faces.

271.] As a simple example, let the plane of yz be so chosen as to bisect the thickness of the plate: so that

$$d = d' = \tfrac{1}{2}l, \quad r = s = 1, \quad p = 0.$$

The series (74) will then split up into two, which will respectively include (75) and (76). The most general solution in this case may thus be written

$$\begin{aligned} \phi = &\sum_{i=0}^{i=\infty} \mathfrak{B}_i \sin \frac{2i\pi x}{l} \sin \frac{2i\pi(\Omega t - \beta_i)}{l} \\ &+ \sum_{i=0}^{i=\infty} \mathfrak{C}_i \cos \frac{(2i+1)\pi x}{l} \sin \frac{(2i+1)\pi(\Omega t - \gamma_i)}{l} \end{aligned} \qquad \text{.........(77)}$$

Since this is a case of free motion, we may apply the formulæ of § 262 to determine the energy possessed by each prismatic portion of the plate having its generators parallel to Ox, and its transverse section of unit area.

Writing first of all in (45)

$$\left.\begin{aligned} i &= 2i\pi\Omega/l \\ u_i &= \frac{2i\pi}{l} \mathfrak{B}_i \cos \frac{2i\pi x}{l} \cos \frac{2i\pi\beta_i}{l} \\ u_i' &= -\frac{2i\pi}{l} \mathfrak{B}_i \cos \frac{2i\pi x}{l} \sin \frac{2i\pi\beta_i}{l} \end{aligned}\right\},$$

we get, for the first part of the potential energy,

$$\Sigma\frac{8i^4\pi^4\Omega^2\rho}{l^4}\mathfrak{B}_i^2\sin^2\frac{2i\pi(\Omega t-\beta_i)}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}}\cos^2\frac{2i\pi x}{l}dx$$

$$=\Sigma\frac{4i^4\pi^4\Omega^2\rho}{l^3}\mathfrak{B}_i^2\sin^2\frac{2i\pi(\Omega t-\beta_i)}{l}.$$

Treating the second series in the same way, we find for the total potential energy of the prism

$$W=\frac{4\pi^4\Omega^2\rho}{l^3}\Sigma i^4\mathfrak{B}_i^2\sin^2\frac{2i\pi(\Omega t-\beta_i)}{l}$$

$$+\frac{\pi^4\Omega^2\rho}{4l^3}\Sigma(2i+1)^4\mathfrak{C}_i^2\sin^2\frac{(2i+1)\pi(\Omega t-\gamma_i)}{l}.$$

And similarly we may deduce from (46) for the kinetic energy of the prism

$$\mathfrak{T}=\frac{4\pi^4\Omega^2\rho}{l^3}\Sigma i^4\mathfrak{B}_i^2\cos^2\frac{2i\pi(\Omega t-\beta_i)}{l}$$

$$+\frac{\pi^4\Omega^2\rho}{4l^3}\Sigma(2i+1)^4\mathfrak{C}_i^2\cos^2\frac{(2i+1)\pi(\Omega t-\gamma_i)}{l}.$$

Thus the total energy of the prism will be

$$\mathfrak{E}=\frac{\pi^4\Omega^2\rho}{4l^3}\sum_{i=0}^{i=\infty}\left\{(2i)^4\mathfrak{B}_i^2+(2i+1)^4\mathfrak{C}_i^2\right\} \quad \text{.............. (78)}$$

and in order that this series may be convergent, it is necessary that $\mathfrak{B}_i^2$ should vary inversely as some power of i higher* than the 5th; and similarly for $\mathfrak{C}_i^2$. Let us take, for example,

$$\mathfrak{B}_i=\frac{Bl^2}{(2i)^3\Omega}\mathbf{U},\ \mathfrak{C}_i=\frac{Cl^2}{(2i+1)^3\Omega}\mathbf{U},$$

where B, C, $\mathbf{U}$ are constants independent of i. Then

$$\mathfrak{E}=\frac{\pi^4\rho l\mathbf{U}^2}{4}\left\{\frac{B^2}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\right)+C^2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots\right)\right\}.$$

The two infinite series within the brackets are convergent, and their sums† are known to be $\pi^2/6$ and $\pi^2/8$ respectively. Thus

$$\mathfrak{E}=\frac{\pi^6}{16}\left(\frac{B^2}{3}+C^2\right)\cdot\tfrac{1}{2}\rho l\mathbf{U}^2,$$

and if B and C be so related that

$$\frac{B^2}{3}+C^2=\frac{16}{\pi^6},$$

* Todhunter's *Algebra*, Art. 562.

† Todhunter's *Plane Trigonometry*, Ch. xxiii., Ex. 1, 3.

the total energy possessed by the prism—and consequently also that of the whole plate—will be precisely the same if it were moving *bodily* with a velocity of translation **U**.

Substituting in (77) we get for the potential due to this *equivalent state of vibration* (see Chapter IX., below)

$$\phi = \frac{\mathbf{U}l^2}{\Omega}\sqrt{\frac{48}{\pi^2} - 3C^2}\sum_{i=0}^{i=\infty}\frac{1}{(2i)^3}\sin\frac{2i\pi x}{l}\sin\frac{2i\pi(\Omega t - \beta_i)}{l}$$
$$+\frac{\mathbf{U}l^2}{\Omega}C\sum_{i=0}^{i=\infty}\frac{1}{(2i+1)^3}\cos\frac{(2i+1)\pi x}{l}\sin\frac{(2i+1)\pi(\Omega t - \gamma_i)}{l}\;\ldots\ldots\ldots(79)$$

where C is completely arbitrary.

If we wish to impose the further restriction that the origin (and with it the whole median plane of the plate) shall remain at rest, we have only to make

$$\frac{d\phi}{dx} = 0, \text{ when } x = 0.$$

This requires that $C = 4/\pi$, and we then have

$$\phi = \frac{4\mathbf{U}l^2}{\pi\Omega}\sum_{i=0}^{i=\infty}\frac{1}{(2i+1)^3}\cos\frac{(2i+1)\pi x}{l}\sin\frac{(2i+1)\pi(\Omega t - \gamma_i)}{l}\;\ldots\ldots(80)$$

272.] **Solution in terms of Spherical Harmonics.** The general equation (63), when transformed to spherical polars by means of formula (65) of § 243, becomes

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi_i}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi_i}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\phi_i}{\partial\omega^2} + \frac{i^2r^2\phi_i}{\Omega^2} = 0\ldots\ldots(81)$$

Let us assume that

$$\phi_i = \Sigma_s(\Phi_{i,s}\,.\,\mathbf{H}_s) = \Sigma_s\phi_{i,s}$$

where $\mathbf{H}_s$ is a surface harmonic of order s, and $\Phi_{i,s}$ a function of r only. Then $\phi_{i,s}$ must satisfy (81) for all values of i and s, and since $\mathbf{H}_s$ satisfies identically

$$\nabla^2(r^s\mathbf{H}_s) = 0,$$

(81) reduces to

$$\frac{d^2\Phi_{i,s}}{dr^2} + \frac{2}{r}\frac{d\Phi_{i,s}}{dr} + \left[\frac{i^2}{\Omega^2} - \frac{s(s+1)}{r^2}\right]\Phi_{i,s} = 0.$$

This equation may also be written in the form

$$\frac{d^2}{dr^2}(r^{\frac{1}{2}}\Phi_{i,s}) + \frac{1}{r}\frac{d}{dr}(r^{\frac{1}{2}}\Phi_{i,s}) + \left[\frac{i^2}{\Omega^2} - \frac{(s+\frac{1}{2})^2}{r^2}\right]r^{\frac{1}{2}}\Phi_{i,s} = 0,$$

and the solution which gives finite values for $\Phi_{i,s}$ and $d\Phi_{i,s}/dr$ at the origin is

$$\Phi_{i,s} = A_{i,s}\sqrt{\frac{\Omega}{ir}}\,.\,\mathbf{J}_{s+\frac{1}{2}}\left(\frac{ir}{\Omega}\right)$$

where $\mathbf{J}_{s+\frac{1}{2}}$ denotes Bessel's function of the first kind, and of order $(s+\frac{1}{2})$.

Thus finally we have a solution of the form

$$\phi = \Sigma_i \left\{ \sqrt{\frac{\Omega}{ir}} \sin it \,.\, \Sigma_s \left[\mathbf{H}_s \,.\, \mathbf{J}_{s+\frac{1}{2}}\left(\frac{ir}{\Omega}\right)\right] + \sqrt{\frac{\Omega}{ir}} \cos it \,.\, \Sigma_s \left[\mathbf{H}_s' \,.\, \mathbf{J}_{s+\frac{1}{2}}\left(\frac{ir}{\Omega}\right)\right]\right\} \quad \text{......(82)}$$

where $\mathbf{H}_s$, $\mathbf{H}_s'$ represent two surface harmonics of order s.

This solution is adapted to a solid of spherical form, having the origin at its centre. Stokes' solution, suitable for infinite space *outside* the sphere, will be found in Lord Rayleigh's *Theory of Sound*, § 323.

The choice of harmonics is not unrestricted, because the preservation of the *continuity* of the body demands that

$$\frac{\partial \phi}{\partial \theta} = 0, \text{ when } \sin\theta = 0;$$

(see § 287, below). Hence all the harmonics included in the solution must satisfy the condition

$$\frac{\partial \mathbf{H}_s}{\partial \theta} = 0, \text{ when } \sin\theta = 0.$$

At the surface of the sphere we have, by (72) of § 243,

$$\Xi' = P, \ \ \mathrm{H}' = U, \ \ Z' = T;$$

and on substitution from (73) in (68) of that Article, and thence in (46) of § 239, we find, after availing ourselves of (63) above,

$$\left.\begin{aligned} \Xi' &= 2n\frac{\partial^2\phi}{\partial r^2} = \frac{(m-n)i^2}{\Omega^2}\phi \\ \mathrm{H}' &= 2n\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \\ Z' &= \frac{2n}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\omega}\right) \end{aligned}\right\}.$$

Thus, if r be the radius of the surface, the conditions that (82) may represent a form of *free* vibration are

$$\left.\begin{aligned} \left[\frac{d^2}{d\mathrm{r}^2} - \frac{(m-n)i^2}{2n\Omega^2}\right]\mathrm{r}^{-\frac{1}{2}}\mathbf{J}_{s+\frac{1}{2}}\left(\frac{i\mathrm{r}}{\Omega}\right) &= 0 \\ \frac{\partial \mathbf{H}_s}{\partial\theta}\cdot\frac{d}{d\mathrm{r}}\mathrm{r}^{-\frac{3}{2}}\mathbf{J}_{s+\frac{1}{2}}\left(\frac{i\mathrm{r}}{\Omega}\right) &= 0 \\ \frac{\partial \mathbf{H}_s}{\partial\omega}\cdot\frac{d}{d\mathrm{r}}\mathrm{r}^{-\frac{3}{2}}\mathbf{J}_{s+\frac{1}{2}}\left(\frac{i\mathrm{r}}{\Omega}\right) &= 0 \end{aligned}\right\} \quad \text{..................(83)}$$

for all values of i and s. Thus, *in general*, every value of i which can occur in connection with a given value of s must be of the form

$$i = \frac{\Omega \mathrm{i}}{r},$$

where i is any root common to the two simultaneous equations

$$\left.\begin{array}{r}\left(\dfrac{d^2}{d\mathrm{i}^2} - \dfrac{m-n}{2n}\right)\left[\mathrm{i}^{-\frac{1}{2}} J_{s+\frac{1}{2}}(\mathrm{i})\right] = 0 \\ \dfrac{d}{d\mathrm{i}}\left[\mathrm{i}^{-\frac{3}{2}} J_{s+\frac{1}{2}}(\mathrm{i})\right] = 0\end{array}\right\}.$$

The first of these equations may be written

$$J''_{s+\frac{1}{2}}(\mathrm{i}) - \frac{1}{\mathrm{i}} J'_{s+\frac{1}{2}}(\mathrm{i}) + \left(\frac{3}{4\mathrm{i}^2} - \frac{m-n}{2n}\right) J_{s+\frac{1}{2}}(\mathrm{i}) = 0.$$

But, by Bessel's fundamental equation

$$J''_{s+\frac{1}{2}}(\mathrm{i}) + \frac{1}{\mathrm{i}} J'_{s+\frac{1}{2}}(\mathrm{i}) + \left[1 - \frac{(2s+1)^2}{4\mathrm{i}^2}\right] J_{s+\frac{1}{2}}(\mathrm{i}) = 0:$$

and, on eliminating J'' between these two the above equations of condition may be written in the simpler form

$$J'_{s+\frac{1}{2}}(\mathrm{i}) = \left[\frac{(2s+1)^2+3}{8\mathrm{i}} - \frac{\Omega^2 \mathrm{i}}{4\Omega'^2}\right] J_{s+\frac{1}{2}}(\mathrm{i})$$

$$= \frac{3}{2\mathrm{i}} J_{s+\frac{1}{2}}(\mathrm{i}),$$

or

$$\left.\begin{array}{r}\mathrm{i}J'_{s+\frac{1}{2}}(\mathrm{i}) = \frac{3}{2} J_{s+\frac{1}{2}}(\mathrm{i}) \\ \mathrm{i} = \dfrac{\Omega'}{\Omega}\sqrt{2(s-1)(s+2)}\end{array}\right\}.$$

and

The admissible values of s are therefore the positive integral roots of the equation

$$s - 1 - \frac{s+1}{2s+3} \cdot \frac{(s-1)(s+2)}{1\,!} + \frac{s+3}{(2s+3)(2s+5)} \cdot \frac{(s-1)^2(s+2)^2}{2\,!}$$
$$- \frac{s+5}{(2s+3)(2s+5)(2s+7)} \cdot \frac{(s-1)^3(s+2)^3}{3\,!} + \ldots = 0.$$

I am not aware that the roots of this equation have ever been investigated, but it may be observed that it has at least *one* positive integral root, namely

$$s = 1,$$

and that the value of i corresponding to this value of s is zero. Thus no surface harmonic of the first order can enter into a form of free vibration.

273.] **Spherical Sound Waves.** In one particular case—namely, that in which the only harmonics present in the solution are of order zero—the two latter of the conditions (83) are satisfied identically, $\mathbf{H}_0$ being a constant, and the admissible values of i are given by

$$i = \frac{\Omega \mathrm{i}}{\mathrm{r}},$$

where i is a root of $\qquad \mathbf{J}_{\frac{1}{2}}'(\mathrm{i}) = \left[\frac{1}{2\mathrm{i}} - \frac{\Omega^2 \mathrm{i}}{4\Omega'^2}\right]\mathbf{J}_{\frac{1}{2}}(\mathrm{i})$

or the equivalent* equation

$$\mathrm{i} \cot \mathrm{i} = 1 - \frac{\Omega^2}{4\Omega'^2}\mathrm{i}^2 \text{..........................} (84)$$

The solution (82) now takes the form

$$\phi = \Sigma C_i \sqrt{\frac{\mathrm{r}}{\mathrm{i}r}} \cdot \mathbf{J}_{\frac{1}{2}}\left(\frac{\mathrm{i}r}{\mathrm{r}}\right) \sin \frac{\mathrm{i}\Omega}{\mathrm{r}}(t - \gamma_i),$$

or, as it may also be written,*

$$\phi = \sqrt{\frac{2}{\pi}} \Sigma C_i \frac{\mathrm{r}}{\mathrm{i}r} \sin \frac{\mathrm{i}r}{\mathrm{r}} \sin \frac{\mathrm{i}\Omega}{\mathrm{r}}(t - \gamma_i).$$

The corresponding value for the *radial* displacement u may be written in either* of the forms

$$u = -\Sigma C_i \sqrt{\frac{\mathrm{i}}{\mathrm{r}r}} \mathbf{J}_{\frac{3}{2}}\left(\frac{\mathrm{i}r}{\mathrm{r}}\right) \sin \frac{\mathrm{i}\Omega}{\mathrm{r}}(t - \gamma_i),$$

or $$u = \sqrt{\frac{2}{\pi}} \Sigma \frac{C_i}{r}\left[\cos \frac{\mathrm{i}r}{\mathrm{r}} - \frac{\mathrm{r}}{\mathrm{i}r} \sin \frac{\mathrm{i}r}{\mathrm{r}}\right] \sin \frac{\mathrm{i}\Omega}{\mathrm{r}}(t - \gamma_i).$$

This solution evidently represents a series of free spherical waves of radial vibration, propagated inwards and outwards with the same radial velocity Ω.

274.] **Sound Waves in general. Possible Forms.** In order that the family of surfaces represented by the general equation

$$\chi(x, y, z) = \xi,$$

where ξ is a variable parameter, may represent a possible form of sound waves, sustainable *without the aid of Applied Forces*, the parameter ξ must satisfy *two* conditions. For let ϕ be the potential, which is a function of ξ, and let η and ζ be the parameters of the two families† of surfaces orthogonal to the above and to one another.

* Todhunter's *Functions of Laplace, Lamé and Bessel*, end of Article 378.

† Two such families must always exist; for, from the character of the motion, a continuous series of curves can be drawn to cut all the ξ surfaces orthogonally. The two systems of these curves, drawn through the two lines of curvature which intersect at any point of a ξ surface, will define a surface of the η system, and one of the ζ system, respectively.

Taking ξ, η, ζ for orthogonal curvilinear coördinates, we have, by substitution from (13) of § 231 in (63) above

$$h_1 h_2 h_3 \frac{\partial}{\partial \xi}\left(\frac{h_1}{h_2 h_3}\frac{d\phi}{d\xi}\right) + \frac{i^2}{\Omega^2}\phi = 0;$$

or, by (14) of § 231,

$$h_1^2 \frac{d^2\phi}{d\xi^2} + \nabla^2\xi \cdot \frac{d\phi}{d\xi} + \frac{i^2\phi}{\Omega^2} = 0 \qquad (85)$$

Thus, ϕ being a function of ξ only, it follows that

$$\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}$$

and

$$\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2 + \left(\frac{\partial\xi}{\partial z}\right)^2$$

must both be capable of expression as functions of ξ only, or as constants.

The rotational, or u, v, w *solution.*

275.] **General Equations.** If the strain be rotational, but unaccompanied by alterations of density, the component displacements must satisfy the general equations

$$\left.\begin{aligned} \Omega'^2\nabla^2 \mathrm{u} + i^2\mathrm{u} &= 0 \\ \Omega'^2\nabla^2 \mathrm{v} + i^2\mathrm{v} &= 0 \\ \Omega'^2\nabla^2 \mathrm{w} + i^2\mathrm{w} &= 0 \end{aligned}\right\} \qquad (65)$$

and

$$\frac{\partial \mathrm{u}}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} + \frac{\partial \mathrm{w}}{\partial z} = 0; \qquad (64)$$

all such solutions being excluded as make

$$\mathrm{u}dx + \mathrm{v}dy + \mathrm{w}dz$$

a perfect differential.

The boundary conditions (42) may, in virtue of (64), be thrown into the form

$$\left.\begin{aligned} 2\lambda\frac{\partial \mathrm{u}_i}{\partial x} + \mu\left(\frac{\partial \mathrm{v}_i}{\partial x} + \frac{\partial \mathrm{u}_i}{\partial y}\right) + \nu\left(\frac{\partial \mathrm{u}_i}{\partial z} + \frac{\partial \mathrm{w}_i}{\partial x}\right) &= F_i/n \\ 2\mu\frac{\partial \mathrm{v}_i}{\partial y} + \nu\left(\frac{\partial \mathrm{w}_i}{\partial y} + \frac{\partial \mathrm{v}_i}{\partial z}\right) + \lambda\left(\frac{\partial \mathrm{v}_i}{\partial x} + \frac{\partial \mathrm{u}_i}{\partial y}\right) &= G_i/n \\ 2\nu\frac{\partial \mathrm{w}_i}{\partial z} + \lambda\left(\frac{\partial \mathrm{u}_i}{\partial z} + \frac{\partial \mathrm{w}_i}{\partial x}\right) + \mu\left(\frac{\partial \mathrm{w}_i}{\partial y} + \frac{\partial \mathrm{v}_i}{\partial z}\right) &= H_i/n \end{aligned}\right\} \qquad (86)$$

276.] **Plane Waves of Transverse, Tangential, or Distortional Vibrations.** Let us suppose that v is a function of x only, while u and w are both zero: (64) is then satisfied identically, while (65) gives

$$\frac{\partial^2 \mathrm{v}}{\partial x^2} + \frac{i^2}{\Omega'^2}\mathrm{v} = 0.$$

Thus $$\mathrm{v}_i = A_i \sin\frac{ix}{\Omega'} + B_i \cos\frac{ix}{\Omega'}$$

and the full solution is of the form

$$\mathrm{v} = \Sigma C_i \sin\frac{\mathrm{i}}{\Omega'}(x - \Omega' t - \beta_i) + \Sigma C_i' \sin\frac{\mathrm{i}}{\Omega'}(x + \Omega' t - \beta_i').$$

This represents a series of plane waves, of vibrations which are transverse to the direction of propagation, or *in* the wave fronts, propagated with the same velocity Ω' independent of their periods, in the positive and negative directions of Ox.

These vibrations are of the same character as those by which light is propagated through the luminiferous ether. Thus if the ether were composed of homogeneous and isotropic "continuous" matter, the velocity of light would be the same whatever its colour. Moreover, it is easy to shew that the same result would hold for light of all colours, propagated *in any given direction*, if the ether were crystalline, but still "continuous." Now the dispersion of white light into its coloured constituents, by ordinary refraction at the bounding surface of any two transparent media of different densities, is proved to be due to the different velocities with which light of various colours is propagated in either medium. This familiar phænomenon is consequently sufficient in itself to prove that the luminiferous ether —at least, as it exists in the interior of solid and liquid bodies— cannot possess the properties of "continuous" matter.

The fascinating problem of the structure and properties of the ether is too wide and too difficult to be more than alluded to in this place. The student who wishes to follow up the subject should consult Sir William Thomson's *Lectures on Molecular Dynamics*, delivered at the John Hopkins University, Baltimore, U.S.A., in 1884. These lectures contain a most interesting summary of the various hypotheses which have been framed to account for the phænomena of dispersion, polarisation, double refraction, etc., with the grounds on which each has failed, together with a fuller development of Sir William Thomson's own remarkable conception.

277.] **The General Solution.** The problem, as stated in § 275, appears rather complicated, but it is easy to present it in a form which is of the utmost admissible generality, and yet satisfies all the conditions identically.

Thus let us assume

$$\left.\begin{aligned} u &= \frac{\partial\psi_3}{\partial y} - \frac{\partial\psi_2}{\partial z} \\ v &= \frac{\partial\psi_1}{\partial z} - \frac{\partial\psi_3}{\partial x} \\ w &= \frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_1}{\partial y} \end{aligned}\right\} \qquad (87)$$

where ψ_1, ψ_2, ψ_3 are *any three solutions whatever* of the equation

$$\Omega'^2\nabla^2\psi + i^2\psi = 0 \qquad (88)$$

It is obvious that equations (64) and (65) are satisfied identically by these values of u, v, w; and it is also easy to shew that they cannot possibly make

$$u\,dx + v\,dy + w\,dz$$

a perfect differential. For in that case we should have

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \text{ etc.},$$

or the equivalent relations between ψ_1, ψ_2, ψ_3

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_3}{\partial z}\right) &= \nabla^2\psi_1 = -\frac{i^2}{\Omega'^2}\psi_1 \\ \frac{\partial}{\partial y}\left(\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_3}{\partial z}\right) &= \nabla^2\psi_2 = -\frac{i^2}{\Omega'^2}\psi_2 \\ \frac{\partial}{\partial z}\left(\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_3}{\partial z}\right) &= \nabla^2\psi_3 = -\frac{i^2}{\Omega'^2}\psi_3 \end{aligned}\right\},$$

which make

$$\psi_1 dx + \psi_2 dy + \psi_3 dz = -\frac{\Omega'^2}{i^2}\,d\left(\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_3}{\partial z}\right)$$

a perfect differential; and this would require that

$$u = 0, \quad v = 0, \quad w = 0.$$

Thus (87) and (88) constitute a solution which satisfies identically all the conditions imposed, while, since it involves three arbitrary solutions of (88) which is of the same form as (65) it is of the utmost possible generality.

The problem is now reduced to the solution of the fundamental equation (88), which is similar to (63) and does not therefore need further illustration.

Poisson's Integrals.

278.] **Having given any one partial solution of (63) or (88), to express the complete solution as the sum of two definite integrals.** Equation (63), which should properly be written

$$\Omega^2\nabla^2\phi_i + i^2\phi_i = 0, \qquad (89)$$

is only the equation satisfied by the partial component of ϕ of order i, and results [compare the general equations (32) and (41) of § 260] from the decomposition of the perfectly general equation

$$\Omega^2\nabla^2\phi - \ddot{\phi} = 0 \qquad (90)$$

satisfied by the resultant potential, as a whole.

Writing this latter in the form

$$\left(\frac{\partial^2}{\partial t^2} - \Omega^2\nabla^2\right)\phi = 0,$$

and remembering that the operator ∇, being independent of t, behaves as a constant in combination with functions of t or the operator $\partial/\partial t$, we obtain the symbolical solution

$$\phi = \cos(\iota\Omega t\nabla)\Phi' + \frac{\sin(\iota\Omega t\nabla)}{\iota\Omega\nabla}\Phi,$$

where $\iota = \sqrt{-1}$, and Φ, Φ' are perfectly arbitrary functions of x, y, z.

Expanding the operators,

$$\phi = \left[1 + \frac{\Omega^2 t^2}{2\,!}\nabla^2 + \frac{\Omega^4 t^4}{4\,!}\nabla^4 + \ldots\right]\Phi' + t\left[1 + \frac{\Omega^2 t^2}{3\,!}\nabla^2 + \frac{\Omega^4 t^4}{5\,!}\nabla^4 + \ldots\right]\Phi;$$

thus, when $t = 0$,

$$\phi = \Phi', \quad \dot{\phi} = \dot{\Phi}.$$

Now, with the notation of (39), § 260,

$$\left.\begin{aligned}\phi &= \Sigma(\phi_i \sin it + \phi_i' \cos it)\\ \dot{\phi} &= \Sigma i(\phi \ \cos it - \phi_i' \sin it)\end{aligned}\right\},$$

and consequently, when $t = 0$,

$$\phi = \Sigma(\phi_i'), \quad \dot{\phi} = \Sigma(i\phi_i).$$

Thus, we must make

$$\Phi = \Sigma(i\phi_i), \quad \Phi' = \Sigma(\phi_i'),$$

and the symbolical solution becomes

$$\phi = \cos(\iota\Omega t\nabla)\Sigma(\phi_i') + \frac{\sin(\iota\Omega t\nabla)}{\iota\Omega\Delta}\Sigma(i\phi_i) \qquad (91)$$

Again, by the symbolical expression of Maclaurin's Theorem, if χ be any continuous function of (x, y, z),

$$\chi(x+\xi,\ y+\eta,\ z+\zeta) = e^{\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\zeta\frac{\partial}{\partial z}}\,.\,\chi(x,\ y,\ z).$$

Thus, if we describe a sphere,

$$(x+\xi)^2+(y+\eta)^2+(z+\zeta)^2=r^2,$$

with radius r, and centre at (x, y, z), the mean value of χ taken over the surface of the sphere will be

$$\frac{1}{4\pi r^2}\iint e^{\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\zeta\frac{\partial}{\partial z}}\chi(x,\ y,\ z)\,.\,dS.$$

Since x, y, z are independent of position on the surface, the function χ may be taken from under the sign of integration, and the integral considered altogether as an operator upon it. It thus becomes

$$\frac{1}{4\pi r^2}\iint e^{\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\zeta\frac{\partial}{\partial z}}dS\,.\,\chi(x,\ y,\ z).$$

The axes of ξ, η, ζ are at present parallel to Ox, Oy, Oz, but the operators $\partial/\partial x, \partial/\partial y, \partial/\partial z$ behaving as constants within the integral, and the surface being symmetrical as to ξ, η, ζ, we may transform to new axes of ξ', η', ζ' through the centre, such that

$$\xi'=\frac{\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\zeta\frac{\partial}{\partial z}}{\sqrt{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}}}=\frac{1}{\nabla}\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\zeta\frac{\partial}{\partial z}\right).$$

The integral thus becomes

$$\frac{1}{4\pi r^2}\iint e^{\xi'\nabla}dS\,.\,\chi(x,\ y,\ z)$$

$$=\frac{1}{2r}\int_{-r}^{r} e^{\xi'\nabla}d\xi'\,.\,\chi(x,\ y,\ z)$$

$$=\frac{1}{2r\nabla}\left(e^{r\nabla}-e^{-r\nabla}\right)\chi(x,\ y,\ z)$$

$$=\frac{\sin(\iota r\nabla)}{\iota r\nabla}\chi(x,\ y,\ z).$$

Thus, writing $r=\Omega t$, the symbolical expression

$$\frac{\sin(\iota\Omega t\nabla)}{\iota\Omega t\nabla}\chi(x,\ y,\ z)$$

represents the mean value of the function χ, taken over a sphere of radius Ωt with its centre at (x, y, z). Consequently we may write

$$\frac{\sin(\iota\Omega t\nabla)}{\iota\Omega\nabla}\chi(x, y, z) = \frac{t}{4\pi}\int_0^\pi\int_0^{2\pi}\chi(x + \Omega t\sin\theta\cos\omega,$$
$$y + \Omega t\sin\theta\sin\omega,\ z + \Omega t\cos\theta)\sin\theta d\theta d\omega,$$

where θ, ω are the spherical angles of § 243. Differentiating both sides as to t,

$$\cos(\iota\Omega t\nabla)\chi(x, y, z) = \frac{1}{4\pi}\frac{\partial}{\partial t}t\int_0^\pi\int_0^{2\pi}\chi(x + \Omega t\sin\theta\cos\omega,$$
$$y + \Omega t\sin\theta\sin\omega,\ z + \Omega t\cos\theta)\sin\theta d\theta d\omega.$$

Substituting these integrals in (91) we have for the full solution of (90)

$$\phi = \frac{1}{4\pi}\sum\left\{it\int_0^\pi\int_0^{2\pi}\phi_i(x + \Omega t\sin\theta\cos\omega,\ y + \Omega t\sin\theta\sin\omega,\right.$$
$$z + \Omega t\cos\theta)\sin\theta d\theta d\omega$$
$$+\frac{\partial}{\partial t}t\int_0^\pi\int_0^{2\pi}\phi_i'(x + \Omega t\sin\theta\cos\omega,\ y + \Omega t\sin\theta\sin\omega,$$
$$\left. z + \Omega t\cos\theta)\sin\theta d\theta d\omega\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(92)$$

Thus having obtained from (63) any partial solution, of the form

$$\phi_i(x, y, z) . \sin it + \phi_i'(x, y, z) . \cos it,$$

we can at once deduce the complete solution, as the sum of two definite integrals.

These integrals may also be regarded as giving the value of ϕ at any time t in terms of the values of $\phi[=\Sigma(\phi_i')]$ and $\dot{\phi}[=\Sigma(i\phi_i)]$ when $t=0$.

As a simple example, the potential for plane sound waves (§ 268), travelling parallel to Ox, may be written

$$\phi = \frac{1}{4\pi}\sum\left\{A_it\int_0^\pi\int_0^{2\pi}\sin\frac{i}{\Omega}(x + \Omega t\sin\theta\cos\omega - \alpha_i)\sin\theta d\theta d\omega\right.$$
$$\left.+B_i\frac{\partial}{\partial t}t\int_0^\pi\int_0^{2\pi}\sin\frac{i}{\Omega}(x + \Omega t\sin\theta\cos\omega - \beta_i)\sin\theta d\theta d\omega\right\},$$

giving, when $t=0$,

$$\left.\begin{aligned}\phi &= \Sigma B_i \sin\frac{i}{\Omega}(x-\beta_i)\\ \dot\phi &= \Sigma A_i \sin\frac{i}{\Omega}(x-\alpha_i)\end{aligned}\right\}.$$

The solution for ψ (§ 277) is of precisely the same form, the sole distinction being the substitution of Ω' for Ω.

The Problem of Forced Vibrations under Periodic Surface Tractions, and Periodic Applied Forces derivable from a Potential.

279.] **General Equations.** The only Applied Forces with which we have to deal under natural conditions are forces of attraction and repulsion, whose components are always derivable by differentiation from a Force Potential.

We shall therefore always assume, except when the contrary is expressly stated, that a function Ψ exists such that

$$X=\frac{\partial\Psi}{\partial x},\quad Y=\frac{\partial\Psi}{\partial y},\quad Z=\frac{\partial\Psi}{\partial z}\quad\quad(93)$$

at every point of the body. It may easily be verified that the components of the same force, referred to any system of curvilinear coördinates, are, with the notation of Chapter V.,

$$\Xi=h_1\frac{\partial\Psi}{\partial\xi},\quad \mathrm{H}=h_2\frac{\partial\Psi}{\partial\eta},\quad \mathrm{Z}=h_3\frac{\partial\Psi}{\partial\zeta}\quad\quad(94)$$

If the Applied Forces be strictly periodic, their potential must be of the form

$$\Psi=\Sigma(\Psi_s\sin st+\Psi'_s\cos st)\quad\quad(95)$$

and on substitution in the general equations of motion (29) and (29*a*) of § 258, we see that the partial displacement-amplitudes of order i must satisfy

$$\left.\begin{aligned}m\frac{\partial}{\partial x}\left(\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}+\frac{\partial w_i}{\partial z}\right)+n\nabla^2u_i+\rho\left(i^2u_i+\frac{\partial\Psi_i}{\partial x}\right)&=0\\ m\frac{\partial}{\partial y}\left(\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}+\frac{\partial w_i}{\partial z}\right)+n\nabla^2v_i+\rho\left(i^2v_i+\frac{\partial\Psi_i}{\partial y}\right)&=0\\ m\frac{\partial}{\partial z}\left(\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}+\frac{dw_i}{\partial z}\right)+n\nabla^2w_i+\rho\left(i^2w_i+\frac{\partial\Psi_i}{\partial z}\right)&=0\end{aligned}\right\}\quad\quad(96)$$

where Ψ is to be supposed zero, unless the value of i coincides with any one of the values of s in the series (95).

The boundary conditions will still be expressed by equations (55) of § 264.

280.] **The forced vibrations constitute a pure strain.** Omitting the suffix from equations (96), and writing them in the form

$$\left.\begin{aligned}\frac{\partial}{\partial x}[\Omega^2\Delta+\Psi]-2\Omega'^2\left[\frac{\partial\theta_3}{\partial y}-\frac{\partial\theta_2}{\partial z}\right]+i^2u=0\\ \frac{\partial}{\partial y}[\Omega^2\Delta+\Psi]-2\Omega'^2\left[\frac{\partial\theta_1}{\partial z}-\frac{\partial\theta_3}{\partial x}\right]+i^2v=0\\ \frac{\partial}{\partial z}[\Omega^2\Delta+\Psi]-2\Omega'^2\left[\frac{\partial\theta_2}{\partial x}-\frac{\partial\theta_1}{\partial y}\right]+i^2w=0\end{aligned}\right\}$$

we may eliminate their first terms by cross-differentiation, and we thus obtain

$$\left.\begin{aligned}\Omega'^2\nabla^2\theta_1+i^2\theta_1=0\\ \Omega'^2\nabla^2\theta_2+i^2\theta_2=0\\ \Omega'^2\nabla^2\theta_3+i^2\theta_3=0\end{aligned}\right\}.$$

Now these are precisely the same equations that would be obtained by cross-differentiation of (65) in § 266 (*ii.*). Hence we conclude that the rotational part of the vibrations is of the same form as if there were no Applied Forces.

Or, in other words, the forced vibrations due to a system of Applied Forces having a potential are such as to produce dilatation and shear, and any distribution of rotations which may exist is due to *superposed free vibrations* independent of Ψ.

281.] **Dilatation and Shear.** Expressing the strain components in terms of the displacement potential ϕ, equations (96) become

$$\left.\begin{aligned}\frac{\partial}{\partial x}[\Omega^2\nabla^2\phi_i+i^2\phi_i+\Psi_i]=0\\ \frac{\partial}{\partial y}[\Omega^2\nabla^2\phi_i+i^2\phi_i+\Psi_i]=0\\ \frac{\partial}{\partial z}[\Omega^2\nabla^2\phi_i+i^2\phi_i+\Psi_i]=0\end{aligned}\right\},$$

whence we deduce*

$$\Omega^2\nabla^2\phi_i+i^2\phi_i+\Psi_i=0 \qquad \text{(97)}$$

* Since we have to deal only with the derivatives of our potentials (displacements, forces, etc.), they are always indeterminate to the extent of an additive constant.

Thus, by (61) of § 124,

$$\left.\begin{aligned}\Omega^2\nabla^2\Delta_i + i^2\Delta_i + \nabla^2\Psi_i &= 0\\ \Omega^2\nabla^2 a_i + i^2 a_i + 2\frac{\partial^2\Psi_i}{\partial y\partial z} &= 0\\ \Omega^2\nabla^2 b_i + i^2 b_i + 2\frac{\partial^2\Psi_i}{\partial z\partial x} &= 0\\ \Omega^2\nabla^2 c_i + i^2 c_i + 2\frac{\partial^2\Psi_i}{\partial x\partial y} &= 0\end{aligned}\right\},$$

and consequently:

(*i.*) If Ψ satisfies

$$\nabla^2\Psi = 0, \qquad (98)$$

which is equivalent to

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0,$$

the dilatation is independent of the form of Ψ, and the *forced* vibrations are purely distortional.

(*ii.*) If Ψ satisfies

$$\frac{\partial^2\Psi}{\partial y\partial z} = \frac{\partial^2\Psi}{\partial z\partial x} = \frac{\partial^2\Psi}{\partial x\partial y} = 0 \qquad (99)$$

which are equivalent to

$$\frac{\partial X}{\partial y} = \frac{\partial X}{\partial z} = \frac{\partial Y}{\partial z} = \frac{\partial Y}{\partial x} = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial y} = 0,$$

the shears are independent of the form of Ψ, and the *forced* vibrations are purely dilatational.

282.] **Example. Radial Force.** If the force at every point be in the direction of the radius from the origin to the point, we deduce from (94) that (with the notation of § 243) Ψ is a function of r only, and

$$\Xi = \frac{d\Psi}{dr}.$$

Thus Ξ is symmetrical about the origin, and the forced vibrations will clearly be radial, so that ϕ also will be independent of θ, ω.

Thus (97) may be written

$$\frac{\Omega^2}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi_i}{dr}\right) + i^2\phi_i + \Psi_i = 0,$$

or
$$\left(\frac{d^2}{dr^2} + \frac{i^2}{\Omega^2}\right)(r\phi_i) + r\Psi_i = 0.$$

We need only concern ourselves with the Particular Integral, as the Complementary Function gives *free* vibrations. Thus

$$r\phi_i = -\frac{1}{\left(\frac{d}{dr}\right)^2 + \frac{i^2}{\Omega^2}}(r\Psi_i),$$

the symbolical solution of which gives

$$\phi_i = \frac{\Omega}{ir}\left\{\cos\frac{ir}{\Omega}\int_0^r r\sin\frac{ir}{\Omega}\Psi_i dr - \sin\frac{ir}{\Omega}\int_0^r r\cos\frac{ir}{\Omega}\Psi_i dr\right\}.$$

Hence corresponding to the force potential

$$\Psi = \Sigma(\Psi_s \sin st + \Psi_s' \cos st)$$

we have the displacement potential of forced vibrations

$$\phi = \Sigma\frac{\Omega}{sr}\left\{\cos\frac{sr}{\Omega}\int^r r\sin\frac{sr}{\Omega}(\Psi_s \sin st + \Psi_s' \cos st)dr\right.$$

$$\left. - \sin\frac{sr}{\Omega}\int^r r\cos\frac{sr}{\Omega}(\Psi_s \sin st + \Psi_s' \cos st)dr\right\}.$$

283.] **General Solution.** Equation (97), satisfied by the partial component ϕ_i, results from the decomposition of the more general equation

$$\Omega^2\nabla^2\phi - \ddot{\phi} + \Psi = 0 \qquad (100),$$

which represents the relation existing at each point of the body between the resultant displacement potential and the resultant force potential at the point. The most general solution of this equation, consistent with the assumed form (95) of Ψ, may be found as follows:—

The function Ψ is finite and continuous in value (§§ 223-228) throughout the body, though not necessarily continuous in form. Let (x', y', z') represent the coordinates of *any* point within the body, and let

$$\chi(x' - x, y' - y, z' - z, t)$$

represent any continuous function which never becomes infinite, except when

$$x' - x = y' - y = z' - z = 0.$$

Then if we assume

$$\phi = \iiint \Psi(x', y', z') \cdot \chi \cdot dx'dy'dz',$$

the integral being taken throughout the volume of the body, we may apply Boussinesq's Theorem (see § 310, below) to the differentiation of ϕ, and write

$$\frac{\partial\phi}{\partial x}=\iiint\Psi(x', y', z')\,.\,\frac{\partial\chi}{\partial x}\,.\,dx'dy'dz'$$
$$+\Psi(x, y, z)\int_0^{2\pi}\int_0^{\kappa}\{\chi(-\sqrt{\kappa^2-\eta^2},\ \eta\cos\omega,\ \eta\sin\omega,\ t)$$
$$-\chi(\sqrt{\kappa^2-\eta^2},\ \eta\cos\omega,\ \eta\sin\omega,\ t)\}\,\eta d\eta d\omega\ ;$$

where the double integral is ultimately to receive the limiting value which it assumes when $\kappa=0$.

Now assume that χ is of the form

$$\chi=\frac{1}{r}\mathrm{F}(r, t),$$

where $$r=\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}.$$

The double integral then becomes

$$\int_0^{2\pi}\int_0^{\kappa}\frac{1}{\kappa}\{\mathrm{F}(\kappa, t)-\mathrm{F}(\kappa, t)\}\,\eta d\eta d\omega$$
$$=\pi\kappa\{\mathrm{F}(\kappa, t)-\mathrm{F}(\kappa, t)\}=0\ ;$$

and consequently

$$\frac{\partial\phi}{\partial x}=\iiint\Psi(x', y', z')\,.\,\frac{\partial}{\partial x}\left[\frac{1}{r}\mathrm{F}(r, t)\right]dx'dy'dz'$$
$$=\iiint\Psi(x', y', z')\,.\,\frac{x-x'}{r^3}\left[r\frac{\partial\mathrm{F}(r, t)}{\partial r}-\mathrm{F}(r, t)\right]dx'dy'dz'.$$

Applying the same theorem to the second differentiation of ϕ, we have

$$\frac{\partial^2\phi}{\partial x^2}=\iiint\Psi(x', y', z')\,.\,\frac{\partial^2}{\partial x^2}\left[\frac{1}{r}\mathrm{F}(r, t)\right].\,dx'dy'dz'$$
$$+\Psi(x, y, z)\int_0^{2\pi}\int_0^{\kappa}\left\{\frac{\sqrt{\kappa^2-\eta^2}}{\kappa^3}\left[\kappa\frac{\partial}{\partial\kappa}\mathrm{F}(\kappa, t)-\mathrm{F}(\kappa, t)\right]\right.$$
$$\left.-\frac{-\sqrt{\kappa^2-\eta^2}}{\kappa^3}\left[\kappa\frac{\partial}{\partial\kappa}\mathrm{F}(\kappa, t)-\mathrm{F}(\kappa, t)\right]\right\}\eta d\eta d\omega.$$

The double integral is in this case

$$\frac{2}{\kappa^3}\left[\kappa\frac{\partial}{\partial\kappa}\mathrm{F}(\kappa, t)-\mathrm{F}(\kappa, t)\right]\int_0^{2\pi}\int_0^{\kappa}\sqrt{\kappa^2-\eta^2}\,.\,\eta d\eta d\omega$$
$$=\frac{4\pi}{3}\left[\kappa\frac{\partial}{\partial\kappa}\mathrm{F}(\kappa, t)-\mathrm{F}(\kappa, t)\right].$$

Thus, if we also differentiate ϕ twice as to y, and as to z, and add the symmetrical results, we have ultimately

$$\nabla^2\phi = \iiint \Psi(x', y', z')\nabla^2\left[\frac{1}{r}F(r, t)\right]dx'dy'dz'$$
$$+ 4\pi\Psi(x, y, z)\left[\kappa\frac{\partial}{\partial\kappa}F(\kappa, t) - F(\kappa, t)\right].$$

Now let

$$F(r, t) = \frac{1}{4\pi\Omega^2}\sin i\left(t - \frac{r}{\Omega}\right),$$

so that

$$\phi = \frac{1}{4\pi\Omega^2}\iiint \Psi(x', y', z')\sin i\left(t - \frac{r}{\Omega}\right)\frac{dx'dy'dz'}{r}.$$

Then, by formula (65) of § 243,

$$\nabla^2\left[\frac{1}{r}F(r, t)\right] = \frac{1}{4\pi\Omega^2 r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial}{\partial r}\left[\frac{1}{r}\sin i\left(t - \frac{r}{\Omega}\right)\right]\right\}$$
$$= -\frac{i^2}{4\pi\Omega^4}\cdot\frac{1}{r}\sin i\left(t - \frac{r}{\Omega}\right).$$

Also

$$F(\kappa, t) = \frac{1}{4\pi\Omega^2}\sin i\left(t - \frac{\kappa}{\Omega}\right),$$

and

$$\kappa\frac{\partial}{\partial\kappa}F(\kappa, t) = -\frac{i\kappa}{4\pi\Omega^3}\cos i\left(t - \frac{\kappa}{\Omega}\right).$$

Thus, proceeding to the limit in which $\kappa = 0$, we have finally

$$\nabla^2\phi = -\frac{i^2}{4\pi\Omega^4}\iiint \Psi(x', y', z')\sin i\left(t - \frac{r}{\Omega}\right)\frac{dx'dy'dz'}{r} - \frac{1}{\Omega^2}\Psi(x, y, z)\sin it.$$

But

$$\ddot{\phi} = -\frac{i^2}{4\pi\Omega^2}\iiint \Psi(x', y', z')\sin i\left(t - \frac{r}{\Omega}\right)\frac{dx'dy'dz'}{r},$$

and therefore

$$\Omega^2\nabla^2\phi - \ddot{\phi} + \Psi(x, y, z)\sin it = 0.$$

Similarly, if we assume

$$\phi = \frac{1}{4\pi\Omega^2}\iiint \Psi'(x', y', z')\cos i\left(t - \frac{r}{\Omega}\right)\frac{dx'dy'dz'}{r},$$

we find

$$\Omega^2\nabla^2\phi - \ddot{\phi} + \Psi'(x, y, z)\cos it = 0.$$

The complete solution of (100), corresponding to the force potential

$$\Psi = \Sigma(\Psi_i \,.\, \sin it + \Psi_i' \,.\, \cos it),$$

is therefore

$$\phi = \frac{1}{4\pi\Omega^2}\iiint \Sigma \left\{ \Psi_i(x', y', z') \,.\, \sin i\left(t - \frac{r}{\Omega}\right) \right.$$
$$\left. + \Psi_i'(x', y', z') \,.\, \cos i\left(t - \frac{r}{\Omega}\right) \right\} \frac{dx'dy'dz'}{r}; \quad \ldots\ldots\ldots (101)$$

and the partial components of ϕ, of order i, are hence easily shown to be

$$\left.\begin{aligned} \phi_i &= \frac{1}{4\pi\Omega^2}\iiint \left\{ \Psi_i(x', y', z') \,.\, \cos\frac{ir}{\Omega} + \Psi_i'(x', y', z') \,.\, \sin\frac{ir}{\Omega} \right\} \frac{dx'dy'dz'}{r} \\ \phi_i' &= \frac{1}{4\pi\Omega^2}\iiint \left\{ \Psi_i'(x', y', z') \,.\, \cos\frac{ir}{\Omega} - \Psi_i(x', y', z') \,.\, \sin\frac{ir}{\Omega} \right\} \frac{dx'dy'dz'}{r} \end{aligned}\right\} \ldots (102)$$

The triple integrals are in every case to be taken within the limits of the body.

284.] **Return to the Preceding Problem.** When the point (x, y, z) lies altogether outside the limits of integration: that is, when $x' - x$, $y' - y$, $z' - z$ can never vanish, and consequently χ can never become infinite: Boussinesq's formula reduces to

$$\frac{\partial}{\partial x}\iiint \Psi(x', y', z') \,.\, \chi \,.\, dx'dy'dz'$$
$$= \iiint \Psi(x', y', z') \,.\, \frac{\partial\chi}{\partial x} \,.\, dx'dy'dz'.$$

Hence we easily deduce that if the triple integrals in formulæ (101) and (102) be taken throughout any regions of space *wholly external to the body*, Ψ, Ψ' being, as before, finite and continuous functions of position, these formulæ will represent a general solution of the equations (90), (89) of irrotational free vibrations

$$\left.\begin{aligned} \Omega^2\nabla^2\phi - \ddot{\phi} &= 0 \\ \Omega^2\nabla^2\phi_i + i^2\phi_i &= 0 \end{aligned}\right\}.$$

The student will find no difficulty in proving, by direct differentiation, that the integrals (101) and (102) do satisfy these equations.

The Problem of Equilibrium under Surface Tractions only.

285.] **General Equations.** When the body is in equilibrium in a state of strain maintained by surface tractions only, equations (6), (7), (8) take the simple forms

$$\left.\begin{aligned}\frac{\partial P}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial T}{\partial z}&=0\\ \frac{\partial U}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial S}{\partial z}&=0\\ \frac{\partial T}{\partial x}+\frac{\partial S}{\partial y}+\frac{\partial R}{\partial z}&=0\end{aligned}\right\}\qquad(103)$$

$$\left.\begin{aligned}m\frac{\partial \Delta}{\partial x}+n\nabla^2 u&=0\\ m\frac{\partial \Delta}{\partial y}+n\nabla^2 v&=0\\ m\frac{\partial \Delta}{\partial z}+n\nabla^2 w&=0\end{aligned}\right\}\qquad(104)$$

$$\left.\begin{aligned}(m+n)\frac{\partial \Delta}{\partial x}-2n\left(\frac{\partial \theta_3}{\partial y}-\frac{\partial \theta_2}{\partial z}\right)&=0\\ (m+n)\frac{\partial \Delta}{\partial y}-2n\left(\frac{\partial \theta_1}{\partial z}-\frac{\partial \theta_3}{\partial x}\right)&=0\\ (m+n)\frac{\partial \Delta}{\partial z}-2n\left(\frac{\partial \theta_2}{\partial x}-\frac{\partial \theta_1}{\partial y}\right)&=0\end{aligned}\right\}\qquad(105)$$

The conditions to be satisfied over the bounding surface will take the form

$$\left.\begin{aligned}u&=u_0\\ v&=v_0\\ w&=w_0\end{aligned}\right\}\qquad(106)$$

or the form

$$\left.\begin{aligned}\lambda P+\mu U+\nu T&=F\\ \lambda U+\mu Q+\nu S&=G\\ \lambda T+\mu S+\nu R&=H\end{aligned}\right\}\qquad(107)$$

according as the values of the surface displacements or of the surface tractions are given.

286.] **The Solution Determinate.** We know from § 255 that the problem of finding a solution of (103), (104), or (105), which will satisfy (107) over the whole surface, is quite determinate as regards the strain, and therefore also as regards the

stress; while the solution in terms of the displacements is only indeterminate to the extent of an arbitrary translation and rotation of the body as a whole.

The solution of (103), (104), or (105), which satisfies (106) at all points of the surface—or, indeed, which assigns given displacements to *any three* points in the body, or on its surface, which are *not in the same straight line*—is consequently absolutely unique.

Thus, in seeking the solution of any given problem, we may avail ourselves with perfect confidence of considerations of symmetry, and all other devices which may simplify the forms of the equations, knowing that from *any solution* which satisfies all the conditions of the problem *all* other possible solutions can be deduced—even in the most general and unrestricted case—by superposition of an arbitrary displacement of the body as a whole.

287.] **Preservation of Continuity.** Finally, we may observe that the necessity of preserving the continuity of the substance of the body imposes certain restrictions upon our choice of a solution—even when continuous* in form—by which it may gain in definiteness.

For example, the radial displacement u of § 243 must vanish with r, if the origin be contained within the substance of the body; while the displacement v of § 243 must vanish with $\sin\theta$,† and the displacement u of § 244 with r, if any portion of Oz lies within the substance of the body.

It is obvious that these precautions are necessary to guard against *spherical*, *conical*, and *cylindrical* ruptures, respectively.

Example I.

288.] **Circular Cylindrical Tube under uniform internal and external normal pressures.** A shell bounded by infinitely long coaxial circular cylinders, of radii A (internal) and B (external), is subjected to a uniform normal pressure Π over the whole of its inner surface, and a uniform normal pressure Π' over the whole of its outer surface. Required the distribution of strain.

The symmetry of the conditions leads us to expect that the displacement of every point in the shell will be wholly radial: that is, in the direction of the straight line drawn from the point perpendicular to the axis; and also that the magnitude and sign

* See §§ 223-228 for the restrictions imposed upon discontinuous solutions.
† See § 272 for an example.

of this displacement will be the same for all points situated on any circular cylindrical surface coaxial with the bounding surfaces of the tube.

Taking the axis of the tube for the axis of z, and choosing arbitrarily the origin and axes of x and y, we will then assume, with the notation of § 244, that $v=w=0$, and that u is independent of θ and z.

On this assumption we have

$$\left.\begin{aligned}\Delta &= \frac{1}{r}\frac{d}{dr}(ur)\\ \Theta_1 &= \Theta_2 = \Theta_3 = 0\end{aligned}\right\},$$

and, on substitution in equations (88) of § 244,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ur)\right] = 0.$$

Integrating this equation twice,

$$u = Cr + \frac{C'}{r};$$

where C, C' are arbitrary constants. In this case both terms are admissible (§ 287), because Oz is not within the substance of the body.

At the inner surface we have

$$r = A,\ \frac{d\Phi}{dr} = -1,\ \Xi' = \Pi;$$

and at the outer surface

$$r = B,\ \frac{d\Phi}{dr} = 1,\ \Xi' = -\Pi'.$$

Hence by equations (89) of § 244,

$$\left.\begin{aligned}P &= -\Pi, \text{ when } r = A\\ P &= -\Pi', \text{ when } r = B\end{aligned}\right\}.$$

But

$$\left.\begin{aligned}e &= \frac{du}{dr} = C - \frac{C'}{r^2}\\ f &= \frac{u}{r} \ \ = C + \frac{C'}{r^2}\end{aligned}\right\},$$

and therefore

$$\begin{aligned}P &= (m+n)\left(C - \frac{C'}{r^2}\right) + (m-n)\left(C + \frac{C'}{r^2}\right)\\ &= 2mC - \frac{2nC'}{r^2}.\end{aligned}$$

Thus the boundary conditions become

$$\left.\begin{aligned}\frac{2nC'}{A^2} - 2mC &= \Pi \\ \frac{2nC'}{B^2} - 2mC &= \Pi'\end{aligned}\right\},$$

and consequently

$$\left.\begin{aligned}C &= \frac{A^2\Pi - B^2\Pi'}{2m(B^2 - A^2)} \\ C' &= \frac{A^2B^2(\Pi - \Pi')}{2n(B^2 - A^2)}\end{aligned}\right\}.$$

Substituting for C and C', we have finally

$$u = \frac{(A^2\Pi - B^2\Pi')r}{2m(B^2 - A^2)} + \frac{A^2B^2(\Pi - \Pi')}{2n(B^2 - A^2)r} \quad\text{.................(108)}$$

If $\Pi - \Pi'$ and $A^2\Pi - B^2\Pi'$ be of opposite signs: that is, if

$$B^2/A^2 > \Pi/\Pi' > 1:$$

u will be zero when

$$r = \sqrt{\frac{m}{n} \cdot \frac{A^2B^2(\Pi - \Pi')}{B^2\Pi' - A^2\Pi}} \quad\text{.........................(109)}$$

In order however that u may vanish at any points *within the substance of the tube*, we must impose the further restriction that this value of r shall be between A and B. The necessary and sufficient conditions are

$$\frac{mA^2 + nB^2}{(m+n)A^2} > \frac{\Pi}{\Pi'} > \frac{(m+n)B^2}{mB^2 + nA^2};$$

and if these be fulfilled, the cylindrical surface described in the body with the above radius will retain its form and dimensions unaltered; the inner and outer shells into which it divides the tube being compressed upon it from either side.

If

$$\frac{\Pi}{\Pi'} = \frac{(m+n)B^2}{mB^2 + nA^2}$$

the inner surface of the tube retains its natural dimensions, and if

$$\frac{\Pi}{\Pi'} = \frac{mA^2 + nB^2}{(m+n)A^2}$$

the outer surface does so.

289.] **Principal Stresses. Lines of Stress.** It is obvious that equations (91) of § 244 are satisfied identically; so that r, θ, z are the principal coördinates of the strain. The principal stresses are by (55) of § 241

$$\left.\begin{aligned} N_1 &= -\frac{B^2\Pi'(r^2-A^2)+A^2\Pi(B^2-r^2)}{(B^2-A^2)r^2} \\ N_2 &= -\frac{(B^2\Pi'-A^2\Pi)r^2-A^2B^2(\Pi-\Pi')}{(B^2-A^2)r^2} \\ N_3 &= -\frac{(m-n)(B^2\Pi'-A^2\Pi)}{m(B^2-A^2)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(110)$$

The corresponding Lines of Stress (§ 216) are respectively—

(1) those portions of the radii drawn perpendicular to the axis which are intercepted within the substance of the tube;

(2) circles in planes perpendicular to the axis, and having their centres in the axis;

(3) straight lines parallel to the axis.

These three systems we shall refer to as the *radial, circular*, and *longitudinal* systems respectively.

Since $B>r>A$, it is evident that N_1 is always negative, and consequently all the *radial* stress lines are *Struts* (§ 216) throughout their length. The pressure transmitted by these lines increases or decreases continuously from the limit Π at the inner surface to the limit Π' at the outer surface.

The stress N_3, transmitted along the longitudinal stress lines, is constant, and its sign depends only on that of $B^2\Pi'-A^2\Pi$. Thus these lines are *Struts* or *Ties* according as

$$\frac{\Pi}{\Pi'} \lessgtr \frac{B^2}{A^2}.$$

In the limiting case, in which

$$\frac{\Pi}{\Pi'} = \frac{B^2}{A^2},$$

these are *lines of zero stress*, and the stress, as well as the strain, is in two dimensions.

Since dN_2/dr is negative, the third principal stress *regarded as a pressure* increases continuously with r. Thus, if N_2 is a pressure at the inner surface, it will be a pressure everywhere; while, if it is a traction at the outer surface, it will be a traction everywhere.

Hence we deduce that, if

$$\frac{\Pi}{\Pi'} < \frac{2B^2}{A^2+B^2},$$

the *circular* lines of stress are *Struts* throughout, transmitting a pressure which increases with their radius. But, if

$$\frac{\Pi}{\Pi'} > \frac{A^2 + B^2}{2A^2},$$

these lines are *Ties* throughout the body, transmitting a traction which diminishes as their radius increases.

Finally, if the pressure-ratio falls within these limits: that is, if

$$\frac{A^2 + B^2}{2A^2} > \frac{\Pi}{\Pi'} > \frac{2B^2}{A^2 + B^2}:$$

the stress transmitted along the circular lines of stress will be a traction at the inner surface, and a pressure at the outer surface, vanishing and changing sign when

$$r = \sqrt{\frac{A^2 B^2 (\Pi - \Pi')}{B^2 \Pi' - A^2 \Pi}}\,;\qquad\qquad(111)$$

so that all the circular stress lines with this radius will be *lines of zero stress.*

It may be observed that the limits for the existence of a *cylinder of zero circular stress*, fall within the limits (108), for the existence of a *cylinder of zero radial displacement;* the two cylinders do not however coincide, as will appear on comparing their radii, given by (111) and (109).

290.] **Strength of the Tube.** It would be interesting to discuss the various ways in which the perfect elasticity of the tube may be endangered, by approach of one or other of the principal stresses, at the point where it is greatest, to the elastic strength of the material under tension or compression. We must confine ourselves here however to a single example.

Let $\Pi/\Pi' > B^2/A^2$. Then the *radial pressure* has its maximum value Π when $r = A$, the *circular traction* has its maximum value

$$\frac{(A^2 + B^2)\Pi - 2B^2\Pi'}{B^2 - A^2}$$

when $r = A$, and the *longitudinal traction* has the uniform value

$$\frac{(m - n)(A^2\Pi - B^2\Pi')}{m(B^2 - A^2)}.$$

The second of these is the greatest, so that if the elastic strength of the material be about the same for tension and compression [Table C (*bis*), p. 202], the first yielding of the tube will take the form of transverse stretching, or increase of its diameter beyond its power of elastic recovery.

If $\mathbf{T}$ be the *elastic* strength of the material under tension, the condition for elastic safety is

$$\frac{(A^2+B^2)\Pi - 2B^2\Pi'}{B^2-A^2} < \mathbf{T};$$

so that we have, as guides for the proper dimensions of the tube, when the pressures to which it is to be subjected are known,

$$\frac{\Pi}{\Pi'} > \frac{B^2}{A^2} > \frac{\Pi+\mathbf{T}}{2\Pi'-\Pi+\mathbf{T}}.$$

291.] **Application to cylindrical boilers.** The case which we have just considered—when the ratio Π/Π' is considerable, especially in comparison with B^2/A^2—may be taken fairly to represent the strain suffered by a long cylindrical boiler (except in the neighbourhood of its ends). Thus, if $\mathbf{T}$ represents the *working strength* of the material [Table (D), p. 203] which allows for a large "factor of safety," the proper thickness t for a boiler of internal radius A, to be worked at steam pressure Π under atmospheric pressure Π' will, with due regard to economy of material, be given by

$$\frac{(t+A)^2}{A^2} = \frac{\Pi+\mathbf{T}}{2\Pi'+\mathbf{T}-\Pi}.$$

Example. It is required to determine the proper thickness for a cylindrical wrought iron boiler, 4 feet in diameter, to be worked at a maximum pressure of 120 pounds to the square inch in the open air.

The working strength of wrought iron is given in Table (D) at 4·5 tons to the square inch, and the atmospheric pressure may be taken at about 15 pounds per square inch. Thus, reducing lengths to inches, and stresses to pounds per square inch, we have

$$\begin{aligned} A &= 24 \\ \Pi &= 120 \\ \Pi' &= 15 \\ \mathbf{T} &= 10080, \end{aligned}$$

and consequently the thickness in inches is given by

$$(t+24)^2 = (24)^2 \cdot \frac{120+10080}{30+10080-120}$$

or

$$t = 24\left(\sqrt{\frac{340}{333}} - 1\right) = \cdot 509.$$

The employment of "half-inch plate" for the construction of such a boiler will therefore allow an ample factor of safety, to guard against the danger of accidental rise of pressure. In fact, a boiler so constructed would not begin to *give* until the steam pressure had risen to over 500 pounds per square inch: always supposing that the portions near the ends were able to sustain as great a stress as the middle portion.

Example II.

292.] **Circular Cylindrical Shear.** A body, bounded by two coaxial circular cylinders of infinite length, has its inner surface (radius A) rigidly attached to an immoveable cylinder of the same radius; while its external surface (radius B) is subjected to a uniform *tangential* traction $\mathbf{F}$, everywhere perpendicular to the axis of the cylinder. Required the nature of the strain produced.

In this example, as in the last, the conditions present complete symmetry about the axis, and complete uniformity in the direction of the axis. It is therefore natural to assume that the resultant displacement of each point is in the plane, perpendicular to the axis, which contains the point, and that the amount of this displacement depends only on the distance of the point from the axis.

Thus, with the notation of § 244, we shall assume that $w=0$, and that u and v (and therefore also β) are independent of θ and z. We have then

$$\left.\begin{aligned} \Delta &= \frac{1}{r}\frac{d}{dr}(ur) \\ \Theta_1 &= \Theta_2 = 0 \\ 2\Theta_3 &= \frac{1}{r}\frac{d}{dr}(vr) \end{aligned}\right\},$$

and, on substitution in (88) of § 244,

$$\frac{d\Delta}{dr} = \frac{d\Theta_3}{dr} = 0.$$

Integrating, we get

$$\left.\begin{aligned} u &= Cr + \frac{C'}{r} \\ v &= Dr + \frac{D'}{r} \end{aligned}\right\}.$$

The given boundary conditions are partly of the one type, and partly of the other [§ 285, (106), (107)]; for when $r=A$, we are to have $u=v=0$: and when $r=B$, $P=0$, $U=\mathbf{F}$.

The first two conditions give

$$CA + \frac{C'}{A} = 0,\ DA + \frac{D'}{A} = 0\ ;$$

and on substitution from (85) of § 244 in (46) of § 239, the latter conditions become

$$2mCB - \frac{2nC'}{B} = 0,\ -\frac{2nD'}{B^2} = \mathbf{F}.$$

Thus

$$\left.\begin{aligned} C = C' = 0 \\ A^2D = -D' = B^2\mathbf{F}/2n \end{aligned}\right\},$$

and, finally, $u = 0$ and

$$v = \frac{B^2r}{2n}\left(\frac{1}{A^2} - \frac{1}{r^2}\right)\mathbf{F}.$$

Each cylindrical surface in the body coaxial with the bounding surfaces is therefore simply rotated about the axis through an angle

$$\beta = \frac{B^2}{2n}\left(\frac{1}{A^2} - \frac{1}{r^2}\right)\mathbf{F},$$

where r is its radius, without any change in its form or dimensions. The amount of rotation increases from within outwards, and the strain amounts to a *circular shearing motion* (in planes perpendicular to the axis) of cylindrical layers of the body, without any changes of density.

Each line in the body parallel to the axis is shifted as a whole, parallel to itself, while each radial line is distorted into a hyperbolic form. For instance, the radius of the shell which initially coincides with the axis of x assumes the curve

$$xy = \frac{B^2\mathbf{F}}{2n}\left(\frac{x^2}{A^2} - 1\right)$$

which is a hyperbola, having for its asymptotes the lines

$$x = 0,\ y = B^2\mathbf{F}x/2n.$$

Since the strain is supposed small, $\mathbf{F}$ will be very small compared with n, and the hyperbolas will be nearly rectangular, as well as of very small curvature in the portion intercepted by the shell.

In Figure 34, the dotted lines represent the above hyperbola and its asymptotes, the portion distinguished by an unbroken line being the strained form of the radius of the shell initially coinciding with Ox. This figure is drawn for an exaggerated case, in which $\mathbf{F} = {\cdot}00523\ n$.

293.] **Lines of Stress.** Since

$$P = Q = R = S = T = 0,$$

and
$$U = nr\frac{d\beta}{dr} = \frac{B^2\mathbf{F}}{r^2},$$

all lines in the body parallel to the axis are *Lines of zero stress*, and the two principal stresses in any plane perpendicular to the axis are the remaining roots of the discriminating cubic (21) of § 163, which here reduces to

$$\phi^2 - U^2 = 0.$$

Thus
$$N_1 = -N_2 = U = B^2\mathbf{F}/r^2.$$

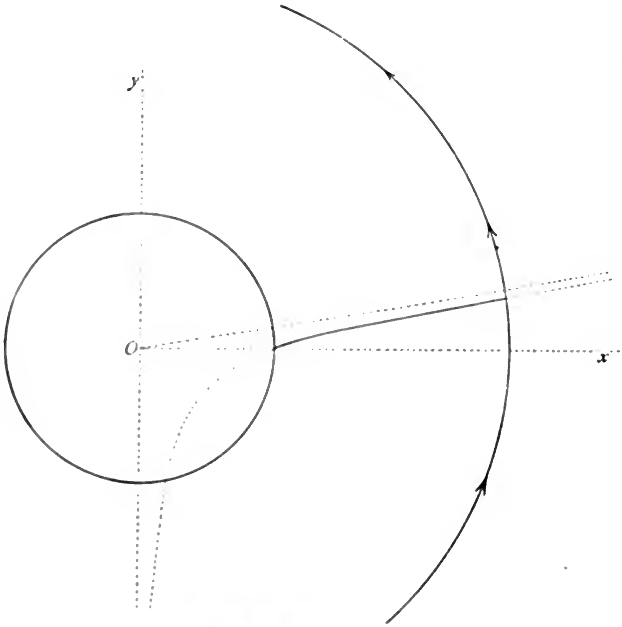


Fig.34

In the system of coördinates which we are now employing, the directions of the "axes of reference" of § 163, at each point of the body, are those of the elementary lines dr, $rd\theta$, dz: thus, if ds be an element of a Line of Stress, and λ, μ, ν the cosines of the angles which it makes with the coördinate elements, we have

$$\lambda ds = dr, \quad \mu ds = rd\theta, \quad \nu ds = dz,$$

X

and the differential equations of the Line of Stress corresponding to the principal stress N (see note on § 241, at end of the volume) are

$$\frac{Urd\theta}{dr}=\frac{Udr}{rd\theta}=N;\ dz=0.$$

The differential equation of the *Tie Lines*, transmitting the traction U, is therefore

$$dr=rd\theta,$$

and that of the *Strut Lines* transmitting the pressure U is

$$dr=-rd\theta.$$

Thus the Ties are the equiangular spirals

$$r=C\mathrm{e}^{\theta},$$

and the Struts the similar spirals

$$r=C\mathrm{e}^{-\theta},$$

each system cutting all radii at the constant angle $\pi/4$, while the traction or pressure transmitted along each diminishes, as the inverse square of the distance from the pole, from $B^2\mathbf{F}/A^2$ at the inner surface to $\mathbf{F}$ at the outer surface of the shell.

In Figure 35, the whole lines represent the Ties, and the dotted lines the Struts; if these are studied in connection with the direction of the Surface Tractions (indicated by the arrows), the simultaneous *dragging* and *squeezing* effects of the latter will readily be understood.

The traction exerted on the inner surface by the fixed cylindrical core is equal to the value of U when $r=A$; it is therefore

$$\mathbf{F}'=\frac{B^2}{A^2}\mathbf{F}.$$

This is otherwise obvious; for, in order that equilibrium may be possible, the external couples on the body must balance one another, precisely as if it were rigid (§ 146). Thus, considering a unit length of the shell, we must have

$$\mathbf{F}'\,.\,A\,.\,2\pi A=\mathbf{F}\,.\,B\,.\,2\pi B,$$

or

$$\mathbf{F}'\,.\,A^2=\mathbf{F}\,.\,B^2.$$

Example III.

294.] **Spherical Shell under internal and external normal pressures whose intensities vary directly as the**

distance of the point of application from a given diametral plane. A spherical shell suffers a normal pressure $\Pi \cos\theta$ over its inner surface (radius A), and a normal pressure $\Pi' \cos\theta$ over its outer surface (radius B); θ being the angle which the radius vector of any point makes with the given diameter Oz, and Π, Π' being constants. Required the conditions of equilibrium, and the nature of the strain produced.

Adopting the notation of § 243, it is evident that the conditions are symmetrical about Oz, so that the displacement of every point will take place in the plane which contains Oz and the point, and the strain will be altogether independent of ω.

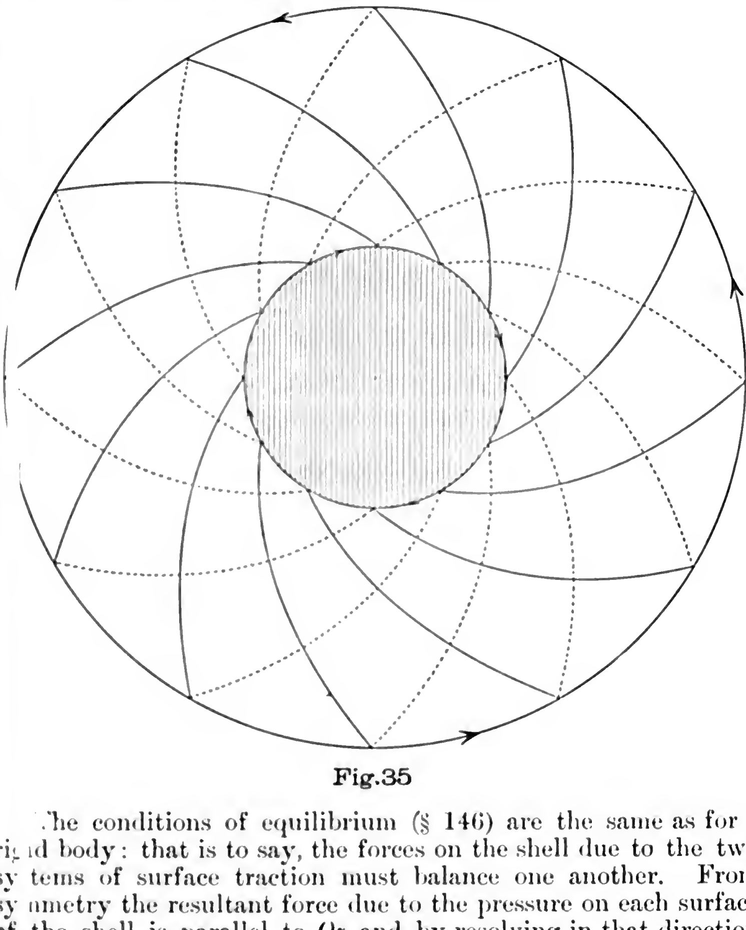


Fig.35

The conditions of equilibrium (§ 146) are the same as for a rigid body: that is to say, the forces on the shell due to the two systems of surface traction must balance one another. From symmetry the resultant force due to the pressure on each surface of the shell is parallel to Oz, and by resolving in that direction

the force on each element of surface we find the condition of equilibrium to be

$$\int_0^\pi \Pi \cos\theta \,.\, \cos\theta \,.\, 2\pi A^2 \sin\theta d\theta = \int_0^\pi \Pi' \cos\theta \,.\, \cos\theta \,.\, 2\pi B^2 \sin\theta d\theta$$

or

$$\Pi A^2 = \Pi' B^2 \quad\text{.................................}\quad (112)$$

Assuming that this necessary condition is satisfied, we have by (68) and (69) of § 243

$$\left.\begin{aligned} \Delta &= \frac{1}{r^2}\frac{\partial}{\partial r}(ur^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta) \\ \Theta_1 &= \Theta_2 = 0 \\ 2\Theta_3 &= \frac{1}{r}\frac{\partial}{\partial r}(vr) - \frac{1}{r}\frac{\partial u}{\partial\theta} \end{aligned}\right\} \quad\text{..... ....}\quad (113)$$

while the general equations of equilibrium (71) become

$$\left.\begin{aligned} (m+n)\frac{\partial\Delta}{\partial r} - \frac{2n}{r\sin\theta}\frac{\partial}{\partial\theta}(\Theta_3\sin\theta) = 0 \\ (m+n)\frac{\partial\Delta}{\partial\theta} + 2n\frac{\partial}{\partial r}(\Theta_3 r) = 0 \end{aligned}\right\} \quad\text{...............}\quad (114)$$

The boundary conditions (72) reduce to

$$P = -\Pi\cos\theta,\ U = 0,\ \text{when } r = A,$$
$$P = -\Pi'\cos\theta,\ U = 0,\ \text{when } r = B;$$

and on substitution from (68) these become

$$\left.\begin{aligned} (m-n)\Delta + 2n\frac{\partial u}{\partial r} + \Pi\cos\theta = 0 \\ \text{when } r = A \end{aligned}\right\} \quad\text{..................}\quad (115)$$

$$\left.\begin{aligned} m-n)\Delta + 2n\frac{\partial u}{\partial r} + \Pi'\cos\theta = 0 \\ \text{when } r = B \end{aligned}\right\} \quad\text{..................}\quad (116)$$

and

$$\left.\begin{aligned} r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial\theta} = 0 \\ \text{when } r = A \text{ or } B \end{aligned}\right\} \quad\text{.........................}\quad (117)$$

Finally, by § 287, we must have

$$v = 0,\ \text{when } \sin\theta = 0 \quad\text{.......................}\quad (118)$$

Now equations (114) may be written

$$\left.\begin{aligned}(m+n)r^2\sin\theta\frac{\partial\Delta}{\partial r}-2n\frac{\partial}{\partial\theta}(\Theta_3 r\sin\theta)=0\\ (m+n)\sin\theta\frac{\partial\Delta}{\partial\theta}+2n\frac{\partial}{\partial r}(\Theta_3 r\sin\theta)=0\end{aligned}\right\},$$

and on elimination of Θ_3 we have

$$\frac{\partial}{\partial r}\left[r^2\sin\theta\frac{\partial\Delta}{\partial r}\right]+\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Delta}{\partial\theta}\right]=0,$$

or

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Delta}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Delta}{\partial\theta}\right)=0.$$

Comparing this equation with (65) of § 243 we see that

$$\nabla^2\Delta=0^*\text{..............................}(119)$$

It also appears from the boundary conditions (115) and (116) that at either surface Δ must be equal to $\cos\theta$, multiplied by a constant factor. But $\cos\theta$ is a surface harmonic of order 1, and thus the solution of (119) must be the sum of two solid harmonics of orders $+1$ and -2.

Let us assume

$$\Delta=\left(Cr+\frac{D}{r^2}\right)\cos\theta\text{ ;..........................}(120)$$

then, on substitution in (114) we have

$$\left.\begin{aligned}\frac{\partial}{\partial\theta}(2\Theta_3\,.\sin\theta)&=\frac{m+n}{n}\left(Cr-\frac{2D}{r^2}\right)\sin\theta\cos\theta\\ \frac{\partial}{\partial r}(2\Theta_3\,.\,r)&=\frac{m+n}{n}\left(Cr+\frac{D}{r^2}\right)\sin\theta\end{aligned}\right\},$$

and the solution of these equations is obviously

$$2\Theta_3=\frac{m+n}{n}\left(\frac{Cr}{2}-\frac{D}{r^2}\right)\sin\theta\text{......................}(121)$$

Again, substituting from (120) and (121) in (113),

$$\left.\begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}(ur^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta)&=\left(Cr+\frac{D}{r^2}\right)\cos\theta\\ \frac{1}{r}\frac{\partial}{\partial r}(vr)-\frac{1}{r}\frac{\partial u}{\partial\theta}&=\frac{m+n}{n}\left(\frac{Cr}{2}-\frac{D}{r^2}\right)\sin\theta\end{aligned}\right\},$$

* This equation is satisfied by Δ in all cases in which there are no applied forces, as may be deduced directly from equations (104) above. See Article 295 below.

and the form of these equations, in connection with the boundary conditions (115), (116), (117) suggests that we should assume for the form of u and v

$$u = \mathrm{u} \cos\theta, \quad v = \mathrm{v} \sin\theta,$$

where u and v are functions of r only. The general equations to be satisfied by u and v then become

$$\left.\begin{aligned} \frac{1}{r^2}\frac{d}{dr}(\mathrm{u}r^2) + \frac{2\mathrm{v}}{r} &= Cr + \frac{D}{r^2} \\ \frac{1}{r}\frac{d}{dr}(\mathrm{v}r) + \frac{\mathrm{u}}{r} &= \frac{m+n}{n}\left(\frac{Cr}{2} - \frac{D}{r^2}\right) \end{aligned}\right\}.$$

These are easily put into the form

$$\left.\begin{aligned} 2\mathrm{v}r &= Cr^3 + D - \frac{d}{dr}(\mathrm{u}r^2) \\ \frac{d}{dr}(2\mathrm{v}r) &= \frac{m+n}{n}\left(Cr^2 - \frac{2D}{r}\right) - 2\mathrm{u} \end{aligned}\right\} \quad \text{.....................(122)}$$

and on elimination of $2\mathrm{v}r$, we have

$$\frac{d^2}{dr^2}(\mathrm{u}r^2) - 2\mathrm{u} = \frac{2n-m}{n}Cr^2 + \frac{2(m+n)}{n}\frac{D}{r},$$

the complete integral of which is

$$\mathrm{u} = \frac{2n-m}{10n}Cr^2 - \frac{m+n}{n}\frac{D}{r} + C' - \frac{D'}{r^3},$$

where C' and D' are arbitrary constants. The first of equations (122) then gives at once

$$\mathrm{v} = \frac{2m+n}{10n}Cr^2 + \frac{m+2n}{2n}\frac{D}{r} - C' - \frac{D'}{2r^3}.$$

Thus, finally, the radial and transverse displacements are

$$\left.\begin{aligned} u &= \left[\frac{2n-m}{10n}Cr^2 - \frac{m+n}{n}\frac{D}{r} + C' - \frac{D'}{r^3}\right]\cos\theta \\ v &= \left[\frac{2m+n}{10n}Cr^2 + \frac{m+2n}{2n}\frac{D}{r} - C' - \frac{D'}{2r^3}\right]\sin\theta \end{aligned}\right\} \quad \text{..........(123)}$$

The four arbitrary constants are to be evaluated by means of the four boundary conditions (115), (116), (117); it being obvious that (118) is satisfied identically.

Taking first (117) we have, when $r=A$, and when $r=B$,

$$\frac{3k}{10n}C-\frac{D}{r^3}+\frac{3D'}{r^5}=0.$$

Therefore

$$\left.\begin{aligned}3kA^5C-10n(A^2D-3D')=0\\ 3kB^5C-10n(B^2D-3D')=0\end{aligned}\right\}.$$

Next from (115)

$$3kA^5C+5(3m+n)A^2D+30nD'+5A^4\Pi=0\,;$$

and similarly from (116)

$$3kB^5C+5(3m+n)B^2D+30nD'+5B^4\Pi'=0,$$

where by (112)

$$A^2\Pi=B^2\Pi'.$$

Thus

$$\left.\begin{aligned}C&=\frac{10n(B^2-A^2)A^2\Pi}{9k(m+n)(B^5-A^5)}\\ D&=\frac{A^2\Pi}{3(m+n)}\\ D'&=\frac{(B^3-A^3)A^4B^2\Pi}{9(m+n)(B^5-A^5)}\end{aligned}\right\}\quad\text{.........................(124)}$$

It will be observed that C' does not appear in the boundary equations, and is consequently indeterminate. The reason is that the displacement whose components are

$$u=C'\cos\theta,\ v=-C'\sin\theta$$

amounts merely to a bodily translation of the shell through a distance C' in the positive direction of Oz. This term consequently contributes nothing to the strain, and we may put $C'=0$.

Substituting from (124) in (123), we have finally

$$\left.\begin{aligned}u&=\left[\frac{(2n-m)(B^2-A^2)A^2\Pi r^2}{9k(m+n)(B^5-A^5)}-\frac{A^2\Pi}{3nr}-\frac{(B^3-A^3)A^4B^2\Pi}{9(m+n)(B^5-A^5)r^3}\right]\cos\theta\\ v&=\left[\frac{(2m+n)(B^2-A^2)A^2\Pi r^2}{9k(m+n)(B^5-A^5)}+\frac{(m+2n)A^2\Pi}{6n(m+n)r}-\frac{(B^3-A^3)A^4B^2\Pi}{18(m+n)(B^5-A^5)r^3}\right]\sin\theta\end{aligned}\right\}(125)$$

To investigate the deformation suffered by the body: we have

$$u=\mathrm{u}\cos\theta,\ v=\mathrm{v}\sin\theta,$$

where u, v are functions of r, of the order Π/n, so that u^2 and v^2

are negligible in comparison with x^2, etc. If (x', y', z') be the strained position of the point initially at (x, y, z)

$$\begin{aligned}\sqrt{x'^2+y'^2} &= \sqrt{x^2+y^2} + u\sin\theta + v\cos\theta\\ &= \sqrt{x^2+y^2} + (\mathrm{u}+\mathrm{v})\sin\theta\cos\theta\\ z' &= z + u\cos\theta - v\sin\theta\\ &= z + \mathrm{u}\cos^2\theta - \mathrm{v}\sin^2\theta\\ &= z + \mathrm{u} - (\mathrm{u}+\mathrm{v})\sin^2\theta.\end{aligned}$$

Now to a first approximation (see § 68) we may regard the coefficients of u and v as functions of the initial or final coördinates, indifferently. Thus we may write

$$\sqrt{x^2+y^2} = \sqrt{x'^2+y'^2} - (\mathrm{u}+\mathrm{v})\frac{z'\sqrt{x'^2+y'^2}}{r^2},$$

$$z = z' - \mathrm{u} + (\mathrm{u}+\mathrm{v})\frac{x'^2+y'^2}{r^2},$$

and any spherical surface

$$x^2+y^2+z^2=r^2$$

in the unstrained body, concentric with the bounding surfaces, is strained into the surface

$$(x'^2+y'^2)\left\{1-(\mathrm{u}+\mathrm{v})\frac{z'}{r^2}\right\}^2 + \left\{z' - \mathrm{u} + (\mathrm{u}+\mathrm{v})\frac{x'^2+y'^2}{r^2}\right\}^2 = r^2;$$

or (approximately) into the sphere

$$x'^2+y'^2+(z'-\mathrm{u})^2=r^2.$$

Every such sphere is therefore shifted, without change of *form*, through a distance u in the positive direction of Oz. This distance depends upon the radius of the sphere, being given by

$$\mathrm{u} = \left[\frac{(2n-m)(B^2-A^2)r^2}{9k(m+n)(B^5-A^5)} - \frac{1}{3nr} - \frac{(B^3-A^3)A^2B^2}{9(m+n)(B^5-A^5)r^3}\right]A^2\Pi.$$

It is easily shewn that u will or will not vanish for some value of r between A and B, according as the equation,

$$9(m^2+mn-n^2)x^5 + 5n^2x^3 - 3k(3m+4n) = 0$$

has or has not a real root between A/B and B/A. If there be an odd number of such roots, the bounding surfaces will be displaced in the same direction; if an even number in opposite directions.

Although, however, these spherical surfaces retain their form unaltered, yet their surfaces suffer areal dilatation or contraction (page 61), which varies from point to point so that the cones $\theta=$ constant in the unstrained body become surfaces of revolution of the sixth degree.

*Solution in terms of Spherical Harmonics.**

295.] **The Cubical Dilatation.** The general equations are

$$\left.\begin{aligned} m\frac{\partial\Delta}{\partial x} + n\nabla^2 u &= 0 \\ m\frac{\partial\Delta}{\partial y} + n\nabla^2 v &= 0 \\ m\frac{\partial\Delta}{\partial z} + n\nabla^2 w &= 0 \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(104)$$

Differentiating these as to x, y, z respectively and adding, we obtain

$$\nabla^2\Delta = 0\,;\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(126)$$

an equation which is satisfied in all cases of equilibrium under Surface Tractions only. Thus Δ is in this case always capable of being expanded in a series of Solid Spherical Harmonics.

296.] **Application to Spherical Shell.** Let us suppose the values of Δ to be given at every point of the concentric surfaces of a spherical shell, of which the internal and external radii are A and B. These values must then be capable of expansion in series of Surface Harmonics, so that we shall have, in general:

$$\left.\begin{aligned} &\text{when } r = A,\ \Delta = \sum_{i=0}^{i=\infty}(\mathbf{H}_i)\,; \\ &\text{when } r = B,\ \Delta = \sum_{i=0}^{i=\infty}(\mathbf{H}_i'). \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(127)$$

Here $\mathbf{H}_i$ and $\mathbf{H}_i'$ denote any surface harmonics of order i; and, Δ being always supposed small, these two series are necessarily convergent.

At any point within the substance of the shell, at a distance r from the centre, the value of Δ will be given by

$$\Delta = \sum_0^{\infty}\frac{(B^{i+1}\mathbf{H}_i - A^{i+1}\mathbf{H}_i')r^i - (AB)^{i+1}(A^i\mathbf{H}_i - B^i\mathbf{H}_i')r^{-i-1}}{B^{2i+1} - A^{2i+1}}\quad\ldots\ldots(128)$$

For this series obviously satisfies (126) throughout the shell, and also satisfies (127) at both its surfaces. It is also convergent for all values of r between A and B, for it may be written in the form

$$\Delta = \sum_0^{\infty}\left\{\frac{\dfrac{r^i\mathbf{H}_i}{B^i} - \left(\dfrac{A}{B}\right)^{i+1}\dfrac{r^i\mathbf{H}_i'}{B^i}}{1 - \left(\dfrac{A}{B}\right)^{2i+1}}\right\} - \sum_0^{\infty}\left\{\frac{\left(\dfrac{A}{B}\right)^i\cdot\dfrac{A^{i+1}\mathbf{H}_i}{r^{i+1}} - \dfrac{A^{i+1}\mathbf{H}_i'}{r^{i+1}}}{1 - \left(\dfrac{A}{B}\right)^{2i+1}}\right\}$$

* First worked out by Lamé, *Liouville*, 1854. The method here adopted is that of Thomson and Tait; *Natural Philosophy*, Articles 735-737.

and since $A < r < B$, the terms of these series corresponding to very great values of i are ultimately of the same order as

$$\left(\frac{r}{B}\right)^i . \mathbf{H}_i \text{ and } \left(\frac{A}{r}\right)^{i+1} . \mathbf{H}_i'.$$

Thus, the series (127) being convergent, the series in question are more rapidly convergent than the geometric series

$$\left.\begin{array}{l}\left(\frac{r}{B}\right)^i + \left(\frac{r}{B}\right)^{i+1} + \left(\frac{r}{B}\right)^{i+2} + \ldots\ldots \\ \left(\frac{A}{r}\right)^{i+1} + \left(\frac{A}{r}\right)^{i+2} + \left(\frac{A}{r}\right)^{i+3} + \ldots\ldots\end{array}\right\}$$

respectively.

By a well-known extension of Green's Theorem [Thomson and Tait, *Natural Philosophy*, Appendix A. (*e*)] the solution of (126) throughout any given space, which satisfies given arbitrary conditions all over the bounding surface or surfaces, is absolutely unique; and consequently the value (128) for Δ is perfectly determinate.

297.] **The Component Displacements.** Supposing, for the present, that the value of Δ is known for every point of the body, we obtain by substituting from (128) in (104)

$$\nabla^2 u = -\frac{m}{n}\frac{\partial}{\partial x}\sum_0^\infty \frac{(B^{i+1}\mathbf{H}_i - A^{i+1}\mathbf{H}_i')r^i - (AB)^{i+1}(A^i\mathbf{H}_i - B^i\mathbf{H}_i')r^{-i-1}}{B^{2i+1} - A^{2i+1}} \ldots (129)$$

with symmetrical formulæ for v and w.

Now, if we write

$$\mathbf{H}_i r^i = \mathbf{U}_i, \quad \mathbf{H}_i/r^{i+1} = \mathbf{U}_{-i-1},$$

$\mathbf{U}_i$ and $\mathbf{U}_{-i-1}$ are solid harmonics of orders i and $-(i+1)$ respectively; and the functions

$$\frac{\partial}{\partial x}\mathbf{U}_i, \quad \frac{\partial}{\partial x}\mathbf{U}_{-i-1},$$

involved in the above expression for $\nabla^2 u$, are solid harmonics of the orders $(i-1)$ and $-(i+2)$.

But, if U_s be any solid harmonic of order s, we deduce from formula (65) of § 243 that

$$\left.\begin{array}{l}\nabla^2(\mathbf{U}_s) = \dfrac{1}{r^2}\left\{ s(s+1)\mathbf{U}_s + \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial \mathbf{U}_s}{\partial\theta}\right) + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2\mathbf{U}_s}{\partial\omega^2}\right\} = 0 \\ \nabla^2(r^2\mathbf{U}_s) = (s+2)(s+3)\mathbf{U}_s + \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial \mathbf{U}_s}{\partial\theta}\right) + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2\mathbf{U}_s}{\partial\omega^2}.\end{array}\right\}$$

Thus

$$\nabla^2(r^2\mathbf{U}_s) = [(s+2)(s+3) - s(s+1)]\mathbf{U}_s = 2(2s+3)\mathbf{U}_s,$$

and consequently

$$\left.\begin{aligned} \nabla^2(r^2\mathbf{U}_{i-1}) &= 2(2i+1)\mathbf{U}_{i-1} \\ \nabla^2(r^2\mathbf{U}_{-i-2}) &= -2(2i+1)\mathbf{U}_{-i-2} \end{aligned}\right\}.$$

The solution of (129), and the corresponding equations for v and w, are consequently

$$\left.\begin{aligned} u &= \mathrm{u} - \frac{mr^2}{2n}\frac{\partial}{\partial x}\sum_0^\infty \frac{(B^{i+1}\mathbf{H}_i - A^{i+1}\mathbf{H}_i')r^i + (AB)^{i+1}(A^i\mathbf{H}_i - B^i\mathbf{H}_i')r^{-i-1}}{(2i+1)(B^{2i+1} - A^{2i+1})} \\ v &= \mathrm{v} - \frac{mr^2}{2n}\frac{\partial}{\partial y}\sum_0^\infty \frac{(B^{i+1}\mathbf{H}_i - A^{i+1}\mathbf{H}_i')r^i + (AB)^{i+1}(A^i\mathbf{H}_i - B^i\mathbf{H}_i')r^{-i-1}}{(2i+1)(B^{2i+1} - A^{2i+1})} \\ w &= \mathrm{w} - \frac{mr^2}{2n}\frac{\partial}{\partial z}\sum_0^\infty \frac{(B^{i+1}\mathbf{H}_i - A^{i+1}\mathbf{H}_i')r^i + (AB)^{i+1}(A^i\mathbf{H}_i - B^i\mathbf{H}_i')r^{-i-1}}{(2i+1)(B^{2i+1} - A^{2i+1})} \end{aligned}\right\} \quad (130)$$

where the complementary functions u, v, w, are solutions of the equations

$$\nabla^2\mathrm{u} = 0,\quad \nabla^2\mathrm{v} = 0,\quad \nabla^2\mathrm{w} = 0,$$

which must be so adjusted that the expressions (130) for u, v, w may satisfy identically the equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \Delta \quad \text{.........................(131)}$$

Now, if ϕ_s be any homogeneous function of (x, y, z) of degree s,

$$\frac{\partial}{\partial x}\left(r^2\frac{\partial\phi_s}{\partial x}\right) + \frac{\partial}{\partial y}\left(r^2\frac{\partial\phi_s}{\partial y}\right) + \frac{\partial}{\partial z}\left(r^2\frac{\partial\phi_s}{\partial z}\right) = r^2\nabla^2\phi_s + 2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\phi_s$$
$$= r^2\nabla^2\phi_s + 2s\phi_s$$

by Euler's theorem; and since in our case we have to deal with two homogeneous functions, of degrees i and $-(i+1)$, both of which satisfy $\nabla^2\phi = 0$, we easily deduce from (130) that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial \mathrm{u}}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} + \frac{\partial \mathrm{w}}{\partial z}$$
$$-\frac{m}{n}\sum \frac{i(B^{i+1}\mathbf{H}_i - A^{i+1}\mathbf{H}_i')r^i - (i+1)(AB)^{i+1}(A^i\mathbf{H}_i - B^i\mathbf{H}_i')r^{-i-1}}{(2i+1)(B^{2i+1} - A^{2i+1})}.$$

Substituting this and the value of Δ from (128) in (131), we obtain

$$\frac{\partial \mathrm{u}}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} + \frac{\partial \mathrm{w}}{\partial z} =$$
$$\sum_0^\infty \frac{[mi + n(2i+1)](B^{i+1}\mathbf{H}_i - A^{i+1}\mathbf{H}_i')r^i - [m(i+1) + n(2i+1)](AB)^{i+1}(A^i\mathbf{H}_i - B^i\mathbf{H}_i')r^{-i-1}}{n(2i+1)(B^{2i+1} - A^{2i+1})}$$

and consequently, if we expand u, v, w in series of solid harmonics

$$\left.\begin{aligned} \mathrm{u} &= \sum_0^\infty (\mathrm{u}_i + \mathrm{u}'_{-i-1}) \\ \mathrm{v} &= \sum_0^\infty (\mathrm{v}_i + \mathrm{v}'_{-i-1}) \\ \mathrm{w} &= \sum_0^\infty (\mathrm{w}_i + \mathrm{w}'_{-i-1}) \end{aligned}\right\}$$

we must have

$$\left.\begin{aligned} \frac{(B^{i+1}\mathbf{H}_i - A^{i+1}\mathbf{H}_i')r^i}{B^{2i+1} - A^{2i+1}} &= \frac{n(2i+1)}{mi + n(2i+1)}\left(\frac{\partial \mathrm{u}_{i+1}}{\partial x} + \frac{\partial \mathrm{v}_{i+1}}{\partial y} + \frac{\partial \mathrm{w}_{i+1}}{\partial z}\right) \\ \frac{(AB)^{i+1}(A^i\mathbf{H}_i - B^i\mathbf{H}_i')r^{-i-1}}{B^{2i+1} - A^{2i+1}} &= -\frac{n(2i+1)}{m(i+1) + n(2i+1)}\left(\frac{\partial \mathrm{u}'_{-i}}{\partial x} + \frac{\partial \mathrm{v}'_{-i}}{\partial y} + \frac{\partial \mathrm{w}'_{-i}}{\partial z}\right) \end{aligned}\right\}$$

Substituting in (130), we finally obtain, for the general harmonic solution of (104)

$$\left.\begin{aligned} w &= \sum_0^\infty \left\{ \mathrm{u}_i + \mathrm{u}'_{-i-1} - \frac{mr^2}{2}\frac{\partial}{\partial x}\left[\frac{\dfrac{\partial \mathrm{u}_i}{\partial x} + \dfrac{\partial \mathrm{v}_i}{\partial y} + \dfrac{\partial \mathrm{w}_i}{\partial z}}{m(i-1) + n(2i-1)} - \frac{\dfrac{\partial \mathrm{u}'_{-i-1}}{\partial x} + \dfrac{\partial \mathrm{v}'_{-i-1}}{\partial y} + \dfrac{\partial \mathrm{w}'_{-i-1}}{\partial z}}{m(i+2) + n(2i+3)}\right]\right\} \\ v &= \sum_0^\infty \left\{ \mathrm{v}_i + \mathrm{v}'_{-i-1} - \frac{mr^2}{2}\frac{\partial}{\partial y}\left[\frac{\dfrac{\partial \mathrm{u}_i}{\partial x} + \dfrac{\partial \mathrm{v}_i}{\partial y} + \dfrac{\partial \mathrm{w}_i}{\partial z}}{m(i-1) + n(2i-1)} - \frac{\dfrac{\partial \mathrm{u}'_{-i-1}}{\partial x} + \dfrac{\partial \mathrm{v}'_{-i-1}}{\partial y} + \dfrac{\partial \mathrm{w}'_{-i-1}}{\partial z}}{m(i+2) + n(2i+3)}\right]\right\} \\ u &= \sum_0^\infty \left\{ \mathrm{w}_i + \mathrm{w}'_{-i-1} - \frac{mr^2}{2}\frac{\partial}{\partial z}\left[\frac{\dfrac{\partial \mathrm{u}_i}{\partial x} + \dfrac{\partial \mathrm{v}_i}{\partial y} + \dfrac{\partial \mathrm{w}_i}{\partial z}}{m(i-1) + n(2i-1)} - \frac{\dfrac{\partial \mathrm{u}'_{-i-1}}{\partial x} + \dfrac{\partial \mathrm{v}'_{-i-1}}{\partial y} + \dfrac{\partial \mathrm{w}'_{-i-1}}{\partial z}}{m(i+2) + n(2i+3)}\right]\right\} \end{aligned}\right\} \quad (132$$

where $\mathrm{u}_i \ldots\ldots \mathrm{w}'_{-i-1}$ may be any solid spherical harmonics of their respective orders which satisfy the conditions of § 287, viz.:—

$$\frac{ux + vy}{\sqrt{x^2 + y^2}} = 0, \text{ when } x = y = 0\,;$$

$$\frac{vy + wz}{\sqrt{y^2 + z^2}} = 0, \text{ when } y = z = 0\,;$$

$$\frac{wz + ux}{\sqrt{z^2 + x^2}} = 0, \text{ when } z = x = 0.$$

This complete solution is of course only adapted to a solid of finite extent which does not include the origin—such as the spherical shell with which we started.

For a solid sphere with its centre at the origin we must retain only the harmonics of positive orders, and for an infinitely extended body with such a sphere hollowed out in its substance only the harmonics of negative orders.

Finally, we may add to the solution (132) for u, v, w complementary terms of the form

$$\frac{\partial\phi}{\partial x}, \quad \frac{\partial\phi}{\partial y}, \quad \frac{\partial\phi}{\partial z},$$

respectively, where ϕ is any solution whatever of the equation

$$\nabla^2\phi = 0$$

that satisfies the conditions of § 287; for obviously any such function will disappear on substitution in equations (104).

The most general and complete forms of the solution, for a spherical shell with either the surface *displacements* or the surface *tractions* given over both its surfaces, will be found in Thomson and Tait's *Natural Philosophy*, §§ 736, 737. We shall confine ourselves here to the simpler case of the solid sphere.

298.] **Complete Solution for a Solid Sphere with surface displacements given.** Let A be the radius of the sphere, and let the component displacements at each point of its surface be given by

$$u_0 = \Sigma(\mathbf{H}_i), \quad v_0 = \Sigma(\mathbf{H}_i'), \quad w_0 = \Sigma(\mathbf{H}_i'').$$

Selecting from the general solution (132) the positive harmonic terms, and adding the arbitrary complementary functions, we have

$$\left.\begin{aligned} u &= \frac{\partial\phi}{\partial x} + \Sigma\left[\mathrm{u}_i - M_i r^2 \frac{\partial\psi_{i-1}}{\partial x}\right] \\ v &= \frac{\partial\phi}{\partial y} + \Sigma\left[\mathrm{v}_i - M_i r^2 \frac{\partial\psi_{i-1}}{\partial y}\right] \\ w &= \frac{\partial\phi}{\partial z} + \Sigma\left[\mathrm{w}_i - M_i r^2 \frac{\partial\psi_{i-1}}{\partial z}\right] \end{aligned}\right\} \quad \text{.....................(133)}$$

where

$$\left.\begin{aligned} M_i &= \frac{m}{2[m(i-1) + n(2i-1)]} \\ \psi_{i-1} &= \frac{\partial \mathrm{u}_i}{\partial x} + \frac{\partial \mathrm{v}_i}{\partial y} + \frac{\partial \mathrm{w}_i}{\partial z} \end{aligned}\right\} \quad \text{.....................(134)}$$

and ϕ satisfies $\nabla^2\phi = 0$.

Thus we may legitimately assume

$$\phi = A^2 \Sigma(M_i \psi_{i-1});$$

and we shall then have at the surface of the sphere ($r = A$)

$$\left.\begin{aligned} u &= \left(\frac{A}{r}\right)^i \mathrm{u}_i \\ v &= \left(\frac{A}{r}\right)^i \mathrm{v}_i \\ w &= \left(\frac{A}{r}\right)^i \mathrm{w}_i \end{aligned}\right\}.$$

Thus all the conditions of the problem are satisfied by making

$$\left.\begin{aligned} \mathrm{u}_i &= \left(\frac{r}{A}\right)^i \mathbf{H}_i \\ \mathrm{v}_i &= \left(\frac{r}{A}\right)^i \mathbf{H}_i' \\ \mathrm{w}_i &= \left(\frac{r}{A}\right)^i \mathbf{H}_i'' \end{aligned}\right\},$$

and consequently

$$\psi_{i-1} = \frac{1}{A^i}\left[\frac{\partial}{\partial x}(r^i \mathbf{H}_i) + \frac{\partial}{\partial y}(r^i \mathbf{H}_i') + \frac{\partial}{\partial z}(r^i \mathbf{H}_i'')\right];$$

and on substituting these values in the general formulæ we finally obtain the complete solution

$$\left.\begin{aligned} \mathrm{u} &= \Sigma\left\{\left(\frac{r}{A}\right)^i \mathbf{H}_i + \frac{m(A^2-r^2)\dfrac{\partial}{\partial x}\left[\dfrac{\partial}{\partial x}(r^i\mathbf{H}_i) + \dfrac{\partial}{\partial y}(r^i\mathbf{H}_i') + \dfrac{\partial}{\partial z}(r^i\mathbf{H}_i'')\right]}{2[m(i-1)+n(2i-1)]A^i}\right\} \\ \mathrm{v} &= \Sigma\left\{\left(\frac{r}{A}\right)^i \mathbf{H}_i' + \frac{m(A^2-r^2)\dfrac{\partial}{\partial y}\left[\dfrac{\partial}{\partial x}(r^i\mathbf{H}_i) + \dfrac{\partial}{\partial y}(r^i\mathbf{H}_i') + \dfrac{\partial}{\partial z}(r^i\mathbf{H}_i'')\right]}{2[m(i-1)+n(2i-1)]A^i}\right\} \\ \mathrm{w} &= \Sigma\left\{\left(\frac{r}{A}\right)^i \mathbf{H}_i'' + \frac{m(A^2-r^2)\dfrac{\partial}{\partial z}\left[\dfrac{\partial}{\partial x}(r^i\mathbf{H}_i) + \dfrac{\partial}{\partial y}(r^i\mathbf{H}_i') + \dfrac{\partial}{\partial z}(r^i\mathbf{H}_i'')\right]}{2[m(i-1)+n(2i-1)]A^i}\right\} \end{aligned}\right\} \quad (135)$$

299.] **Complete solution for a Solid Sphere with surface tractions given.** The components F, G, H, parallel to the coördinate axes, of the stress across any concentric spherical surface of radius r are by (107)

$$F = \frac{x}{r}P + \frac{y}{r}U + \frac{z}{r}T, \text{ etc.}$$

Thus

$$\begin{aligned} Fr &= x\left[(m-n)\Delta + 2n\frac{\partial u}{\partial x}\right] + ny\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + nz\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ &= (m-n)x\Delta + n\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) + n\left(x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x} + z\frac{\partial w}{\partial x}\right) \\ &= (m-n)x\Delta + n\left[\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - 1\right)u + \frac{\partial}{\partial x}(xu+yv+zw)\right] \\ &= (m-n)x\Delta + n\left[\left(r\frac{\partial}{\partial r} - 1\right)u + \frac{\partial \xi}{\partial x}\right], \end{aligned}$$

where

$$\xi = xu + yv + zw.$$

Thus, if F, G, H be expanded in solid harmonics, we shall have

$$\left.\begin{aligned} rF_{i-1} &= (m-n)x\Delta_{i-1} + n(i-1)u_i + n\frac{\partial \xi_{i+1}}{\partial x} \\ rG_{i-1} &= (m-n)y\Delta_{i-1} + n(i-1)\,v_i + n\frac{\partial \xi_{i+1}}{\partial y} \\ rH_{i-1} &= (m-n)\,z\Delta_{i-1} + n(i-1)w_i + n\frac{\partial \xi_{i+1}}{\partial z} \end{aligned}\right\} \qquad (136)$$

where

$$\left.\begin{aligned} \Delta_{i-1} &= \frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial z} \\ \xi_{i+1} &= xu_i + yv_i + zw_i \end{aligned}\right\} \qquad (137)$$

Hence, omitting the complementary function ϕ from the general solution (133),

$$\begin{aligned} \Delta_{i-1} &= \frac{\partial \mathrm{u}_i}{\partial x} + \frac{\partial \mathrm{v}_i}{\partial y} + \frac{\partial \mathrm{w}_i}{\partial z} - M_i\left[\frac{\partial}{\partial x}\left(r^2\frac{\partial \psi_{i-1}}{\partial x}\right) + \frac{\partial}{\partial y}\left(r^2\frac{\partial \psi_{i-1}}{\partial y}\right) + \frac{\partial}{\partial z}\left(r^2\frac{\partial \psi_{i-1}}{\partial z}\right)\right] \\ &= [1 - 2(i-1)M_i]\psi_{i-1}, \qquad (138) \end{aligned}$$

where M_i and ψ_{i-1} are given by (134).

Again

$$\begin{aligned} \xi_{i+1} &= x\mathrm{u}_i + y\mathrm{v}_i + z\mathrm{w}_i - M_i r^3\frac{\partial \psi_{i-1}}{\partial r} \\ &= x\mathrm{u}_i + y\mathrm{v}_i + z\mathrm{w}_i - (i-1)M_i r^2\psi_{i-1} \qquad (139) \end{aligned}$$

The first three terms may be reduced to harmonics as follows. Let ϕ_i be any solid harmonic of order i, and $\mathbf{S}_i$ the corresponding surface harmonic. Then

$$\phi_i = r^i\mathbf{S}_i,$$

and the twin solid harmonic is

$$\phi_{-i-1} = r^{-i-1}\mathbf{S}_i.$$

Thus

$$\left.\begin{aligned} \frac{\partial \phi_i}{\partial x} &= ixr^{i-2}\mathbf{S}_i + r^i\frac{\partial \mathbf{S}_i}{\partial x} \\ \frac{\partial \phi_{-i-1}}{\partial x} &= -(i+1)xr^{-i-3}\mathbf{S}_i + r^{-i-1}\frac{\partial \mathbf{S}_i}{\partial x} \end{aligned}\right\},$$

or

$$\left.\begin{aligned} \frac{\partial \mathbf{S}_i}{\partial x} &= r^{-i}\frac{\partial \phi_i}{\partial x} - ixr^{-i-2}\phi_i \\ &= r^{i+1}\frac{\partial \phi_{-i-1}}{\partial x} + (i+1)xr^{-i-2}\phi_i \end{aligned}\right\},$$

and consequently

$$x\phi_i = \frac{1}{2i+1}\left[r^2\frac{\partial\phi_i}{\partial x} - r^{2i+3}\frac{\partial\phi_{-i-1}}{\partial x}\right] \dots\dots\dots\dots\dots\dots (140)$$

Applying this result to (139), and bearing in mind (134),

$$\xi_{i+1} = \frac{1}{2i+1}\{[1-(i-1)(2i+1)M_i]r^2\psi_{i-1} - \phi_{i+1}\},$$

where

$$\phi_{i+1} = r^{2i+3}\left\{\frac{\partial}{\partial x}\left(\frac{\mathrm{u}_i}{r^{2i+1}}\right) + \frac{\partial}{\partial y}\left(\frac{\mathrm{v}_i}{r^{2i+1}}\right) + \frac{\partial}{\partial z}\left(\frac{\mathrm{w}_i}{r^{2i+1}}\right)\right\} \dots\dots (141)$$

Differentiating, and again making use of (140),

$$\frac{\partial\xi_{i+1}}{\partial x} = \frac{1-(i-1)(2i+1)M_i}{2i-1}\left[r^2\frac{\partial\psi_{i-1}}{\partial x} - \frac{2r^{2i+1}}{2i+1}\frac{\partial}{\partial x}\left(\frac{\psi_{i-1}}{r^{2i-1}}\right)\right]$$

$$- \frac{1}{2i+1}\frac{\partial\phi_{i+1}}{\partial x} \dots\dots\dots\dots\dots\dots\dots\dots (142)$$

Substituting from (138) and (142) in 136, and once more availing ourselves of (140) we obtain

$$rF_{i-1} = n(i-1)\left[\mathrm{u}_i - M_i r^2\frac{\partial\psi_{i-1}}{\partial x}\right]$$

$$+ \frac{(m-n)[1-2(i-1)M_i]}{2i-1}\left[r^2\frac{\partial\psi_{i-1}}{\partial x} - r^{2i+1}\frac{\partial}{\partial x}\left(\frac{\psi_{i-1}}{r^{2i-1}}\right)\right]$$

$$+ \frac{n[1-(i-1)(2i+1)M_i]}{2i-1}\left[r^2\frac{\partial\psi_{i-1}}{\partial x} - \frac{2r^{2i+1}}{2i+1}\frac{\partial}{\partial x}\left(\frac{\psi_{i-1}}{r^{2i-1}}\right)\right]$$

$$- \frac{n}{2i+1}\frac{\partial\phi_{i+1}}{\partial x};$$

and finally

$$\left.\begin{aligned}
rF_{i-1} &= n\left\{(i-1)\mathrm{u}_i - 2(i-2)M_i r^2\frac{\partial\psi_{i-1}}{\partial x} - E_i r^{2i+1}\frac{\partial}{\partial x}\left(\frac{\psi_{i-1}}{r^{2i-1}}\right) - \frac{1}{2i+1}\frac{\partial\phi_{i+1}}{\partial x}\right\}\\
rG_{i-1} &= n\left\{(i-1)\mathrm{v}_i - 2(i-2)M_i r^2\frac{\partial\psi_{i-1}}{\partial y} - E_i r^{2i+1}\frac{\partial}{\partial y}\left(\frac{\psi_{i-1}}{r^{2i-1}}\right) - \frac{1}{2i+1}\frac{\partial\phi_{i+1}}{\partial y}\right\}\\
rH_{i-1} &= n\left\{(i-1)\mathrm{w}_i - 2(i-2)M_i r^2\frac{\partial\psi_{i-1}}{\partial z} - E_i r^{2i+1}\frac{\partial}{\partial z}\left(\frac{\psi_{i-1}}{r^{2i-1}}\right) - \frac{1}{2i+1}\frac{\partial\phi_{i+1}}{\partial z}\right\}
\end{aligned}\right\} (143)$$

where

$$E_i = \frac{m(i+2) - n(2i-1)}{(2i+1)[m(i-1) + n(2i-1)]} \dots\dots\dots\dots\dots (144)$$

Now let the radius of the sphere be A, and let the components of the surface traction be given at every point of the surface by

$$F = \Sigma(\mathbf{H}_i),\ G = \Sigma(\mathbf{H}_i'),\ H = \Sigma(\mathbf{H}_i'') \dots\dots\dots\dots\dots (145)$$

The expressions in (143) consist of solid harmonics depending on surface harmonics of the orders i and $i-2$ respectively. Thus, picking out the terms which involve surface harmonics of order i only, we must have

$$\frac{n}{A}\left\{(i-1)\mathrm{u}_i - 2iM_{i+2}r^2\frac{\partial\psi_{i+1}}{\partial x} - E_i r^{2i+1}\frac{\partial}{\partial x}\left(\frac{\psi_{i-1}}{r^{2i-1}}\right) - \frac{1}{2i+1}\frac{\partial\phi_{i+1}}{\partial x}\right\} = \mathbf{H}_i,$$

etc., when $r=A$.

Thus

$$\left.\begin{aligned}
(i-1)\,\mathrm{u}_i - 2iM_{i+2}A^2\frac{\partial\psi_{i+1}}{\partial x} - E_i r^{2i+1}\frac{\partial}{\partial x}\left(\frac{\psi_{i-1}}{r^{2i-1}}\right) - \frac{1}{2i+1}\frac{\partial\phi_{i+1}}{\partial x} &= \frac{r^i\mathbf{H}_i}{nA^{i-1}}\\
(i-1)\,\mathrm{v}_i - 2iM_{i+2}A^2\frac{\partial\psi_{i+1}}{\partial y} - E_i r^{2i+1}\frac{\partial}{\partial y}\left(\frac{\psi_{i-1}}{r^{2i-1}}\right) - \frac{1}{2i+1}\frac{\partial\phi_{i+1}}{\partial y} &= \frac{r^i\mathbf{H}_i'}{nA^{i-1}}\\
(i-1)\,\mathrm{w}_i - 2iM_{i+2}A^2\frac{\partial\psi_{i+1}}{\partial z} - E_i r^{2i+1}\frac{\partial}{\partial z}\left(\frac{\psi_{i-1}}{r^{2i-1}}\right) - \frac{1}{2i+1}\frac{\partial\phi_{i+1}}{\partial z} &= \frac{r^i\mathbf{H}_i''}{nA^{i-1}}
\end{aligned}\right\}\ldots(146)$$

Differentiating these as to x, y, z, respectively, and adding the results,

$$[i-1+i(2i+1)E_i]\psi_{i-1} = \frac{1}{nA^{i-1}}\left[\frac{\partial}{\partial x}(r^i\mathbf{H}_i) + \frac{\partial}{\partial y}(r^i\mathbf{H}_i') + \frac{\partial}{\partial z}(r^i\mathbf{H}_i'')\right]\ldots\ldots\ldots\ldots\ldots\ldots(147)$$

Also substituting for u_i, v_i, w_i in (141) their values as given by (146),

$$2i\phi_{i+1} + 2i(i+1)(2i+1)A^2M_{i+2}\psi_{i+1} = \frac{r^{2i+3}}{nA^{i-1}}\left[\frac{\partial}{\partial x}\left(\frac{\mathbf{H}_i}{r^{i+1}}\right) + \frac{\partial}{\partial y}\left(\frac{\mathbf{H}_i'}{r^{i+1}}\right) + \frac{\partial}{\partial z}\left(\frac{\mathbf{H}_i''}{r^{i+1}}\right)\right]\ldots\ldots\ldots(148)$$

Thus we obtain ψ from (147), and then ϕ from (148), and lastly u, v, w from (146); and we have only to substitute the values so obtained in the general solution

$$\left.\begin{aligned}
u &= \Sigma\left[\mathrm{u}_i - M_i r^2\frac{\partial\psi_{i-1}}{\partial x}\right]\\
v &= \Sigma\left[\mathrm{v}_i - M_i r^2\frac{\partial\psi_{i-1}}{\partial y}\right]\\
w &= \Sigma\left[\mathrm{w}_i - M_i r^2\frac{\partial\psi_{i-1}}{\partial z}\right]
\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(133)$$

If we write

$$\left.\begin{aligned}
X_{i-1} &= \frac{\partial}{\partial x}(r^i\mathbf{H}_i) + \frac{\partial}{\partial y}(r^i\mathbf{H}_i') + \frac{\partial}{\partial z}(r^i\mathbf{H}_i'')\\
\Phi_{i+1} &= r^{2i+2}\left[\frac{\partial}{\partial x}\left(\frac{\mathbf{H}_i}{r^{2i+1}}\right) + \frac{\partial}{\partial y}\left(\frac{\mathbf{H}_i'}{r^{2i+1}}\right) + \frac{\partial}{\partial z}\left(\frac{\mathbf{H}_i''}{r^{2i+1}}\right)\right]
\end{aligned}\right\}\ldots\ldots\ldots(149)$$

the final form of the solution is

$$u = \frac{1}{n}\Sigma\frac{1}{(i-1)A^{i-1}}\left\{r^i\mathrm{H}_i + \frac{1}{2i(2i+1)}\frac{\partial\Phi_{i+1}}{\partial x}\right.$$
$$+\frac{(i-1)m(A^2-r^2)}{2[(2i^2+1)m-(2i-1)n]}\frac{\partial \mathrm{X}_{i-1}}{\partial x}$$
$$\left.+\frac{[(i+2)m-(2i-1)n]r^{2i+1}}{(2i+1)[(2i^2+1)m-(2i-1)n]}\frac{\partial}{\partial x}\left(\frac{\mathrm{X}_{i-1}}{r^{2i-1}}\right)\right\} \qquad (150)$$

with symmetrical expressions for v and w.

300.] **Conditions of Equilibrium.** The components (145) of the surface traction must of course satisfy the conditions for equilibrium of the body as a whole [§ 146, equations (6) and (7)]. Thus we must have

$$\iint \mathrm{H}_i dS = 0, \iint \mathrm{H}'_i dS = 0, \iint \mathrm{H}''_i dS = 0\,; \qquad (151)$$

and

$$\iint (y\mathrm{H}_i'' - z\mathrm{H}_i')dS = 0, \iint (z\mathrm{H}_i - x\mathrm{H}_i'')dS = 0,$$
$$\iint (x\mathrm{H}_i' - y\mathrm{H}_i)dS = 0 \qquad (152)$$

identically, for all values of i.

Equations (151) are satisfied identically by all surface harmonics of orders other than zero; so that we have only to make

$$\mathrm{H}_0 = \mathrm{H}_0' = \mathrm{H}_0'' = 0 \qquad (151a)$$

Also, since x/A, y/A, z/A are surface harmonics of the first order, equations (152) are satisfied identically by all surface harmonics of orders other than 1. Thus the only further conditions required for equilibrium are

$$\left.\begin{aligned}\iint (y\mathrm{H}_1'' - z\mathrm{H}_1')dS = 0\\ \iint (z\mathrm{H}_1 - x\mathrm{H}_1'')dS = 0\\ \iint (x\mathrm{H}_1' - y\mathrm{H}_1)dS = 0\end{aligned}\right\}.$$

But $A\mathrm{H}_1$, $A\mathrm{H}_1'$, $A\mathrm{H}_1''$ are linear functions of x, y, z; so that, if we assume

$$\left.\begin{aligned}A\mathrm{H}_1 &= Lx + My + Nz\\ A\mathrm{H}_1' &= L'x + M'y + N'z\\ A\mathrm{H}_1'' &= L''x + M''y + N''z\end{aligned}\right\},$$

and remember that, since yz, zx, xy are harmonics,

$$\iint yz dS = \iint zx dS = \iint xy dS = 0,$$

we shall have

$$\left.\begin{aligned} M''\iint y^2 dS &= N'\iint z^2 dS \\ N\iint z^2 dS &= L''\iint x^2 dS \\ L'\iint x^2 dS &= M\iint y^2 dS \end{aligned}\right\},$$

and, since

$$\iint x^2 dS = \iint y^2 dS = \iint z^2 dS = \tfrac{4}{3}\pi A^4,$$

equations (152) will be completely satisfied if only

$$M'' = N', \quad N = L'', \quad L' = M;$$

that is if

$$\mathbf{H}_1 = \frac{1}{r}\frac{\partial \mathbf{U}_2}{\partial x}, \quad \mathbf{H}_1{}' = \frac{1}{r}\frac{\partial \mathbf{U}_2}{\partial y}, \quad \mathbf{H}_1{}'' = \frac{1}{r}\frac{\partial \mathbf{U}_2}{\partial z} \quad \text{............(152}a\text{)}$$

where $\mathbf{U}_2$ is any solid harmonic of degree 2.

General Solutions.

301.] **Application of Sir William Thomson's Method to express the component displacements in the form of potentials.** We have already seen (§ 295) that, in all cases of the present problem, the cubical dilatation satisfies the equation

$$\nabla^2\Delta = 0 \quad \text{..................................(126)}$$

Again, by successive differentiations of equations (105), we deduce

$$\left.\begin{aligned} \nabla^2\theta_1 &= \frac{\partial}{\partial x}\left(\frac{\partial\theta_1}{\partial x} + \frac{\partial\theta_2}{\partial y} + \frac{\partial\theta_3}{\partial z}\right) \\ \nabla^2\theta_2 &= \frac{\partial}{\partial y}\left(\frac{\partial\theta_1}{\partial x} + \frac{\partial\theta_2}{\partial y} + \frac{\partial\theta_3}{\partial z}\right) \\ \nabla^2\theta_3 &= \frac{\partial}{\partial z}\left(\frac{\partial\theta_1}{\partial x} + \frac{\partial\theta_2}{\partial y} + \frac{\partial\theta_3}{\partial z}\right) \end{aligned}\right\};$$

but it follows at once from equations (1) of § 253 that

$$\frac{\partial\theta_1}{\partial x} + \frac{\partial\theta_2}{\partial y} + \frac{\partial\theta_3}{\partial z} = 0, \quad \text{.........................(153)}$$

so that we also have

$$\nabla^2\theta_1 = \nabla^2\theta_2 = \nabla^2\theta_3 = 0 \quad \text{......................(126 }a\text{)}$$

If now we resolve the strain, after the method of §§ 265, 266, 275, 277, into its dilatational and rotational elements, writing

$$\left.\begin{aligned} u &= \frac{\partial\phi}{\partial x} + \frac{\partial\psi_3}{\partial y} - \frac{\partial\psi_2}{\partial z} \\ v &= \frac{\partial\phi}{\partial y} + \frac{\partial\psi_1}{\partial z} - \frac{\partial\psi_3}{\partial x} \\ w &= \frac{\partial\phi}{\partial z} + \frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_1}{\partial y} \end{aligned}\right\} \qquad (154)$$

and assuming that ψ_1, ψ_2, ψ_3 satisfy the condition

$$\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_3}{\partial z} = 0, \qquad (155)$$

we have, on differentiation,

$$\left.\begin{aligned} \Delta &= \nabla^2\phi \\ 2\theta_1 &= -\nabla^2\psi_1 \\ 2\theta_2 &= -\nabla^2\psi_2 \\ 2\theta_3 &= -\nabla^2\psi_3 \end{aligned}\right\} \qquad (156)$$

Hence it follows from (126) and (126 a) that

$$\nabla^4\phi = \nabla^4\psi_1 = \nabla^4\psi_2 = \nabla^4\psi_3 = 0 \qquad (157)$$

Equations (156) are satisfied by the assumptions

$$\left.\begin{aligned} \phi &= -\frac{1}{4\pi}\iiint \frac{\Delta'}{r}\,dx'dy'dz' \\ \psi_1 &= \frac{1}{2\pi}\iiint \frac{\theta'_1}{r}\,dx'dy'dz' \\ \psi_2 &= \frac{1}{2\pi}\iiint \frac{\theta'_2}{r}\,dx'dy'dz' \\ \psi_3 &= \frac{1}{2\pi}\iiint \frac{\theta'_3}{r}\,dx'dy'dz' \end{aligned}\right\} \qquad (158)$$

where the notation is similar to that of § 266; and since by equation (216) of § 311, below

$$\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_3}{\partial z} = \frac{1}{2\pi}\iiint \left(\frac{\partial\theta_1'}{\partial x'} + \frac{\partial\theta_2'}{\partial y'} + \frac{\partial\theta_3'}{\partial z'}\right)\frac{dx'dy'dz'}{r},$$

it follows from (153) that (155) is also satisfied.

Thus if Δ, θ_1, θ_2, θ_3 are any solutions of (126) and (126 a) which satisfy (105) and (153), equations (154) and (158) will represent a general solution of the problem of equilibrium under surface tractions only.

302.] **Sir G. B. Airy's solution for the components of Strain and Stress.** It will be seen at once on substitution that the general equations

$$\left.\begin{aligned}\frac{\partial P}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial T}{\partial z}&=0\\ \frac{\partial U}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial S}{\partial z}&=0\\ \frac{\partial T}{\partial x}+\frac{\partial S}{\partial y}+\frac{\partial R}{\partial z}&=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(103)$$

are satisfied identically by the assumptions

$$\left.\begin{aligned}P&=\frac{\partial^2\chi_3}{\partial y^2}+\frac{\partial^2\chi_2}{\partial z^2}\\ Q&=\frac{\partial^2\chi_1}{\partial z^2}+\frac{\partial^2\chi_3}{\partial x^2}\\ R&=\frac{\partial^2\chi_2}{\partial x^2}+\frac{\partial^2\chi_1}{\partial y^2}\\ S&=-\frac{\partial^2\chi_1}{\partial y\partial z}\\ T&=-\frac{\partial^2\chi_2}{\partial z\partial x}\\ U&=-\frac{\partial^2\chi_3}{\partial x\partial y}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(159)$$

The functions χ_1, χ_2, χ_3 may be continuous or discontinuous in form, but their second derivatives must of course satisfy the conditions (63) of § 226, as well as (126) of § 295.

This method of solution is particularly useful in cases of stress in one plane (§§ 175-184), as everything is then made to depend upon a single arbitrary function. Taking the plane of xy so as to coincide with the plane of the stress, the solution is in this case

$$R=S=T=0\,;$$

$$\left.\begin{aligned}P&=\frac{\partial^2\chi}{\partial y^2}\\ Q&=\frac{\partial^2\chi}{\partial x^2}\\ U&=-\frac{\partial^2\chi}{\partial x\partial y}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(160)$$

Illustrations will be found in §§ 307-309 below. The method is originally due to Sir G. B. Airy,* who arrived at it in a much less direct way, by an application of the Calculus of Variations.

* *Report of the British Association;* Cambridge, 1862: p. 82.

The Problem of Equilibrium under a Conservative System of Applied Forces, with or without Surface Tractions.

303.] **General Equations.** For reasons stated in § 279, we shall confine ourselves to the consideration of bodies influenced by such systems of Applied Forces as are derivable by differentiation from a Potential.

Let Ψ denote the Force Potential, as in § 279, so that the component forces per unit mass on any element of the body are given as before by equations (93) and (94).

The equations (6), (7), (8) of equilibrium then take the forms

$$\left.\begin{aligned}\frac{\partial[P+\rho\Psi]}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial T}{\partial z}&=0\\ \frac{\partial U}{\partial x}+\frac{\partial[Q+\rho\Psi]}{\partial y}+\frac{\partial S}{\partial z}&=0\\ \frac{\partial T}{\partial x}+\frac{\partial S}{\partial y}+\frac{\partial[R+\rho\Psi]}{\partial z}&=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(161)$$

$$\left.\begin{aligned}\frac{\partial}{\partial x}[m\Delta+\rho\Psi]+n\nabla^2 u&=0\\ \frac{\partial}{\partial y}[m\Delta+\rho\Psi]+n\nabla^2 v&=0\\ \frac{\partial}{\partial z}[m\Delta+\rho\Psi]+n\nabla^2 w&=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(162)$$

$$\left.\begin{aligned}\frac{\partial}{\partial x}[(m+n)\Delta+\rho\Psi]-2n\left(\frac{\partial\theta_3}{\partial y}-\frac{\partial\theta_2}{\partial z}\right)&=0\\ \frac{\partial}{\partial y}[(m+n)\Delta+\rho\Psi]-2n\left(\frac{\partial\theta_1}{\partial z}-\frac{\partial\theta_3}{\partial x}\right)&=0\\ \frac{\partial}{\partial z}[(m+n)\Delta+\rho\Psi]-2n\left(\frac{\partial\theta_2}{\partial x}-\frac{\partial\theta_1}{\partial y}\right)&=0\end{aligned}\right\}\quad\ldots\ldots(163)$$

and from the latter set we easily deduce

$$\nabla^2[(m+n)\Delta+\rho\Psi]=0\ldots\ldots\ldots\ldots(164)$$

Similarly, the general equations (48) of § 239 become

$$\left.\begin{aligned}\frac{\partial}{\partial s_1}\Big[(m+n)\Delta+\rho\Psi\Big]-2n\left[h_3\frac{\partial}{\partial s_2}\left(\frac{\Theta_3}{h_3}\right)-h_2\frac{\partial}{\partial s_3}\left(\frac{\Theta_2}{h_2}\right)\right]&=0\\ \frac{\partial}{\partial s_2}\Big[(m+n)\Delta+\rho\Psi\Big]-2n\left[h_1\frac{\partial}{\partial s_3}\left(\frac{\Theta_1}{h_1}\right)-h_3\frac{\partial}{\partial s_1}\left(\frac{\Theta_3}{h_3}\right)\right]&=0\\ \frac{\partial}{\partial s_3}\Big[(m+n)\Delta+\rho\Psi\Big]-2n\left[h_2\frac{\partial}{\partial s_1}\left(\frac{\Theta_2}{h_2}\right)-h_1\frac{\partial}{\partial s_2}\left(\frac{\Theta_1}{h_1}\right)\right]&=0\end{aligned}\right\}\ldots(165)$$

The boundary conditions will be given, as before, by (106) or (107): and the considerations of §§ 286, 287 apply equally to this problem.

The simplest method of proceeding to solve this problem is, in general, to obtain the particular integrals of the above equations, depending upon the form of Ψ, and then to add complementary functions satisfying the conditions of the last problem, and so chosen that the complete solutions may satisfy the boundary conditions (106) or (107).

Case in which Ψ can be expressed in a series of Solid Spherical Harmonics.

304.] **Solution of the General Equations.** Let the Force Potential be expanded in the series of solid harmonics

$$\Psi = \Sigma(\Psi_i)\,; \qquad (166)$$

then $\nabla^2\Psi = 0$, and we at once deduce from (164) that $\nabla^2\Delta = 0$.

Thus Δ must also be capable of expansion in a series of solid harmonics: let this series be represented by

$$\Delta = \Sigma(\Delta_i) \qquad (167)$$

These series may be supposed in general to include all values of i positive and negative.

Substituting in (162), they become

$$\left.\begin{aligned} \nabla^2 u &= -\frac{1}{n}\Sigma\frac{\partial}{\partial x}(m\Delta_i + \rho\Psi_i) \\ \nabla^2 v &= -\frac{1}{n}\Sigma\frac{\partial}{\partial y}(m\Delta_i + \rho\Psi_i) \\ \nabla^2 w &= -\frac{1}{n}\Sigma\frac{\partial}{\partial z}(m\Delta_i + \rho\Psi_i) \end{aligned}\right\} \qquad (168)$$

and, on solving these equations as we did (129) of § 297, we obtain

$$\left.\begin{aligned} u &= \mathrm{u} - \frac{r^2}{2n}\Sigma\frac{1}{2i+1}\frac{\partial}{\partial x}(m\Delta_i + \rho\Psi_i) \\ v &= \mathrm{v} - \frac{r^2}{2n}\Sigma\frac{1}{2i+1}\frac{\partial}{\partial y}(m\Delta_i + \rho\Psi_i) \\ w &= \mathrm{w} - \frac{r^2}{2n}\Sigma\frac{1}{2i+1}\frac{\partial}{\partial z}(m\Delta_i + \rho\Psi_i) \end{aligned}\right\} \qquad (169)$$

where u, v, w are solutions of

$$\nabla^2 \mathrm{u} = 0, \quad \nabla^2 \mathrm{v} = 0, \quad \nabla^2 \mathrm{w} = 0.$$

In order that (131) may be satisfied we must have

$$\frac{\partial \mathrm{u}}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} + \frac{\partial \mathrm{w}}{\partial z} - \frac{1}{n}\Sigma\frac{i}{2i+1}(m\Delta_i + \rho\Psi_i) = \Sigma(\Delta_i)\,;$$

and on picking out all the terms corresponding to the harmonics of order i in the solution (169)

$$\Delta_{i-1} = \frac{(2i-1)n\left(\frac{\partial \mathrm{u}_i}{\partial x} + \frac{\partial \mathrm{v}_i}{\partial y} + \frac{\partial \mathrm{w}_i}{\partial z}\right) - (i-1)\rho\Psi_{i-1}}{m(i-1) + n(2i-1)}.$$

Substituting in (169) and adopting the abridged notation of (134) § 298, we have finally

$$\left.\begin{aligned} u &= \Sigma\left[\mathrm{u}_i - M_i r^2 \frac{\partial}{\partial x}\left(\psi_{i-1} + \frac{\rho}{m}\Psi_{i-1}\right)\right] \\ v &= \Sigma\left[\mathrm{v}_i - M_i r^2 \frac{\partial}{\partial y}\left(\psi_{i-1} + \frac{\rho}{m}\Psi_{i-1}\right)\right] \\ w &= \Sigma\left[\mathrm{w}_i - M_i r^2 \frac{\partial}{\partial z}\left(\psi_{i-1} + \frac{\rho}{m}\Psi_{i-1}\right)\right] \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (170)$$

The complementary functions in (170) are of course the complete solutions (133)* of the problem of § 297. The particular integrals given by this method of solution are

$$\left.\begin{aligned} u &= -\frac{\rho r^2}{m}\Sigma\left(M_i \frac{\partial \Psi_{i-1}}{\partial x}\right) \\ v &= -\frac{\rho r^2}{m}\Sigma\left(M_i \frac{\partial \Psi_{i-1}}{\partial y}\right) \\ w &= -\frac{\rho r^2}{m}\Sigma\left(M_i \frac{\partial \Psi_{i-1}}{\partial z}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (171)$$

We may however adopt another mode of solution,† and obtain particular integrals of a different form, as follow :—

Assuming that

$$u_i = \frac{\partial \phi_{i+1}}{\partial x}, \quad v_i = \frac{\partial \phi_{i+1}}{\partial y}, \quad w_i = \frac{\partial \phi_{i+1}}{\partial z},$$

* The *arbitrary* complements of (133) appear as the complementary solutions of the equations obtained below for ϕ, by the second method.

† See Thomson and Tait's *Natural Philosophy*, Articles 733, 834.

and substituting in the general equations (168), we obtain

$$\left.\begin{aligned}\frac{\partial}{\partial x}\left[(m+n)\nabla^2\phi_{i+1}+\rho\Psi_{i-1}\right]&=0\\ \frac{\partial}{\partial y}\left[(m+n)\nabla^2\phi_{i+1}+\rho\Psi_{i-1}\right]&=0\\ \frac{\partial}{\partial z}\left[(m+n)\nabla^2\phi_{i+1}+\rho\Psi_{i-1}\right]&=0\end{aligned}\right\},$$

and the particular integral of these equations is obviously

$$\phi_{i+1}=-\frac{\rho r^2}{2(m+n)}\cdot\frac{\Psi_{i-1}}{2i+1},$$

giving for the corresponding particular integrals of (168)

$$\left.\begin{aligned}u&=-\frac{\rho}{2(m+n)}\sum\frac{\partial}{\partial x}\left(\frac{r^2\Psi_{i-1}}{2i+1}\right)\\ v&=-\frac{\rho}{2(m+n)}\sum\frac{\partial}{\partial y}\left(\frac{r^2\Psi_{i-1}}{2i+1}\right)\\ w&=-\frac{\rho}{2(m+n)}\sum\frac{\partial}{\partial z}\left(\frac{r^2\Psi_{i-1}}{2i+1}\right)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(172)$$

Thus it is clear that both of the particular integrals, (171) and (172), are partial; and in fact, if we assume the particular integrals to be of the form

$$\left.\begin{aligned}u_i=-C_i r^2\frac{\partial\Psi_{i+1}}{\partial x}-C_i'\frac{\partial}{\partial x}(r^2\Psi_{i-1})\\ \text{etc., etc.}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(173)$$

and substitute in (168), we obtain the single relation

$$mC_i/M_i+2(2i+1)(m+n)C_i'=\rho\ldots\ldots\ldots\ldots\ldots\ldots(174)$$

between C_i and C_i', so that one of these constants is altogether arbitrary so long as we are concerned only with the general equations

The former solution (171) satisfies (174) by making

$$C_i=\rho M_i/m,\quad C_i'=0;$$

and the solution (172) also satisfies (174) by making

$$C_i=0,\quad C_i'=\rho/2(2i+1)(m+n).$$

305.] **Complete Solution for a Solid Sphere with surface displacements given.** The complete harmonic solution in its most general form is

$$\left.\begin{aligned} u &= \sum\left[\mathrm{u}_i + \frac{\partial \phi_{i+1}}{\partial x} - r^2\frac{\partial}{\partial x}(M_i\psi_{i-1} + C_i\Psi_{i-1}) - C_i'\frac{\partial}{\partial x}(r^2\Psi_{i-1})\right] \\ v &= \sum\left[\mathrm{v}_i + \frac{\partial \phi_{i+1}}{\partial y} - r^2\frac{\partial}{\partial y}(M_i\psi_{i-1} + C_i\Psi_{i-1}) - C_i'\frac{\partial}{\partial y}(r^2\Psi_{i-1})\right] \\ w &= \sum\left[\mathrm{w}_i + \frac{\partial \phi_{i+1}}{\partial z} - r^2\frac{\partial}{\partial z}(M_i\psi_{i-1} + C_i\Psi_{i-1}) - C_i'\frac{\partial}{\partial z}(r^2\Psi_{i-1})\right] \end{aligned}\right\}\ldots(175)$$

where the constants C, C' (one of which is arbitrary) are connected by the relation (174), and M, ψ are given by (134): u, v, w, ϕ being any solid harmonics which allow u, v, w to satisfy the conditions of § 287 (see § 297, page 332). This form of the solution corresponds to (133) of § 298. In applying it to the case of a solid sphere we must of course assume that the series (175) include only positive values of i.

Let A be the radius of the sphere, and let the components of the surface displacements be given by

$$u_0 = \Sigma(\mathbf{H}_i),\ v_0 = \Sigma(\mathbf{H}_i'),\ w_0 = \Sigma(\mathbf{H}_i'').$$

Then we must have, when $r = A$,

$$\mathrm{u}_i + \frac{\partial \phi_{i+1}}{\partial x} - r^2\frac{\partial}{\partial x}(M_i\psi_{i-1} + C_i\Psi_{i-1}) - C_i'\frac{\partial}{\partial x}(r^2\Psi_{i-1}) = \mathbf{H}_i, \text{ etc.};$$

or, differentiating the term involving C', and making use of (140) above,

$$\mathrm{u}_i + \frac{\partial \phi_{i+1}}{\partial x} - M_i r^2\frac{\partial \psi_{i-1}}{\partial x} = \frac{(2i-1)C_i + (2i+1)C_i'}{2i-1} r^2\frac{\partial \Psi_{i-1}}{\partial x}$$
$$- \frac{2C_i'}{2i-1} r^{2i+1}\frac{\partial}{\partial x}\left(\frac{\Psi_{i-1}}{r^{2i-1}}\right) + \left(\frac{r}{A}\right)^i \mathbf{H}_i.$$

The solution is obviously (compare that of § 298) determined by

$$\left.\begin{aligned} \phi_{i-1} &= A^2\left[M_i\psi_{i-1} + \frac{(2i-1)C_i + (2i+1)C_i'}{2i-1}\Psi_{i-1}\right] \\ \mathrm{u}_i &= \left(\frac{r}{A}\right)^i \mathbf{H}_i - \frac{2C_i'}{2i-1} r^{2i+1}\frac{\partial}{\partial x}\left(\frac{\Psi_{i-1}}{r^{2i-1}}\right) \end{aligned}\right\}\ldots\ldots\ldots(176)$$

In order to simplify the reduction of this result we will first of all eliminate C_i from (176) by means of (174). We have

$$C_i = \frac{\rho M_i}{m} - \frac{2(2i+1)(m+n)M_i C_i'}{m},$$

so that

$$\frac{(2i-1)C_i+(2i+1)C_i'}{2i-1}=\frac{\rho M_i}{m}-\frac{2i(2i+1)M_iC_i'}{2i-1};$$

and, on substituting in the first of equations (176),

$$\phi_{i-1}=A^2M_i(\psi_{i-1}+\rho\Psi_{i-1}/m)-\frac{2i(2i+1)A^2M_iC_i'\Psi_{i-1}}{2i-1}.$$

But the second of these equations gives us

$$\psi_{i-1}=\frac{X_{i-1}}{A^i}+\frac{2i(2i+1)C_i'\Psi_{i-1}}{2i-1},$$

where X has the same meaning as in (149). Consequently

$$\phi_{i-1}=A^2M_i(X_{i-1}/A^i+\rho\Psi_{i-1}/m)$$

and is therefore independent of C_i'.

Substituting these values of ϕ and ψ in (175), and performing the differentiations indicated in the last terms of those expressions, we have finally

$$\left.\begin{aligned}u&=\Sigma\left[\left(\frac{r}{A}\right)^i\mathbf{H}_i+(A^2-r^2)M_i\frac{\partial}{\partial x}\left(\frac{X_{i-1}}{A^i}+\frac{\rho\Psi_{i-1}}{m}\right)\right]\\v&=\Sigma\left[\left(\frac{r}{A}\right)^i\mathbf{H}_i'+(A^2-r^2)M_i\frac{\partial}{\partial y}\left(\frac{X_{i-1}}{A^i}+\frac{\rho\Psi_{i-1}}{m}\right)\right]\\w&=\Sigma\left[\left(\frac{r}{A}\right)^i\mathbf{H}_i''+(A^2-r^2)M_i\frac{\partial}{\partial z}\left(\frac{X_{i-1}}{A^i}+\frac{\rho\Psi_{i-1}}{m}\right)\right]\end{aligned}\right\}\quad\text{........(177)}$$

Thus the arbitrary constant C' disappears from the complete solution for the displacement: or, in other words, it is indifferent whether we take for the particular integrals of our general equations the form (171), or the form (172), or any combination of these which satisfies the condition (174). This is an excellent illustration of the statement made in §§ 255 and 286—that the solution is absolutely determinate when the surface displacements are given.

306.] **Complete Solution for a Solid Sphere with surface tractions given.** Let A be the radius of the sphere, and let the components of the surface traction parallel to the coordinate axes Ox, Oy, Oz be given by

$$F=\Sigma(\mathbf{H}_i),\quad G=\Sigma(\mathbf{H}_i'),\quad H=\Sigma(\mathbf{H}_i'').$$

Let us first consider the stress due to the particular integrals (173). We know from §§ 255, 286 that any two distributions of displacement which satisfy the general equations of equilibrium throughout the body, and also the stress conditions all over its

surface, can only differ by terms amounting to a displacement of the body as a whole. Thus, if the arbitrary element of (173) appears at all in the solution of the present problem, it must be in such a way as to contribute nothing to the strain, and it will therefore be unessential.

Eliminating it then, and taking the simpler form (171), we have for the particular integrals

$$\left.\begin{aligned} u_i &= -\frac{\rho M_i}{m} r^2 \frac{\partial \Psi_{i-1}}{\partial x} \\ v_i &= -\frac{\rho M_i}{m} r^2 \frac{\partial \Psi_{i-1}}{\partial y} \\ w_i &= -\frac{\rho M_i}{m} r^2 \frac{\partial \Psi_{i-1}}{\partial z} \end{aligned}\right\} \qquad (178)$$

Let F', G', H' denote the components of the stress, due to these displacements, across a concentric spherical surface of radius r. Then, by (136) and (137) of § 299

$$\left.\begin{aligned} rF'_{i-1} = (i-1)nu_i + (m-n)x\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} + \frac{\partial w_i}{\partial z}\right) + n\frac{\partial}{\partial x}(xu_i + yv_i + zw_i) \\ \text{etc., etc.} \end{aligned}\right\}$$

Substituting from (178), and making use of (140) when necessary, we deduce

$$rF'_{i-1} = -\frac{2(i-1)\rho M_i}{(2i-1)m}\left\{\Big[m + (2i-1)n\Big] r^2 \frac{\partial \Psi_{i-1}}{\partial x} - mr^{2i+1}\frac{\partial}{\partial x}\left(\frac{\Psi_{i-1}}{r^{2i-1}}\right)\right\}.$$

Thus, on picking out the terms which involve surface harmonics of order i, we find that the system of displacements

$$\left.\begin{aligned} u = -\frac{\rho r^2}{m}\Sigma\left(M_{i+2}\frac{\partial \Psi_{i+1}}{\partial x}\right) \\ \text{etc., etc.} \end{aligned}\right\} \qquad (179)$$

gives rise to the surface tractions

$$\left.\begin{aligned} F' = -\frac{2\rho}{m}\Sigma\left\{\frac{(i+1)M_{i+2}}{2i+3}\Big[m + (2i+3)n\Big]\frac{A^{i+1}}{r^i}\frac{\partial \Psi_{i+1}}{\partial x} \right. \\ \left. - \frac{(i-1)mM_i}{2i-1}A^{i-1}r^{i+1}\frac{\partial}{\partial x}\left(\frac{\Psi_{i-1}}{r^{2i-1}}\right)\right\} \\ \text{etc., etc.} \end{aligned}\right\} \qquad (180)$$

Again, the solution corresponding to the complementary functions

$$u = \Sigma\left(\mathrm{u}_i - M_i r^2 \frac{\partial \Psi_{i-1}}{\partial x}\right) \text{ etc.,}$$

of (170) has been worked out in § 299. It appears that, if $\mathbf{S}_i$, $\mathbf{S}_i'$, $\mathbf{S}_i''$ be surface harmonics of order i, and if

$$\left.\begin{aligned} X'_{i-1} &= \frac{\partial}{\partial x}(r^i\mathbf{S}_i) + \frac{\partial}{\partial y}(r^i\mathbf{S}_i') + \frac{\partial}{\partial z}(r^i\mathbf{S}_i'') \\ \Phi'_{i+1} &= r^{2i+3}\left[\frac{\partial}{\partial x}\left(\frac{\mathbf{S}_i}{r^{2i+1}}\right) + \frac{\partial}{\partial y}\left(\frac{\mathbf{S}_i'}{r^{2i+1}}\right) + \frac{\partial}{\partial z}\left(\frac{\mathbf{S}_i''}{r^{2i+1}}\right)\right] \end{aligned}\right\} \text{.........(181)}$$

the system of displacements

$$\left.\begin{aligned} u = \frac{1}{n}\Sigma\frac{1}{(i-1)A^{i-1}}\Bigg\{ & r^i\mathbf{S}_i + \frac{1}{2i(2i+1)}\frac{\partial\Phi'_{i+1}}{\partial x} \\ & + \frac{(i-1)m(A^2-r^2)}{2[(2i^2+1)m-(2i-1)n]}\frac{\partial X'_{i-1}}{\partial x} + \frac{[(i+2)m-(2i-1)n]r^{2i+1}}{(2i+1)[(2i^2+1)m-(2i-1)n]}\frac{\partial}{\partial x}\left(\frac{X'_{i-1}}{r^{2i-1}}\right)\Bigg\} \\ & \text{etc., etc.,} \end{aligned}\right\} \text{(182)}$$

which represents a particular form of the solution (150), gives rise to the distribution of surface traction

$$F = \Sigma(\mathbf{S}_i),\quad G = \Sigma(\mathbf{S}_i'),\quad H = \Sigma(\mathbf{S}_i'').$$

Thus the system of displacements compounded of (179) and (182) will satisfy the general equations (as particular integrals and complementary functions, respectively), and will give to the components of the surface traction the form

$$\begin{aligned} F = \Sigma\Bigg\{ & \mathbf{S}_i - \frac{2(i+1)\rho M_{i+2}}{(2i+3)m}\left[m+(2i+3)n\right]\frac{A^{i+1}}{r^i}\frac{\partial\Psi'_{i+1}}{\partial x} \\ & + \frac{2(i-1)\rho M_i}{2i-1}A^{i-1}r^{i+1}\frac{\partial}{\partial x}\left(\frac{\Psi'_{i-1}}{r^{2i-1}}\right)\Bigg\} \\ & \text{etc., etc.;} \end{aligned}$$

so that to complete our solution we have only to make

$$\begin{aligned} \mathbf{S}_i = \mathbf{H}_i & + \frac{2(i+1)\rho M_{i+2}}{(2i+3)m}\left[m+(2i+3)n\right]\frac{A^{i+1}}{r^i}\frac{\partial\Psi'_{i+1}}{\partial x} \\ & - \frac{2(i-1)\rho M_i}{2i-1}A^{i-1}r^{i+1}\frac{\partial}{\partial x}\left(\frac{\Psi'_{i-1}}{r^{2i-1}}\right) \text{.....................(183)} \end{aligned}$$

with symmetrical expressions for $\mathbf{S}_i'$ and $\mathbf{S}_i''$.

Substituting from (183) in (181),

$$\left.\begin{aligned} X'_{i-1} &= X_{i-1} + \frac{2i(i-1)(2i+1)\rho M_i A^{i-1}}{2i-1}\Psi'_{i-1} \\ \Phi'_{i+1} &= \Phi_{i+1} - \frac{2(i+1)^2(2i+1)\rho M_{i+2}A^{i+1}}{(2i+3)m}\left[m+(2i+3)n\right]\Psi'_{i+1} \end{aligned}\right\} \text{...(184)}$$

where Φ and X are given by (149).

Compounding (179) and (182), and substituting from (183) and (184), we have finally for the complete solution *

$$
\begin{aligned}
u = \frac{1}{n}\sum\frac{1}{(i-1)A^{i-1}}\Bigg\{ & r^i\mathrm{H}_i + \frac{1}{2i(2i+1)}\frac{\partial \Phi_{i+1}}{\partial x} \quad \cdot\ \cdot \\
& + \frac{(i-1)m(A^2-r^2)}{2[(2i^2+1)m-(2i-1)n]}\frac{\partial \mathrm{X}_{i-1}}{\partial x} + \frac{[(i+2)m-(2i-1)n]r^{2i+1}}{(2i+1)[(2i^2+1)m-(2i-1)n]}\frac{\partial}{\partial x}\left(\frac{\mathrm{X}_{i-1}}{r^{2i-1}}\right)\Bigg\} \\
+ \frac{\rho}{n}\sum\frac{i-1}{(2i^2+1)m-(2i-1)n}\Bigg\{ & \frac{(i+1)m-n}{2(i-2)}A^2\frac{\partial \Psi_{i-1}}{\partial x} \\
& - \frac{i(2i+1)m^2}{2(2i-1)[(i-1)m+(2i-1)n]}r^2\frac{\partial \Psi_{i-1}}{\partial x} - \frac{mr^{2i+1}}{2i-1}\frac{\partial}{\partial x}\left(\frac{\Psi_{i-1}}{r^{2i-1}}\right)\Bigg\} \quad \text{(185)}
\end{aligned}
$$

Airy's Method.

307.] Airy's Solution for the components of Strain and Stress. It has been shown in § 302 that the general equations (103) admit of a very simple and general solution (159) for the stress components. On comparing (161) with (103) it is at once evident that the same form of solution is applicable, the only changes required being the substitution of $P+\rho\Psi$, $Q+\rho\Psi$, $R+\rho\Psi$ for P, Q, R.

Thus, corresponding to the solution (159) we have the more general form

$$
\left.
\begin{aligned}
P &= \frac{\partial^2\chi_3}{\partial y^2} + \frac{\partial^2\chi_2}{\partial z^2} - \rho\Psi \\
Q &= \frac{\partial^2\chi_1}{\partial z^2} + \frac{\partial^2\chi_3}{\partial x^2} - \rho\Psi \\
R &= \frac{\partial^2\chi_2}{\partial x^2} + \frac{\partial^2\chi_1}{\partial y^2} - \rho\Psi \\
S &= -\frac{\partial^2\chi_1}{\partial y\partial z} \\
T &= -\frac{\partial^2\chi_2}{\partial z\partial x} \\
U &= -\frac{\partial^2\chi_3}{\partial x\partial y}
\end{aligned}
\right\} \quad \text{(186)}
$$

The principal application of this method is, as already stated in § 302, to cases of *Plane Stress*. For example, take the case of a body in the form of a rectangular parallelepiped of any proportions, placed with its three pairs of opposite faces parallel

* For the conditions of equilibrium in this problem, see Example 20, at the end of this Chapter.

respectively to the three coördinate planes. Let this body be free from surface tractions over the pair of faces perpendicular to Oz, and let it be acted upon by impressed forces and by surface tractions over the remaining two pairs of faces, everywhere perpendicular to Oz, and in magnitude independent of z. Then the stress-components R, S, T will be independent of z, and zero over every face of the body: and consequently they must be zero throughout. The force-potential Ψ will also be independent of z, and so therefore will the remaining stress components P, Q, U.

Thus the stress at every point of the body is wholly in the plane perpendicular to Oz (§§ 175-184), and the solution (186) reduces to the simple form

$$\left.\begin{aligned} P &= \frac{\partial^2\chi}{\partial y^2} - \rho\Psi \\ Q &= \frac{\partial^2\chi}{\partial x^2} - \rho\Psi \\ U &= -\frac{\partial^2\chi}{\partial x\,\partial y} \\ R &= S = T = 0 \end{aligned}\right\} \qquad (187)$$

where χ is a function of x and y, to be so chosen as to give P, Q, and U their proper values over the bounding surfaces.

Two of the examples considered by Sir G. B. Airy in his original paper* will be investigated in the following articles: the remainder will be found amongst the Examples at the end of this Chapter.

308.] **Case of a heavy rectangular beam, with one end clamped to a vertical wall and the other end free; the faces of the beam being horizontal and vertical.** Let L be the length of the beam (horizontal), B its breadth (horizontal), and D its depth (vertical). Take the origin at the centre of the fixed end, Ox in the direction of the length (axis of the beam), and Oy vertically downwards. Then $\Psi = \mathrm{g}y$, and equations (187) become

$$\left.\begin{aligned} P &= \frac{\partial^2\chi}{\partial y^2} - \mathrm{g}\rho y \\ Q &= \frac{\partial^2\chi}{\partial x^2} - \mathrm{g}\rho y \\ U &= -\frac{\partial^2\chi}{\partial x\,\partial y} \end{aligned}\right\} \ldots$$

* *Report of the British Association*; Cambridge, 1862: p. 82.

If we make

$$\psi = \chi - \tfrac{1}{6}g\rho y^3 \text{ ....(188)}$$

these equations take the more manageable form

$$\left.\begin{aligned} P &= \frac{\partial^2\psi}{\partial y^2} \\ Q &= \frac{\partial^2\psi}{\partial x^2} - g\rho y \\ U &= -\frac{\partial^2\psi}{\partial x\partial y} \end{aligned}\right\} \text{ ....(189)}$$

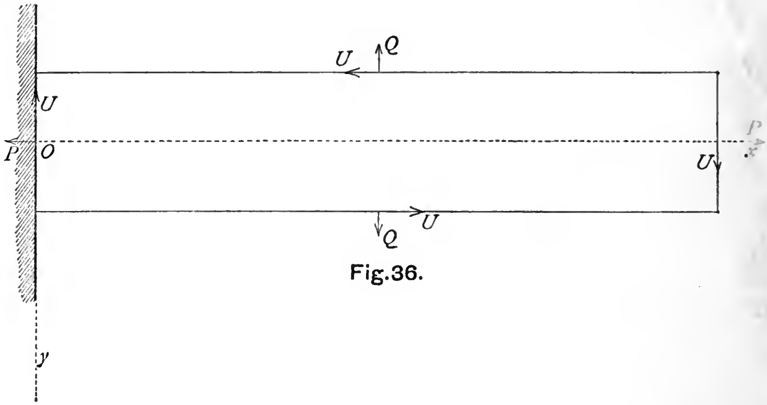


Fig.36.

Since the end $x=0$ is the only portion of the surface in contact with solid matter, the surface tractions must vanish over all the other faces. Thus

$$P=0,\ U=0, \text{ when } x=L;$$

and $$Q=0,\ U=0, \text{ when } y=\pm\tfrac{1}{2}D.$$

Also, since the whole weight of the beam is supported by the integral tangential stress over the fixed end,

$$B\int_{-\frac{1}{2}D}^{\frac{1}{2}D} U dy = g\rho BDL, \text{ when } x=0.$$

Substituting for P, Q and U their values in terms of ψ, the surface conditions become

$$\left.\begin{aligned} &\frac{\partial^2\psi}{\partial y^2}[x=L]=0,\ \frac{\partial^2\psi}{\partial x\partial y}[x=L]=0, \\ &\text{for all values of } y \end{aligned}\right\} \text{ ....(190)}$$

$$\left.\begin{array}{l}\dfrac{\partial^2\psi}{\partial x^2}[y=\pm\tfrac{1}{2}D]=\pm\tfrac{1}{2}g\rho D,\ \dfrac{\partial^2\psi}{\partial x\partial y}[y=\pm\tfrac{1}{2}D]=0,\\ \text{for all values of } x\end{array}\right\}\text{.........(191)}$$

$$\frac{\partial\psi}{\partial x}[x=0,\ y=-\tfrac{1}{2}D]-\frac{\partial\psi}{\partial x}[x=0,\ y=+\tfrac{1}{2}D]=g\rho DL\text{.......(192)}$$

From (190) it appears that both ψ and $\partial\psi/\partial x$ vanish when $x=L$; and from (191) that $\partial^2\psi/\partial x^2$ is independent of x and changes sign with y, while $\partial\psi/\partial y$ vanishes when $y=\pm\tfrac{1}{2}D$.

Hence we infer that ψ only involves x in the form of the factor $(L-x)^2$, that it contains only odd powers of y, and that $(\tfrac{1}{4}D^2-y^2)$ is a factor of $\partial\psi/\partial y$. From these data we deduce without difficulty that ψ is necessarily of the form

$$\psi=(L-x)^2\left[\frac{D^2C_1y}{4\,.\,1}+\frac{(D^2C_3-4C_1)y^3}{4\,.\,3}+\frac{(D^2C_5-4C_3)y^5}{4\,.\,5}+\dots\right]$$

where C_1, C_3, C_5, ... are constants, to be determined by the remaining conditions. These are (192), and the first condition of (191), and it will be found on substitution that both give rise to the same equation: namely—

$$C_1+\tfrac{1}{12}(D^2C_3-4C_1)+\tfrac{1}{80}(D^2C_5-4C_3)+\dots=2g\rho/D^2.$$

Since only one of the arbitrary constants is determinate, we will adopt the simplest hypothesis and assume

$$C_1=3g\rho/D^2;\ C_3=C_5=\dots=0.$$

We shall then have

$$\left.\begin{array}{ll} & \psi=\dfrac{3g\rho}{D^2}(L-x)^2\left(\dfrac{D^2y}{4}-\dfrac{y^3}{3}\right)\\ \text{and} & \\ & \chi=g\rho y\left[(L-x)^2\left(\dfrac{3}{4}-\dfrac{y^2}{D^2}\right)+\dfrac{y^2}{6}\right]\end{array}\right\}\text{...............(193)}$$

giving on substitution in (189)

$$\left.\begin{array}{l}P=-\dfrac{6g\rho}{D^2}y(L-x)^2\\ Q=\dfrac{2g\rho}{D^2}y\left(\dfrac{D^2}{4}-y^2\right)\\ U=\dfrac{6g\rho}{D^2}(L-x)\left(\dfrac{D^2}{4}-y^2\right)\end{array}\right\}\text{....................(194)}$$

and it will be found on trial that these values satisfy all the imposed conditions.

Figure 37 is reproduced from Airy's sketch of the Lines of Stress. Their equations are not integrable in finite terms, but this approximation was arrived at by determining the directions [by means of § 129 (64)] of the lines of stress passing through each of a great number of points, and joining up the elementary

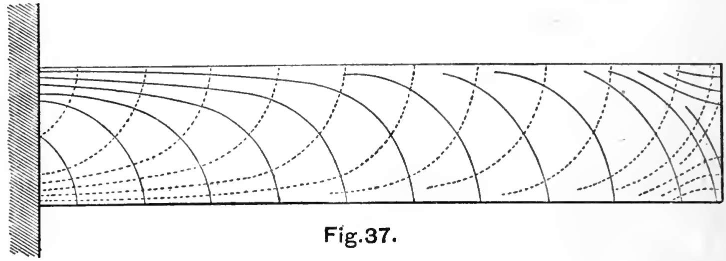


Fig.37.

arcs so obtained. The obliquity of many of the intersections proves that the Lines of Stress are not very accurately represented, but a good idea is doubtless given of their general tendency.

The same remarks apply to Figures 39, 40 and 41, which illustrate examples 29, 30 and 31 at the end of the Chapter.

In all cases the whole curves represent the Ties and the dotted curves the Struts.

309.] **A rectangular beam (placed as in the last example) is supported at both ends, but not clamped—so that no couple acts upon it at either end: while a given load is uniformly distributed over a certain portion of its length.** Take the axes of reference as in the last example;

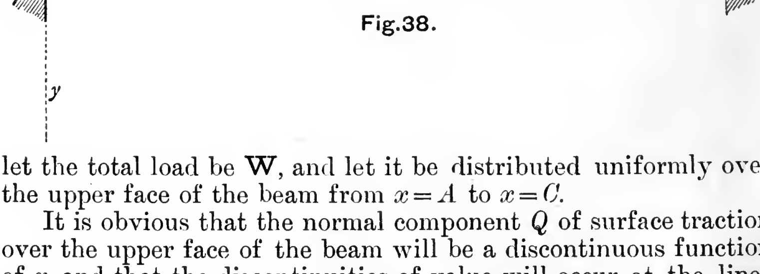


Fig.38.

let the total load be **W**, and let it be distributed uniformly over the upper face of the beam from $x=A$ to $x=C$.

It is obvious that the normal component Q of surface traction over the upper face of the beam will be a discontinuous function of x, and that the discontinuities of value will occur at the lines $x=A$ and $x=C$. [$Q=0$ from $x=0$ to $x=A$; $Q=-\mathbf{W}/B(C-A)$

PLATE III.

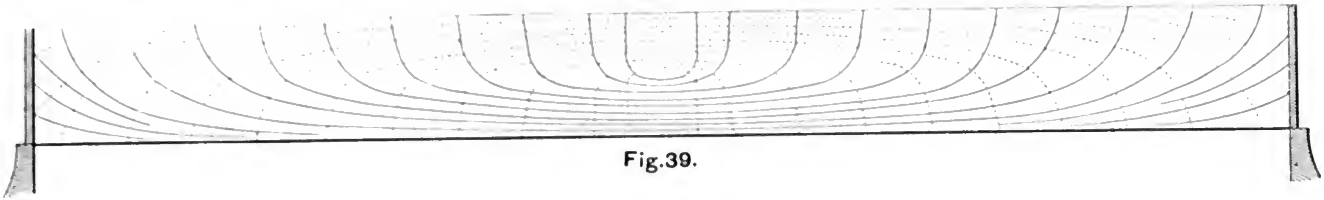


Fig. 39.

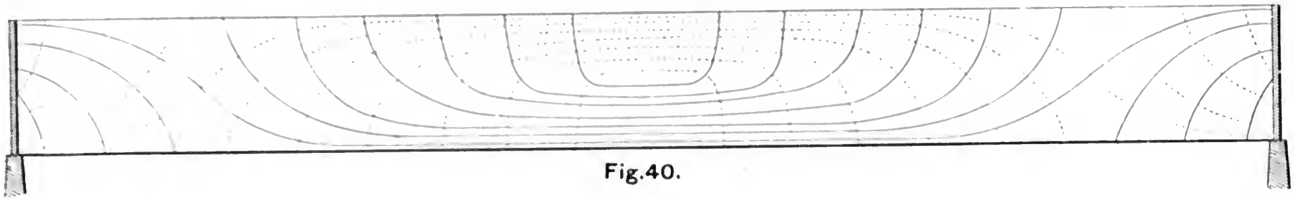


Fig. 40.

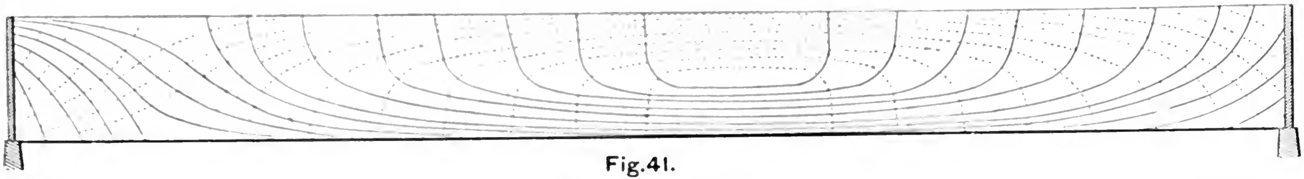


Fig. 41.

AIRY'S EXAMPLES.

(*Page 354.*)

from $x=A$ to $x=C$; $Q=0$ from $x=C$ to $x=L$]. The principles of § 228 therefore lead us to assume that ψ will be a discontinuous function of x, the discontinuities of form and value occurring at the planes $x=A$ and $x=C$, subject to the conditions of stress continuity (63) of § 226.

Let us then assume that

$$\begin{aligned}\psi &= \psi_1, \text{ from } x=0 \text{ to } x=A\,;\\ \psi &= \psi_2, \text{ from } x=A \text{ to } x=C\,;\\ \psi &= \psi_3, \text{ from } x=C \text{ to } x=L.\end{aligned}$$

If P_1, Q_1, U_1; P_2, Q_2, U_2; P_3, Q_3, U_3 be the stress components in the three regions into which we thus divide the beam, as deduced from ψ_1, ψ_2, ψ_3 respectively by means of equations (189), the conditions of stress continuity require that

$$\left.\begin{aligned}P_2 - P_1 = U_2 - U_1 = 0, &\text{ when } x=A\,;\\ P_2 - P_3 = U_2 - U_3 = 0, &\text{ when } x=C.\end{aligned}\right\}$$

The conditions that there may be no couples on the ends of the beam are

$$\left.\begin{aligned}\int_{-\frac12 D}^{\frac12 D} yP_1\,dy = 0, &\text{ when } x=0\,;\\ \int_{-\frac12 D}^{\frac12 D} yP_3\,dy = 0, &\text{ when } x=L.\end{aligned}\right\}$$

Since there is no tangential stress on any portion of the beam's surface except its ends,

$$U_1 = U_2 = U_3 = 0, \text{ when } y = \pm\tfrac12 D.$$

Since there is no normal stress on the sides of the beam within the first and third regions,

$$Q_1 = Q_3 = 0, \text{ when } y = \pm\tfrac12 D\,;$$

and similarly for the lower surface of the loaded region

$$Q_2 = 0, \text{ when } y = +\tfrac12 D.$$

The integral normal *pressure* on the surface of the loaded region is of course equal to the weight of the load, and since this is uniformly distributed

$$B(C-A)Q_2 = -\mathbf{W}.$$

Finally, the integral tangential stresses, reckoned upwards, over the two ends support between them the total weight of the beam and of the load; so that

$$B\int_{-\frac12 D}^{\frac12 D} U_1\,dy[x=0] - B\int_{\frac12 D}^{\frac12 D} U_3\,dy[x=L] = g\rho BDL + \mathbf{W}.$$

Substituting for the stress components their values in terms of ψ_1, ψ_2, ψ_3, these various conditions become

$$\frac{\partial^2}{\partial y^2}(\psi_2-\psi_1)=\frac{\partial^2}{\partial x\partial y}(\psi_2-\psi_1)=0, \text{ when } x=A \qquad (195)$$

$$\frac{\partial^2}{\partial y^2}(\psi_2-\psi_3)=\frac{\partial^2}{\partial x\partial y}(\psi_2-\psi_3)=0, \text{ when } x=C \qquad (196)$$

$$\tfrac{1}{2}D\left\{\frac{\partial\psi_1}{\partial y}[x=0,\ y=\tfrac{1}{2}D]+\frac{\partial\psi_1}{\partial y}[x=0,\ y=-\tfrac{1}{2}D]\right\}-\psi_1[x=0,\ y=\tfrac{1}{2}D]$$
$$+\psi_1[x=0,\ y=-\tfrac{1}{2}D]=0 \qquad (197)$$

$$\tfrac{1}{2}D\left\{\frac{\partial\psi_3}{\partial y}[x=L,\ y=\tfrac{1}{2}D]+\frac{\partial\psi_3}{\partial y}[x=L,\ y=-\tfrac{1}{2}D]\right\}-\psi_3[x=L,\ y=\tfrac{1}{2}D]$$
$$+\psi_3[x=L,\ y=-\tfrac{1}{2}D]=0 \qquad (198)$$

$$\frac{\partial^2\psi_1}{\partial x\partial y}=\frac{\partial^2\psi_2}{\partial x\partial y}=\frac{\partial^2\psi_3}{\partial x\partial y}=0, \text{ when } y=\pm\tfrac{1}{2}D \qquad (199)$$

$$\frac{\partial^2\psi_1}{\partial x^2}=\frac{\partial^2\psi_3}{\partial x^2}=\pm\tfrac{1}{2}\mathrm{g}\rho D, \text{ when } y=\pm\tfrac{1}{2}D \qquad (200)$$

$$\frac{\partial^2\psi_2}{\partial x^2}=\tfrac{1}{2}\mathrm{g}\rho D, \text{ when } y=\tfrac{1}{2}D \qquad (201)$$

$$\frac{\partial^2\psi_2}{\partial x^2}=-\tfrac{1}{2}\mathrm{g}\rho D-\frac{\mathbf{W}}{B(C-A)}, \text{ when } y=-\tfrac{1}{2}D \qquad (202)$$

$$\frac{\partial\psi_3}{\partial x}[x=L,\ y=\tfrac{1}{2}D]-\frac{\partial\psi_3}{\partial x}[x=L,\ y=-\tfrac{1}{2}D]-\frac{\partial\psi_1}{\partial x}[x=0,\ y=\tfrac{1}{2}D]$$
$$+\frac{\partial\psi_1}{\partial x}[x=0,\ y=-\tfrac{1}{2}D]=\mathrm{g}\rho DL+\frac{\mathbf{W}}{B} \qquad (203)$$

From (199) it appears that $\partial\psi_1/\partial y$, $\partial\psi_2/\partial y$, and $\partial\psi_3/\partial y$ must all contain the factor $(\tfrac{1}{4}D^2-y^2)$, and from (200) that ψ_1 and ψ_3 cannot involve even powers of y. From (200), (201) and (202) we deduce that $\partial^2\psi_1/\partial x^2$, $\partial^2\psi_2/\partial x^2$, $\partial^2\psi_3/\partial x^2$ must all be independent of x.

We shall satisfy (qualitatively) all these conditions, and at the same time (197) and (198), if we assume

$$\left.\begin{aligned}\psi_1&=\frac{3\mathrm{g}\rho}{D^2}x(x+\alpha)\left(\frac{D^2y}{4}-\frac{y^3}{3}\right)\\ \psi_2&=\frac{3\mathrm{g}\rho}{D^2}(x^2+\beta x+\gamma)\left[\lambda+\mu\left(\frac{D^2y}{4}-\frac{y^3}{3}\right)\right]\\ \psi_3&=\frac{3\mathrm{g}\rho}{D^2}(x-L)(x+\delta)\left(\frac{D^2y}{4}-\frac{y^3}{3}\right)\end{aligned}\right\}.$$

where α, β, γ, δ, λ, μ are constants, to be determined from the remaining conditions.

If we write for brevity

$$\theta = \mathbf{W}/g\rho BD(C-A), \qquad\qquad (204)$$

we must, in order to satisfy (201) and (202) quantitatively, make

$$\mu = 1 - 12\lambda/D^3 = 12\lambda/D^3 + 2\theta + 1.$$

Hence we find

$$\lambda = -\theta D^3/12, \; \mu = 1 + \theta;$$

and consequently

$$\psi_2 = \frac{3g\rho}{D^2}(x^2 + \beta x + \gamma)\left[(1+\theta)\left(\frac{D^2 y}{4} - \frac{y^3}{3}\right) - \frac{\theta D^3}{12}\right] \qquad\qquad (205)$$

Substitution in (203) gives us

$$\delta - a = L + 2\theta(C - A), \qquad\qquad (206)$$

while (195) and (196) become

$$1 + \theta = \frac{A(A+a)}{A^2 + A\beta + \gamma} = \frac{2A + a}{2A + \beta} = \frac{(C-L)(C+\delta)}{C^2 + C\beta + \gamma} = \frac{2C - L + \delta}{2C + \beta} \qquad (207)$$

The latter group is solved without difficulty, and gives

$$\left.\begin{aligned} a &= -L - \theta(C-A)(2L - C - A)/L \\ \beta &= -[L^2 + \theta(2CL - C^2 + A^2)]/(1+\theta)L \\ \gamma &= \theta A^2/(1+\theta) \\ \delta &= \theta(C^2 - A^2)/L \end{aligned}\right\},$$

and it will be found on trial that these values satisfy (206) identically.*

All the constants are determined, and the complete solution is expressed by

$$\left.\begin{aligned} \psi_1 &= \frac{3g}{D} x\left[x - L - \frac{\theta(C-A)(2L-C-A)}{L}\right]\left(\frac{D^2 y}{4} - \frac{y^3}{3}\right) \\ \psi_2 &= \frac{3g}{D}\left[x(x-L) - \frac{\theta}{1+\theta}\,\frac{A^2(x-L) - (L-C)^2 x}{L}\right]\left[(1+\theta)\left(\frac{D^2 y}{4} - \frac{y^3}{3}\right) - \frac{\theta D^3}{12}\right] \\ \psi_3 &= \frac{3g}{D}(x-L)\left[x + \frac{\theta(C^2 - A^2)}{L}\right]\left(\frac{D^2 y}{4} - \frac{y^3}{3}\right) \end{aligned}\right\} \qquad (208)$$

where θ is defined by (204).

The student should calculate the values of the stress components in the three regions, by means of equations (189), and convince himself that all the boundary conditions are satisfied—especially the condition of continuity of P and U throughout the body.

* This identity simply expresses the fact that the beam passes on the weight of the load, unaltered, to its supports.

307 *bis*.] **Important Addition and Correction.** The solutions of the problems suggested in the last two Articles were given—as has already been stated—on the authority of a paper by the late Astronomer Royal, published in a Report of the British Association. I now observe, however—when the printing of the Articles and engraving of the Figures is already completed—that they cannot be accepted as true solutions, inasmuch as they do not satisfy the general equation (164) of § 303. It is perhaps as well that they should be preserved as a warning to the student against the insidious and comparatively rare error of choosing a solution which satisfies completely all the boundary conditions, without satisfying the fundamental conditions of strain, and which is therefore of course not a solution at all. The indeterminateness of the C constants in Article 308 should have served as a timely warning, by its inconsistency with the general Theorem of Article 255. As for the diagrams of the Lines of Stress, they are only given as approximations, and a little consideration will convince the student—especially when he has mastered Chapter VII.—that they do represent the general character of the distribution of Tension and Thrust.

The remainder of this Article is to be considered as a continuation of § 307.

The functions χ_1, χ_2, χ_3, in terms of which we have expressed the six components of strain, are not wholly arbitrary, nor in general wholly independent. The six strain components being obtained by differentiation from three independent displacements, certain relations must exist between their derivatives in order to ensure the possibility of re-integration. From equations (1) of § 253, combined with (153) of § 301, we easily deduce

$$2\frac{\partial\theta_1}{\partial x}=\frac{\partial b}{\partial y}-\frac{\partial c}{\partial z},\quad 2\frac{\partial\theta_1}{\partial y}=\frac{\partial a}{dy}-2\frac{\partial f}{\partial z},\quad 2\frac{\partial\theta_1}{\partial z}=2\frac{\partial g}{\partial y}-\frac{\partial a}{\partial z};\quad\text{(A)}$$

with similar formulæ for the derivatives of θ_2 and θ_3: these may be verified by substitution, and are *identities*. On eliminating θ_1, θ_2, θ_3 by cross differentiation, in all possible ways, between these nine differential equations of the first order, we obtain the following six of the second order

$$\left.\begin{aligned}
\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}&=\frac{\partial^2 a}{\partial y\partial z}\\
\frac{\partial^2 e}{\partial z^2}+\frac{\partial^2 g}{\partial x^2}&=\frac{\partial^2 b}{\partial z\partial x}\\
\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 e}{\partial y^2}&=\frac{\partial^2 c}{\partial x\partial y}\\
2\left(\frac{\partial^2 e}{\partial y\partial z}+\frac{\partial^2 a}{\partial x^2}\right)&=\frac{\partial}{\partial x}\left(\frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}+\frac{\partial c}{\partial z}\right)\\
2\left(\frac{\partial^2 f}{\partial z\partial x}+\frac{\partial^2 b}{\partial y^2}\right)&=\frac{\partial}{\partial y}\left(\frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}+\frac{\partial c}{\partial z}\right)\\
2\left(\frac{\partial^2 g}{\partial x\partial y}+\frac{\partial^2 c}{\partial z^2}\right)&=\frac{\partial}{\partial z}\left(\frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}+\frac{\partial c}{\partial z}\right)
\end{aligned}\right\}\quad\text{(B)}$$

These six relations must be satisfied *identically* by every system of values that can legitimately be assumed for the six component strains, and every system of functions *which satisfies these and equations* (161) will also satisfy (164) and all other equations deducible from (161) by differentiation.

Substituting from (186) in (B), we have for the six fundamental relations between Airy's functions

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial y^2}[\Phi - (1+\sigma)\nabla^2\chi_3] + \frac{\partial^2}{\partial z^2}[\Phi - (1+\sigma)\nabla^2\chi_2] = 0\\
&\frac{\partial^2}{\partial z^2}[\Phi - (1+\sigma)\nabla^2\chi_1] + \frac{\partial^2}{\partial x^2}[\Phi - (1+\sigma)\nabla^2\chi_3] = 0\\
&\frac{\partial^2}{\partial x^2}[\Phi - (1+\sigma)\nabla^2\chi_2] + \frac{\partial^2}{\partial y^2}[\Phi - (1+\sigma)\nabla^2\chi_1] = 0\\
&\frac{\partial^2}{\partial y\partial z}[\Phi - (1+\sigma)\nabla^2\chi_1] = 0\\
&\frac{\partial^2}{\partial z\partial x}[\Phi - (1+\sigma)\nabla^2\chi_2] = 0\\
&\frac{\partial^2}{\partial x\partial y}[\Phi - (1+\sigma)\nabla^2\chi_3] = 0
\end{aligned}\right\}\ldots\ldots\ldots\ldots(C)$$

where

$$\Phi = \nabla^2(\chi_1 + \chi_2 + \chi_3) - \left(\frac{\partial^2\chi_1}{\partial x^2} + \frac{\partial^2\chi_2}{\partial y^2} + \frac{\partial^2\chi_3}{\partial z^2}\right) - (1-2\sigma)\rho\Psi\ldots\ldots\ldots(D)$$

In all cases of *Plane Stress* under gravity, such as those of §§ 308, 309, and of Examples 25—31, since the second derivatives of Ψ are all zero and χ_3 or χ is independent of z, equations (C) reduce to

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2}\right) = \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2}\right) = \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2}\right) = 0\ldots\ldots(E)$$

Thus

$$\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} = 2\alpha x + 2\beta y + 2\gamma,$$

where α, β, γ are arbitrary constants, and consequently

$$\chi = \tfrac{1}{3}[\alpha(x^3 + 3xy^2) + \beta(3x^2y + y^3) + 3\gamma(x^2 + y^2)] + \xi,\ldots\ldots\ldots\ldots(F)$$

where the complementary function ξ is *any* solution of

$$\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} = 0,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(G)$$

Boussinesq's Potential Solutions

310.] **Boussinesq's Theorem on the Differentiation of Potential Functions.** The ordinary "gravitation potential" of any distribution M of matter, at a given point P (x, y, z) is

$$\int \frac{dM}{r},$$

where r is the distance from P of the element dM of the **Potentiating Matter.*** The same nomenclature may be conveniently extended to the purely analytical function

$$\iiint \frac{\psi(x', y', z')}{r} dx'dy'dz',$$

where $$r = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$$

and the integration is extended throughout any (continuous or discontinuous) regions of space; the function ψ being finite for all values of x', y', z' within these regions. For this integral can always be converted into a true potential, simply by multiplying it by a constant factor of appropriate physical dimensions, and the regions of space throughout which it is integrated then correspond to those occupied by the potentiating masses.

We shall include the above function, and three others intimately connected with it, under the general title of *Potential Functions*, with the notation

$$\left.\begin{aligned} \mathbf{I} &= \iiint \frac{1}{r}\psi(x', y', z')dx'dy'dz' \\ \mathbf{D} &= \iiint r\psi(x', y', z')dx'dy'dz' \\ \mathbf{L}_1 &= \iiint \log(z - z' + r)\psi(x', y', z')dx'dy'dz' \\ \mathbf{L}_2 &= \iiint [(z - z')\log(z - z' + r) - r]\psi(x', y', z')dx'dy'dz' \end{aligned}\right\}\text{...(209)}$$

distinguishing them (with Boussinesq) as the *Inverse, Direct*, and *First* and *Second Logarithmic* Potential Functions. For analytical purposes we shall assume that the function ψ is, for all values of x', y', z', either finite or zero; and that, in approaching the surface which divides any region within which ψ is finite from any region within which it is zero, the value of ψ decreases continuously (however rapidly) till it vanishes at the actual surface. We thus make ψ continuous *in value* throughout

* Beltrami's *masse potenzianti*, Boussinesq's *masses potentiantes*. It would save many tedious periphrases if this most convenient term were universally adopted.

space, without in any way limiting its discontinuity *of form*, and consequently the first partial derivatives of ψ may be strictly regarded as finite throughout space.

We shall regard all space as divided into *Free Space* where ψ is zero, and *Space occupied by potentiating matter* where ψ differs from zero. It is obvious that all the integrals (209) may be indifferently regarded as extending throughout all space, or throughout those regions only which are occupied by matter.

The evaluation of these integrals and their derivatives presents no difficulty when P is situated in free space [*i.e.*, when $\psi(x, y, z)=0$]; but when P is within the potentiating matter [*i.e.*, when $\psi(x, y, z)$ differs from zero], it is possible for $x'-x$, $y'-y$, $z'-z$ to vanish, and in this case the functions to be integrated may include infinite terms.

The problem to be solved, before the Potential Functions can be made available for analytical purposes, is therefore to obtain formulæ of differentiation as to x, y, z of the function

$$\Phi=\iiint\psi(x', y', z')\chi(x'-x, y'-y, z'-z)dx'dy'dz', \qquad\qquad (210)$$

ntegrated throughout space, for all values of x, y, z:—

(i.) in the case where the continuous function χ can only become infinite at one point in space, namely, where

$$x'=x,\ y'=y,\ z'=z;$$

(ii.) in the case where the continuous function χ becomes nfinite for all those points where

$$x'=x,\ y'=y,$$

whatever be the value of z'.

First Case (applicable to **I**, **D** and their derivatives). Boussnesq's investigation rests on the following principle:—

The values of the integral, and of all its derivatives, will be unaltered, if we suppose the limits of integration to include all space except that enclosed by a small sphere with its centre at P, and an elementary radius κ; the value of κ being reduced to zero after evaluation.

If we accept this principle and remember that

$$\frac{\partial\Phi}{\partial x}dx$$

denotes simply the increase in the value of Φ, produced by the translation of the point P through an elementary distance dx in the positive direction of Ox, it is clear that (P being supposed always to carry with it its little enveloping sphere) this increase will consist of two distinct portions:—

First, the gain due to the displacement *of the sphere*, which has left behind it, on the negative side of P, a small volume now to be included within the limits of integration, and has taken from these limits an equal volume on the positive side of P. (*See* Fig. 42.)

Secondly, the increase, due to the displacement of P, in the value of that portion of the integral which is taken throughout all space *not included within the sphere in either of its positions.*

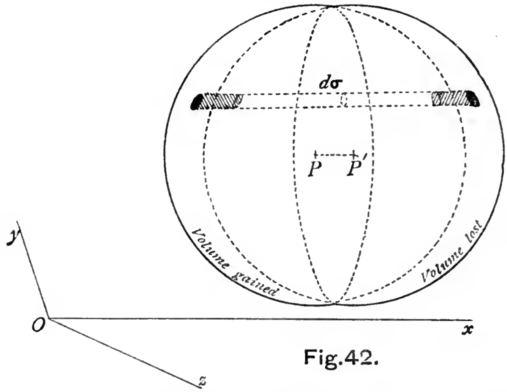


Fig.42.

Figure 42 represents the translation of the sphere, and shows clearly the loss and gain of equal volumes by outside space. To find the net gain of the integral Φ, let a cylinder of elementary section $d\sigma$ be drawn with its sides parallel to Ox, and cutting all four hemispheres. This cylinder will cut off an element of volume $dxd\sigma$ from the space gained and from the space lost. If we take (x', y', z') for the coördinates of the centre of $d\sigma$ the element of

$$\frac{\partial\Phi}{\partial x}dx$$

due to this cylinder will be very nearly

$$\begin{aligned}\{\psi[x-\sqrt{\kappa^2-(y'-y)^2-(z'-z)^2},y',z'].\chi[-\sqrt{\kappa^2-(y'-y)^2-(z'-z)^2},y'-y,z'-z]\\ -\psi[x+\sqrt{\kappa^2-(y'-y)^2-(z'-z)^2},y',z'].\chi[+\sqrt{\kappa^2-(y'-y)^2-(z'-z)^2},y'-y,z'-z]\}d\sigma dx\end{aligned}$$

The whole gain due to the shifting of the sphere is found by integrating this expression as to σ, over the whole area of the circle in which the two spherical surfaces intersect. If we write

$$\left.\begin{aligned} y' &= y+\eta\cos\omega \\ z' &= z+\eta\sin\omega \end{aligned}\right\}$$

then
$$\iint d\sigma = \int_0^{2\pi}\int_0^{\kappa}\eta\, d\eta\, d\omega,$$

and the net gain due to shifting of the sphere is

$$dx\int_0^{2\pi}\int_0^{\kappa}\{\psi[x-\sqrt{\kappa^2-\eta^2},\ y+\eta\cos\omega,\ z+\eta\sin\omega]\,.\,\chi[-\sqrt{\kappa^2-\eta^2},\ \eta\cos\omega,\ \eta\sin\omega]$$

$$-\psi[x+\sqrt{\kappa^2-\eta^2},\ y+\eta\cos\omega,\ z+\eta\sin\omega]\,.\,\chi[+\sqrt{\kappa^2-\eta^2},\ \eta\cos\omega,\ \eta\sin\omega]\}\,\eta d\eta d\omega\,;$$

and since $\kappa>\eta>0$, and κ is ultimately to be made zero, this may be written

$$\psi(x,\ y,\ z)dx\int_0^{2\pi}\int_0^{\kappa}\{\chi[-\sqrt{\kappa^2-\eta^2},\ \eta\cos\omega,\ \eta\sin\omega]$$

$$-\chi[+\sqrt{\kappa^2-\eta^2},\ \eta\cos\omega,\ \eta\sin\omega]\}\,\eta d\eta d\omega.$$

Again, the gain in that portion of the integral which is taken throughout all space excluded by the sphere in both positions is of course

$$dx\frac{\partial}{\partial x}\iiint\psi(x',\ y',\ z')\chi(x'-x,\ y'-y,\ z'-z)dx'dy'dz'$$

or

$$dx\iiint\psi\frac{\partial\chi}{\partial x}dxdydz$$

the limits of integration excluding both spheres.

Adding together these two terms, and proceeding to the limit, we have finally*

$$\frac{\partial\Phi}{\partial x}=\iiint\psi(x',\ y',\ z')\frac{\partial}{\partial x}\chi(x'-x,\ y'-y,\ z'-z)dx'dy'dz'$$

$$+\psi(x,\ y,\ z)\int_0^{2\pi}\int_0^{\kappa}\{\chi[-\sqrt{\kappa^2-\eta^2},\ \eta\cos\omega,\ \eta\sin\omega]$$

$$-\chi[+\sqrt{\kappa^2-\eta^2},\ \eta\cos\omega,\ \eta\sin\omega]\}\,\eta d\eta d\omega\ldots\ldots\ldots\ldots\ldots(212)$$

where the triple integral is taken throughout all space, and the double integral is to be given the limiting value it assumes when $\kappa=0$.

If P is in free space $\psi(x,\ y,\ z)=0$, and†

$$\frac{\partial\Phi}{\partial x}=\iiint\psi(x',\ y',\ z')\,.\,\frac{\partial\chi}{\partial x}\,.\,dx'dy'dz'\ldots\ldots\ldots\ldots\ldots(213)$$

The formulæ for differentiation as to y and z may be deduced by symmetry from (212) and (213).

It may be observed, as a useful general rule, that if χ is a homogeneous function of order n, the *residual* (double) integral is of the dimensions κ^{n+2}, and is therefore ultimately negligible if $n>-2$.

* Quoted in Article 283. † Quoted in Article 284.

Second Case (applicable to L_1, L_2, and their derivatives). When the function χ becomes infinite if $x'=x$, $y'=y$, for all values of z', we must replace the sphere by a circular cylinder of infinite length and radius κ (ultimately zero, as before), with the line $x'=x$, $y'=y$ for axis

Taking
$$\left.\begin{aligned} y'-y &= \eta \\ x'-x &= \sqrt{\kappa^2-\eta^2} \end{aligned}\right\}$$

and remembering that a shifting of the cylinder *parallel to* Oz *does not affect the limits of integration*, we find as before

$$\frac{\partial\Phi}{\partial x}=\iiint\psi(x',y',z')\frac{\partial\chi}{\partial x}dx'dy'dz'+\int_{-\infty}^{\infty}\psi(x,y,z')\int_{-\kappa}^{\kappa}\{\chi[-\sqrt{\kappa^2-\eta^2},\eta,z'-z]$$
$$-\chi[+\sqrt{\kappa^2-\eta^2},\eta,z'-z]\}d\eta dz', \quad\text{.......(214)}$$

with a symmetrical formula for $\partial\Phi/\partial y$; while

$$\frac{\partial\Phi}{\partial z}=\iiint\psi(x',y',z')\frac{\partial\chi}{\partial z}dx'dy'dz', \quad\text{.......(215)}$$

whatever be the position of P.

If $\psi(x',y',z')$ is zero for all values of z', when $x'=x$, $y'=y$, the formula (214) becomes symmetrical with (215), as the residual integral then vanishes identically.

311.] **Alternative Formulæ.** If we shift the origin a distance dx in the negative direction of Ox, we change x, x' into $x+dx$, $x'+dx$, without altering y, y', z, z', or the infinite limits of integration.

Thus, since $(x'+dx)-(x+dx)=x'-x$,

$$\Phi(x+dx,y,z)=\iiint\psi(x'+dx,y',z')\chi(x'-x,y'-y,z'-z)dx'dy'dz',$$

and therefore *

$$\left.\begin{aligned} \frac{\partial\Phi}{\partial x}&=\iiint\frac{\partial\psi}{\partial x'}\cdot\chi\cdot dx'dy'dz' \\ &\text{etc., etc.} \end{aligned}\right\} \quad\text{.......(216)}$$

This formula is independent of the position of P and of the form of the function χ.

312.] **Application of the formulæ of differentiation to the Potential Functions.** It is left as an exercise for the student to show, by successive applications of the formulæ (212), (214) and (215) to the Potential Functions (209), that

$$\frac{\partial L_2}{\partial z}=L_1,\quad \frac{\partial^2 L_2}{\partial z^2}=\frac{\partial L_1}{\partial z}=I; \quad\text{.......(217)}$$

* Quoted in Articles 266 (i) and 301.

$$\left.\begin{aligned}&\nabla^2\mathbf{D} = 2\mathbf{I},\ \nabla^2\mathbf{I} = -4\pi\psi\,;\\ &\nabla^2\mathbf{L}_1 = 4\pi\int_z^\infty \psi(x,\ y,\ z')dz'\,;\\ &\nabla^2\mathbf{L}_2 = 4\pi\int_z^\infty \psi(x,\ y,\ z')(z-z')dz'\,;\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(218)$$

and consequently that

$$\nabla^2\mathbf{I} = \nabla^2(\tfrac{1}{2}\nabla^2\mathbf{D}) = \nabla^2\left(\frac{\partial\mathbf{L}_1}{\partial z}\right) = \nabla^2\left(\frac{\partial^2\mathbf{L}_2}{\partial z^2}\right) = -4\pi\psi.\ \ldots\ldots\ldots(219)$$

313.] **Boussinesq's Applications of the Potential Functions to the solution of problems in Elasticity.** One of the most remarkable applications of these functions is to the investigation of the strain produced in an isotropic solid—bounded by one plane face, but otherwise unlimited in extent—by an arbitrary distribution of surface traction over the whole or certain circumscribed portions of the plane face.

We shall here confine ourselves to the case in which no impressed forces act upon the body, and in which the surface traction is wholly normal, of finite magnitude, and applied only to finite areas of the surface.

Taking the face of the solid for the plane of xy, with the axis of z extending into the body, the surface conditions are

$$\left.\begin{aligned}T &= F = 0\\ S &= G = 0\\ R &= H = \psi(x,\ y)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(220)$$

when $z=0$; ψ being an arbitrary discontinuous function of x and y having finite and continuous values over certain circumscribed areas of the plane of xy, and being elsewhere zero.

Since the integral

$$\iint \psi\, dx'\, dy'$$

taken over the whole plane face is finite, the total stress (§ 131) across any hemispherical portion of the body, standing upon its plane surface as base, must also be finite. Thus for great values of $r\ (=\sqrt{x^2+y^2+z^2})$ the stress components must be—at the greatest—of dimensions r^{-2}, and consequently the displacements of dimensions r^{-1}.

Finally, the general equations (104) must be satisfied at every point of the body.

314.] **First general type of solution.** If ϕ be any function whatever that satisfies Laplace's equation $\nabla^2\phi=0$, we have

$$\nabla^2(z\phi) = 2\frac{\partial\phi}{\partial z} \qquad \text{(221)}$$

identically, and consequently

$$\frac{\partial}{\partial x}\left(2\frac{\partial\phi}{\partial z}\right) = \nabla^2\frac{\partial}{\partial x}(z\phi), \text{ etc.}$$

If therefore we assume

$$\left.\begin{aligned} u &= -\frac{\partial}{\partial x}(z\phi), \quad v = -\frac{\partial}{\partial y}(z\phi) \\ w &= -\frac{\partial}{\partial z}(z\phi) + \mathrm{w} \end{aligned}\right\}$$

we shall have

$$\Delta = \frac{\partial}{\partial z}(\mathrm{w} - 2\phi),$$

and on substitution in (104), we find, as the necessary and sufficient condition that the general equations of equilibrium may be satisfied,

$$\mathrm{w} = \frac{2(m+n)}{m}\phi.$$

Let us then assume for our general solution

$$\left.\begin{aligned} u &= -z\frac{\partial\phi}{\partial x} \\ v &= -z\frac{\partial\phi}{\partial y} \\ w &= -z\frac{\partial\phi}{\partial z} + \frac{m+2n}{m}\phi \end{aligned}\right\} \qquad \text{(222)}$$

where ϕ is some root of Laplace's equation which at a great distance r from the origin is of dimensions r^{-1}, at the most.

From (222) we obtain without difficulty, by the help of (221),

$$\Delta = \frac{2n}{m}\frac{\partial\phi}{\partial z} = \frac{n}{m}\nabla^2(z\phi) \qquad \text{(223)}$$

$$\left.\begin{aligned} T &= 2n\frac{\partial}{\partial x}\left[\frac{n}{m}\phi - z\frac{\partial\phi}{\partial z}\right] \\ S &= 2n\frac{\partial}{\partial y}\left[\frac{n}{m}\phi - z\frac{\partial\phi}{\partial z}\right] \\ R &= 2n\left[\frac{m+n}{m}\frac{\partial\phi}{\partial z} - z\frac{\partial^2\phi}{\partial z^2}\right] \end{aligned}\right\} \qquad \text{(224)}$$

315.] **Second general type of solution.** If again we assume

$$u = \frac{\partial \phi'}{\partial x}, \quad v = \frac{\partial \phi'}{\partial y}, \quad w = \frac{\partial \phi'}{\partial z}, \quad \text{.....................(225)}$$

ϕ' being any root of Laplace's equation which for great values of r is of zero or negative dimensions in r, the general equations and the conditions required at infinite distances will both be satisfied.

We deduce from (225)

$$\Delta = 0, \quad \text{..(226)}$$

$$\left.\begin{aligned} T &= 2n\frac{\partial^2\phi'}{\partial z\partial x} \\ S &= 2n\frac{\partial^2\phi'}{\partial y\partial z} \\ R &= 2n\frac{\partial^2\phi'}{\partial z^2} \end{aligned}\right\} \quad \text{..............................(227)}$$

316.] **Special forms of the Potential functions, when reduced to Surface Integrals.** The imaginary distribution of potentiating matter, to which the functions (209) of § 310 may be supposed due, may of course be confined to a surface layer, of *surface density* ψ per unit area, over the whole or portions of the plane of xy. In this case the first three integrals reduce to the forms*

$$\left.\begin{aligned} \mathbf{I} &= \iint \psi(x', y')\frac{dx'dy'}{r} \\ \mathbf{D} &= \iint \psi(x', y')r\,dx'dy' \\ \mathbf{L}_1 &= \iint \psi(x', y')\log(z + r)dx'dy' \end{aligned}\right\} \quad \text{.....................(228)}$$

where $\qquad r = \sqrt{(x - x')^2 + (y - y')^2 + z^2}\,; \quad$ ....(229)

and, since in this case $\psi = 0$ throughout the interior of the body, we deduce from the equations of § 312

$$\frac{\partial \mathbf{L}_1}{\partial z} = \mathbf{I}, \ \nabla^2\mathbf{D} = 2\mathbf{I}, \ \nabla^2\mathbf{I} = 0, \ \nabla^2\mathbf{L}_1 = 0 \quad \text{................(230)}$$

Also, ψ being finite or zero at all points of the surface, $\mathbf{I}$ is of the dimensions required in § 314, and $\mathbf{L}_1$ of those required in § 315.

* The student will observe the applicability of these surface integrals to the method suggested in Article 284 for the solution of the Problem of Free Vibrations.

317. Special form of the first type of solution. Putting $\phi = \mathrm{I}$ in equations (222), (223), (224), they become

$$\left.\begin{aligned} u &= -z\frac{\partial}{\partial x}\iint\frac{\psi dx'dy'}{r} \\ v &= -z\frac{\partial}{\partial y}\iint\frac{\psi dx'dy'}{r} \\ w &= -z\frac{\partial}{\partial z}\iint\frac{\psi dx'dy'}{r} + \frac{m+2n}{m}\iint\frac{\psi dx'dy'}{r} \end{aligned}\right\}\qquad(231)$$

$$\Delta = \frac{2n}{m}\frac{\partial}{\partial z}\iint\frac{\psi dx'dy'}{r} \qquad(232)$$

$$\left.\begin{aligned} T &= 2n\frac{\partial}{\partial x}\left[\frac{n}{m}\iint\frac{\psi dx'dy'}{r} - z\frac{\partial}{\partial z}\iint\frac{\psi dx'dy'}{r}\right] \\ S &= 2n\frac{\partial}{\partial y}\left[\frac{n}{m}\iint\frac{\psi dx'dy'}{r} - z\frac{\partial}{\partial z}\iint\frac{\psi dx'dy'}{r}\right] \\ R &= 2n\left[\frac{m+n}{m}\frac{\partial}{\partial z}\iint\frac{\psi dx'dy'}{r} - z\frac{\partial^2}{\partial z^2}\iint\frac{\psi dx'dy'}{r}\right] \end{aligned}\right\}\qquad(233)$$

Performing the differentiations as to z, by means of formula (213) of § 310, we obtain

$$\left.\begin{aligned} w &= z\iint\frac{z\psi dx'dy'}{r^3} + \frac{m+2n}{m}\iint\frac{\psi dx'dy'}{r^3} \\ \Delta &= -\frac{2n}{m}\iint\frac{z\psi dx'dy'}{r^3} \\ T &= 2n\frac{\partial}{\partial x}\left[\frac{n}{m}\iint\frac{\psi dx'dy'}{r} + z\iint\frac{z\psi dx'dy'}{r^3}\right] \\ S &= 2n\frac{\partial}{\partial y}\left[\frac{n}{m}\iint\frac{\psi dx'dy'}{r} + z\iint\frac{z\psi dx'dy'}{r^3}\right] \\ R &= -2n\left[\frac{n}{m}\iint\frac{z\psi dx'dy'}{r^3} + 3\iint\frac{z^3\psi dx'dy'}{r^5}\right] \end{aligned}\right\}.$$

The integral

$$\iint\frac{\psi dx'dy'}{r}$$

and its derivatives as to x and y are certainly finite for all values of z, and. in order to determine fully the surface conditions, it only remains to evaluate the integrals

$$\iint\frac{z\psi dx'dy'}{r^3},\quad \iint\frac{z^3\psi dx'dy'}{r^5}.$$

To do this, transform the independent variables by the substitutions

$$\left.\begin{aligned} x' &= x + z\eta\cos\omega \\ y' &= y + z\eta\sin\omega \end{aligned}\right\}$$

so that $\qquad r = z\sqrt{1+\eta^2},\ dx'dy' = z^2\eta d\eta d\omega.$

Since $\psi = 0$, except over certain finite areas of the surface, we may extend the limits of integration so as to include the entire plane of xy; thus the limits of η are 0 and ∞, and those of ω are 0 and 2π.

The transformed integrals are

$$\left.\begin{aligned} \iint \frac{z\psi dx'dy'}{r^3} &= \int_0^{2\pi}\int_0^{\infty} \frac{\psi\eta d\eta d\omega}{(1+\eta^2)^{\frac{3}{2}}} \\ \iint \frac{z^3\psi dx'dy'}{r^5} &= \int_0^{2\pi}\int_0^{\infty} \frac{\psi\eta d\eta d\omega}{(1+\eta^2)^{\frac{5}{2}}} \end{aligned}\right\}.$$

But *at the surface* of the body $z=0$, and consequently $\psi(x', y')$ becomes independent of η and ω and may be written $\psi(x, y)$ in the above integrals. The *surface values* are therefore

$$\left.\begin{aligned} \iint \frac{z\psi dx'dy'}{r^3} &= 2\pi\psi(x, y)\int_0^{\infty} \frac{\eta d\eta}{(1+\eta^2)^{\frac{3}{2}}} = 2\pi\psi(x, y) \\ \iint \frac{z^3\psi dx'dy'}{r^5} &= 2\pi\psi(x, y)\int_0^{\infty} \frac{\eta d\eta}{(1+\eta^2)^{\frac{5}{2}}} = \tfrac{2}{3}\pi\psi(x, y) \end{aligned}\right\} \quad\text{..........(234)}$$

Substituting these values in (231), (232), (233), and putting $z=0$, we obtain for the surface values of u, v, w, Δ, T, S, R

$$\left.\begin{aligned} u_0 &= 0,\ v_0 = 0,\ w_0 = \frac{m+2n}{m}\iint\frac{\psi dx'dy'}{r} \\ \Delta_0 &= -\frac{4\pi n}{m}\psi(x, y) \\ F &= \frac{2n^2}{m}\frac{\partial}{\partial x}\iint\frac{\psi dx'dy'}{r} \\ G &= \frac{2n^2}{m}\frac{\partial}{\partial y}\iint\frac{\psi dx'dy'}{r} \\ H &= -\frac{4\pi n(m+n)}{m}\psi(x, y) \end{aligned}\right\} \quad\text{.............(235)}$$

318.] **Special Form of the second type of solution.** Writing L_1 for ϕ' in the equations of § 315, they give us

$$\left.\begin{aligned} u &= \frac{\partial}{\partial x}\iint \psi \log(z+r)dx'dy' \\ v &= \frac{\partial}{\partial x}\iint \psi \log(z+r)dx'dy' \\ w &= \iint \frac{\psi dx'dy'}{r} \end{aligned}\right\} \qquad (236)$$

$$\Delta = 0 \qquad (237)$$

$$\left.\begin{aligned} T &= 2n\frac{\partial}{\partial x}\iint \frac{\psi dx'dy'}{r} \\ S &= 2n\frac{\partial}{\partial y}\iint \frac{\psi dx'dy'}{r} \\ R &= 2n\frac{\partial}{\partial z}\iint \frac{\psi dx'dy'}{r} \end{aligned}\right\} \qquad (238)$$

and we deduce, with the assistance of formulæ (234) of § 317, that at the surface of the body

$$\left.\begin{aligned} u_0 &= \frac{\partial}{\partial x}\iint \psi \log r dx'dy' \\ v_0 &= \frac{\partial}{\partial y}\iint \psi \log r dx'dy' \\ w_0 &= \iint \frac{\psi dx'dy'}{r} \end{aligned}\right\} \qquad (239)$$

$$\Delta_0 = 0 \qquad (240)$$

$$\left.\begin{aligned} F &= 2n\frac{\partial}{\partial x}\iint \frac{\psi dx'dy'}{r} \\ G &= 2n\frac{\partial}{\partial y}\iint \frac{\psi dx'dy'}{r} \\ H &= -4\pi n\psi(x, y) \end{aligned}\right\} \qquad (241)$$

319.] **Solution compounded of the two simple types, and adapted to the case in which the arbitrarily distributed Surface Traction is wholly normal.** Multiplying the values of § 317 by $-1/4\pi n$, and those of § 318 by $+1/4\pi m$, and compounding (231) with 236, (232) with (237), (233) with (238), and (235) with (239—241), we have the system of displacements

$$\left.\begin{aligned} u &= \frac{z}{4\pi n}\frac{\partial}{\partial x}\iint\frac{\psi dx'dy'}{r} + \frac{1}{4\pi m}\frac{\partial}{\partial x}\iint\psi\log(r+z)dx'dy' \\ v &= \frac{z}{4\pi n}\frac{\partial}{\partial y}\iint\frac{\psi dx'dy'}{r} + \frac{1}{4\pi m}\frac{\partial}{\partial y}\iint\psi\log(r+z)dx'dy' \\ w &= \frac{z}{4\pi n}\frac{\partial}{\partial z}\iint\frac{\psi dx'dy'}{r} - \frac{m+n}{4\pi mn}\iint\frac{\psi dx'dy'}{r} \end{aligned}\right\} \quad \text{.....(242)}$$

giving
$$\Delta = -\frac{1}{2\pi m}\frac{\partial}{\partial z}\iint\frac{\psi dx'dy'}{r} \quad \text{....................(243)}$$

and

$$\left.\begin{aligned} T &= \frac{z}{2\pi}\frac{\partial^2}{\partial z\partial x}\iint\frac{\psi dx'dy'}{r} \\ S &= \frac{z}{2\pi}\frac{\partial^2}{\partial y\partial z}\iint\frac{\psi dx'dy'}{r} \\ R &= \frac{1}{2\pi}\left[z\frac{\partial^2}{\partial z^2}\iint\frac{\psi dx'dy'}{r} - \frac{\partial}{\partial z}\iint\frac{\psi dx'dy'}{r}\right] \end{aligned}\right\} \quad \text{..........(244)}$$

throughout the body: while at the surface

$$\left.\begin{aligned} u_0 &= \frac{1}{4\pi m}\frac{\partial}{\partial x}\iint\psi\log r dx'dy' \\ v_0 &= \frac{1}{4\pi m}\frac{\partial}{\partial y}\iint\psi\log r dx'dy' \\ w_0 &= -\frac{m+n}{4\pi mn}\iint\frac{\psi dx'dy'}{r} \end{aligned}\right\} \quad \text{....................(245)}$$

$$\Delta_0 = \frac{\psi(x, y)}{m}; \quad \text{..............................(246)}$$

$$F = 0, \quad G = 0, \quad H = \psi(x, y) \quad \text{......................(247)}$$

We have therefore obtained a complete and perfectly general solution of the problem proposed in § 313.

Examples.

320.] **Regarding the earth as an infinite isotropic solid, with one plane face, to determine the strain produced by a very small but heavy mass lying on the surface.** The surface of contact being very small, the integrals will each reduce to a single element, which we will suppose to be at the origin. We shall then have $x' = 0$, $y' = 0$, and

$$\psi dx'dy' = H dx'dy' = -\mathbf{W},$$

where $\mathbf{W}$ is the incumbent weight. Substituting in equations (242—245), and performing the indicated differentiations on r^{-1} and $\log(r+z)$, we obtain

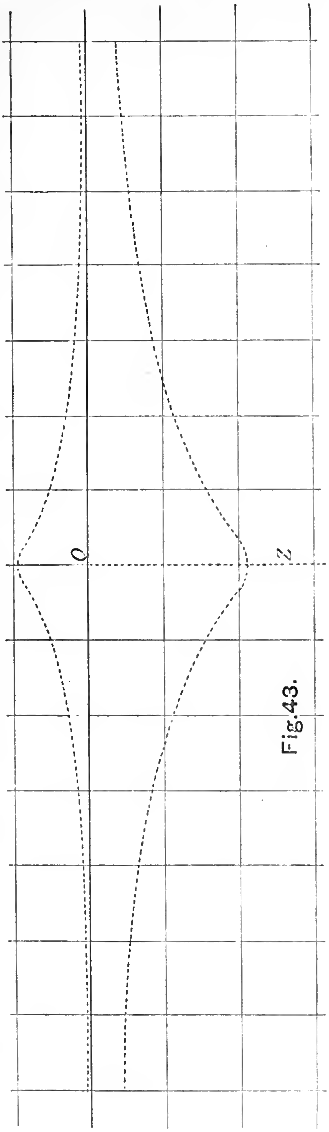


Fig. 43.

$$\left.\begin{aligned} u &= \frac{x}{r}\cdot\frac{W}{4\pi}\left[\frac{z}{nr^2}-\frac{1}{m(r+z)}\right] \\ w &= \frac{y}{r}\cdot\frac{W}{4\pi}\left[\frac{z}{nr^2}-\frac{1}{m(r+z)}\right] \\ v &= \frac{z}{r}\cdot\frac{W}{4\pi}\left[\frac{z}{nr^2}+\frac{m+n}{mnz}\right] \end{aligned}\right\},$$

$$\Delta = -\frac{zW}{2\pi m r^3}\Bigg],$$

$$\left.\begin{aligned} T &= -\frac{x}{r}\cdot\frac{3z^2W}{2\pi r^4} \\ S &= -\frac{y}{r}\cdot\frac{3z^2W}{2\pi r^4} \\ R &= -\frac{z}{r}\cdot\frac{3z^2W}{2\pi r^4} \end{aligned}\right\},$$

throughout the body; and

$$\left.\begin{aligned} u_0 &= -\frac{x}{r}\cdot\frac{W}{4\pi m r} \\ v_0 &= -\frac{y}{r}\cdot\frac{W}{4\pi m r} \\ w_0 &= \frac{(m+n)W}{4\pi m n r} \end{aligned}\right\},$$

over the surface.

W must of course be supposed very small in comparison with the weight moduli of the ground; it must also be remembered that, while W/r represents fairly at distant points the potential of the last Article, yet these formulæ cannot be considered to hold right up to the origin. The above values of u_0, v_0, w_0 represent the strained surface as formed by the revolution about Oz of the hyperbola

$$z'\left(z'+\frac{m+n}{n}x'\right)=\left(\frac{m+n}{n}\right)^2\frac{W}{4\pi m};$$

but the central depression will in fact be rounded instead of conical (see Figure 43), in accordance with the statement of § 55 that discontinuous curvature cannot be *produced* by small strain.

321.] **A right circular cylinder, formed of homogeneous and perfectly rigid material, stands on end upon the ground; required the deformation produced by its weight.** Let **W** be the weight of the cylinder, and A the radius of its base: let the centre of its base be placed at the origin. Since the cylinder is rigid, its base will remain plane, and consequently we must have w_0 constant all over the area of contact. Also the conditions will be symmetrical about the axis of z, and therefore we may write $\psi(\eta)$ for $\psi(x', y')$ and

$$\int_0^{2\pi}\int_0^{A} \Phi\eta d\eta d\omega \text{ for } \iint \Phi dx'dy',$$

where

$$x' = \eta\cos\omega, \quad y' = \eta\sin\omega.$$

Thus the problem reduces itself to the following:—Required a function $\psi(\eta)$, such that

$$\int_0^{2\pi}\int_0^{A} \psi(\eta)\eta d\eta d\omega = -\mathbf{W}, \qquad (248)$$

while

$$\int_0^{2\pi}\int_0^{A} \frac{\psi(\eta)\eta d\eta d\omega}{\sqrt{(x-\eta\cos\omega)^2+(y-\eta\sin\omega)^2}} \qquad (249)$$

is constant for all values of x and y that make $x^2+y^2<A^2$.

Now we know from the theory of electricity * that a free charge E will distribute itself over a circular conducting disc of radius A in such a manner that the surface density on *either side* of the disc is $E/4\pi A\sqrt{A^2-\eta^2}$, while the potential due to the distribution is $\pi E/2A$ at all points *within* the disc, and at all external points is

$$\frac{E}{2}\int_{\xi}^{\infty} \frac{d\lambda}{(A^2+\lambda)\sqrt{\lambda}} = \frac{E}{A}\tan^{-1}\frac{A}{\sqrt{\xi}},$$

where ξ is the greater root of the quadratic

$$\frac{x^2+y^2}{A^2+\xi} + \frac{z^2}{\xi} = 1 \qquad (250)$$

Thus if we make

$$\psi(\eta) = -\frac{\mathbf{W}}{2\pi A\sqrt{A^2-\eta^2}}, \qquad (251)$$

* The formulæ here quoted are easily deduced from those of the ellipsoidal shell, by making (with the notation of Article 252) $B=A$, $C=0$.

from $\eta=0$ to $\eta=A$, (248) will be satisfied identically, while the value of the integral (249) will be $-\pi\mathbf{W}/2A$ all over the area of contact, and for all the rest of space

$$\int_0^{2\pi}\int_0^A \frac{\psi(\eta)\eta d\eta d\omega}{\sqrt{(x-\eta\cos\omega)^2+(y-\eta\sin\omega)^2+z^2}} = -\frac{\mathbf{W}}{A}\tan^{-1}\frac{A}{\sqrt{\xi}}, \text{.......(252)}$$

ξ being given by (250).

Finally we may note that

$$\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^A \log[z+\sqrt{(x-\eta\cos\omega)^2+(y-\eta\sin\omega)^2+z^2}]\psi(\eta)\eta d\eta d\omega$$
$$=\int_0^{2\pi}\int_0^A \frac{\psi(\eta)\eta d\eta d\omega}{\sqrt{(x-\eta\cos\omega)^2+(y-\eta\sin\omega)^2+z^2}},$$

and since the latter function vanishes when $z=\infty$, we may write

$$\int_0^{2\pi}\int_0^A \log[z+\sqrt{(x-\eta\cos\omega)^2+(y-\eta\sin\omega)^2+z}]\psi(\eta)\eta d\eta d\omega$$
$$=-\int_z^\infty\int_0^{2\pi}\int_0^A \frac{\psi(\eta)\eta d\eta d\omega dz}{\sqrt{(x-\eta\cos\omega)^2+(y-\eta\sin\omega)^2+z^2}}$$
$$=\frac{\mathbf{W}}{A}\int_z^\infty \tan^{-1}\frac{A}{\sqrt{\xi}}dz \text{.....................................(253)}$$

Substituting in the equations of § 319, we have at the surface of the body

$$\left.\begin{aligned}
u_0 &= -\frac{\mathbf{W}}{8\pi^2 mA}\int_0^{2\pi}\int_0^A \frac{(x-\eta\cos\omega)\eta d\eta d\omega}{[(x-\eta\cos\omega)^2+(y-\eta\sin\omega)^2]\sqrt{A^2-\eta^2}} \\
v_0 &= -\frac{\mathbf{W}}{8\pi^2 mA}\int_0^{2\pi}\int_0^A \frac{(y-\eta\sin\omega)\eta d\eta d\omega}{[(x-\eta\cos\omega)^2+(y-\eta\sin\omega)^2]\sqrt{A^2-\eta^2}} \\
w_0 &= \frac{(m+n)\mathbf{W}}{8mnA}, \text{ if } x^2+y^2<A^2 \\
&= \frac{(m+n)\mathbf{W}}{8mnA}\left[1-\frac{2}{\pi}\tan^{-1}\frac{\sqrt{x^2+y^2-A^2}}{A}\right], \text{ if } x^2+y^2>A^2
\end{aligned}\right\} \text{..(253)}$$

and throughout its substance

$$\left.\begin{aligned}
u &= -\frac{\mathbf{W}}{4\pi mA}\frac{\partial}{\partial x}\left[\frac{mz}{n}\tan^{-1}\frac{A}{\sqrt{\xi}}-\int_z^\infty \tan^{-1}\frac{A}{\sqrt{\xi}}dz\right] \\
v &= -\frac{\mathbf{W}}{4\pi mA}\frac{\partial}{\partial y}\left[\frac{mz}{n}\tan^{-1}\frac{A}{\sqrt{\xi}}-\int_z^\infty \tan^{-1}\frac{A}{\sqrt{\xi}}dz\right] \\
w &= -\frac{\mathbf{W}}{4\pi mA}\left[\frac{mz}{n}\frac{\partial}{\partial z}-\frac{m+n}{n}\right]\tan^{-1}\frac{A}{\sqrt{\xi}}
\end{aligned}\right\} \text{........(254)}$$

We deduce without difficulty that over the area of contact

$$\left.\begin{aligned} u_0 &= -\frac{\mathbf{W}x}{4\pi m(x^2+y^2)}\left[1-\sqrt{1-\frac{x^2+y^2}{A^2}}\right] \\ v_0 &= -\frac{\mathbf{W}y}{4\pi m(x^2+y^2)}\left[1-\sqrt{1-\frac{x^2+y^2}{A^2}}\right] \end{aligned}\right\};$$

while over the free surface

$$\left.\begin{aligned} u_0 &= -\frac{\mathbf{W}x}{4\pi m(x^2+y^2)} \\ v_0 &= -\frac{\mathbf{W}y}{4\pi m(x^2+y^2)} \end{aligned}\right\}.$$

Thus the horizontal component of the surface displacement is directed towards the axis of z, and its magnitude is

$$\frac{\mathbf{W}}{4\pi m\sqrt{x^2+y^2}},$$

or

$$\frac{\mathbf{W}}{4\pi m\sqrt{x^2+y^2}}\left[1-\sqrt{1-\frac{x^2+y^2}{A^2}}\right],$$

according as the displaced point is on the free surface or within the area of contact.

The student must be referred to Boussinesq's original memoir *Sur l'application des Potentiels à l'étude de l'équilibre et du mouvement des Solides Elastiques* (Gauthier-Villars, Paris; 1885) for more extended applications of this theory, together with some interesting examples.

EXAMPLES.

[*Unless the contrary is expressly stated, it is to be assumed that the body under consideration is free from Applied Forces.*]

Vibrations.

1. The following forms of ϕ all satisfy equation (63), and consequently represent possible forms of free irrotational vibration:

$$(i.)\quad \phi_i = \frac{\sin}{\cos}\left[\frac{iz}{\Omega}\sqrt{a^2+\beta^2+1}\right].\exp\left[\frac{\pm i\alpha x \pm i\beta y}{\Omega}\right].$$

$$(ii.)\quad \phi_i = \frac{\sin}{\cos}\left[\frac{i(\alpha x \pm \beta y)}{\Omega}\right].\exp\left[\pm\frac{iz\sqrt{a^2+\beta^2-1}}{\Omega}\right].$$

$$(iii.)\quad \phi_i = \frac{\sin}{\cos}\left[\frac{i(x \pm \beta y)}{\Omega}\right].\exp\left[\pm\frac{i\beta z}{\Omega}\right].$$

(*Note.* $\exp[\theta]$ is equivalent to e^{θ}).

2. The following is the general solution for symmetrical waves of longitudinal displacement, radiating from or converging to a single centre:

$$\left.\begin{aligned}\phi &= \frac{1}{r}\Sigma A_i \sin i\left(t \mp \frac{r}{\Omega} - a_i\right) \\ u = \frac{\partial\phi}{\partial x}, \quad v &= \frac{\partial\phi}{\partial y}, \quad w = \frac{\partial\phi}{\partial z}.\end{aligned}\right\}.$$

Special cases are considered in § 273, and in the following Example.

3. A spherical shell whose internal and external radii are A and B, vibrates radially, the motion being symmetrical about the centre. Prove that the admissible values of i are given by $i = \mathrm{i}\Omega$, where

$$\tan\left[\mathrm{i}A - \tan^{-1}\left(\frac{4\Omega'^2\mathrm{i}A}{4\Omega'^2 - \Omega^2\mathrm{i}A}\right)\right] = \tan\left[\mathrm{i}B - \tan^{-1}\left(\frac{4\Omega'^2\mathrm{i}B}{4\Omega'^2 - \Omega^2\mathrm{i}A}\right)\right].$$

On making $A = 0$, $B = \mathrm{r}$, this reduces to the formula (84) of § 273. (*N.B.*—The i of the present Example corresponds to i/r in the former case.)

4. A circular cylinder of radius A and infinite length, performs symmetrical radial vibrations. Prove that (with the notation of § 244)

$$u = \Sigma C_i \, \mathbf{J}_1\left(\frac{ir}{\Omega}\right)\sin(it + a_i),$$

the admissible values of i being given by $i = \mathrm{i}\Omega/A$, where i is any root of

$$\mathrm{i}\mathbf{J}_1'(\mathrm{i}) + \left(1 - \frac{2\Omega'^2}{\Omega^2}\right)\mathbf{J}_1(\mathrm{i}) = 0.$$

5. The following is the general solution for waves of transverse vibrations (§§ 275-277) in a given plane radiating symmetrically from a single centre:

$$\left.\begin{aligned}\psi &= \frac{1}{r}\Sigma A_i \sin i\left(t - \frac{r}{\Omega'} - a_i\right) \\ \mathrm{u} = -\frac{\partial\psi}{\partial y}, \quad \mathrm{v} &= \frac{\partial\psi}{\partial x}, \quad \mathrm{w} = 0.\end{aligned}\right\}.$$

Investigate the form assumed by this solution when r is very great in comparison with the wave length $(2\pi\Omega'/i)$.

6. A solution for waves of transverse vibrations may be constructed by making

$$\mathrm{u} = \frac{i^2\psi}{\Omega^2} + \frac{\partial^2\psi}{\partial x^2}, \quad \mathrm{v} = \frac{\partial^2\psi}{\partial x\partial y}, \quad \mathrm{w} = \frac{\partial^2\psi}{\partial x\partial z};$$

ψ having the same form as in the last Example, or being any solution of equation (88) of § 277.

Investigate the form of the motion when r is very great in comparison with the wave length.

7. Prove that the equations of free periodic vibrations, performed symmetrically in planes through Oz, may be written, with the notation of § 243, in the form

$$u_i = \frac{\partial \phi_i}{\partial r} - \frac{1}{r^2 \sin\theta}\frac{\partial \psi_i}{\partial \theta}, \quad v_i = \frac{1}{r}\frac{\partial \phi_i}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \psi_i}{\partial r}, \quad w_i = 0;$$

where ϕ_i and ψ_i are independent functions of r and θ satisfying the equations

$$\left.\begin{aligned} \frac{\partial}{\partial r}\left(r^2\frac{\partial \phi_i}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi_i}{\partial \theta}\right) + \frac{i^2 r^2 \phi_i}{\Omega^2} = 0 \\ r^2\frac{\partial^2 \psi_i}{\partial r^2} + \sin\theta\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \psi_i}{\partial \theta}\right) + \frac{i^2 r^2 \psi_i}{\Omega'^2} = 0 \end{aligned}\right\}$$

8. Equations (53) of § 263 may be put into the form

$$\left(\nabla^2 + \frac{i^2}{\Omega'^2}\right)u = \left(1 - \frac{\Omega^2}{\Omega'^2}\right)\frac{\partial \Delta}{\partial x}, \text{ etc.,}$$

giving
$$\left(\nabla^2 + \frac{i^2}{\Omega^2}\right)\Delta = 0.$$

It is easy to show that

$$u = -\frac{\Omega^2}{i^2}\frac{\partial \Delta}{\partial x}, \quad v = -\frac{\Omega^2}{i^2}\frac{\partial \Delta}{\partial y}, \quad w = -\frac{\Omega^2}{i^2}\frac{\partial \Delta}{\partial z}$$

are particular integrals of these equations; the complementary functions being of course solutions of $(\nabla^2 + i^2/\Omega'^2)u = 0$, etc. Lord Rayleigh* has obtained a solution specially adapted to the case of free vibrations propagated parallel to the plane surface of an infinite solid, which the student will have no difficulty in constructing for himself. Taking $z = 0$ for the plane surface, assuming that Δ_i, u_i, etc., vary as the sine or cosine of $i(\lambda x + \mu y)$, and determining the complementary functions so as to satisfy (131): and then adjusting the arbitrary constants so that R, S, and T may vanish when $z = 0$: the solution finally reduces to the form

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z} + \mathrm{w};$$

* *Proceedings*, Lond. Math. Soc., vol. xvii., p. 4.

where

$$\phi = \Omega^2 \sum \frac{A_i}{i^2} \cos i[pr\cos(\theta - a_i) + t - \beta_i]\{\exp(-iz\sqrt{p^2 - 1/\Omega^2})$$
$$- (1 - 1/2\Omega'^2 p^2) \,.\, \exp(-iz\sqrt{p^2 - 1/\Omega'^2})\},$$

$$\mathrm{w} = \frac{2\Omega\sqrt{\Omega^2 p^2 - 1}}{2\Omega'^2 p^2 - 1} \sum \frac{A_i}{i} \cos i[pr\cos(\theta - a_i) + t - \beta_i] \,.\, \exp(-iz\sqrt{p^2 - 1/\Omega'^2}):$$

p being a root of the bicubic

$$16\Omega'^6(\Omega^2 - \Omega'^2)p^6 - 8\Omega'^4(3\Omega^2 - 2\Omega'^2)p^4 + 8\Omega^2\Omega'^2 p^2 - \Omega^2 = 0,$$

and A, a, β being arbitrary constants. The corresponding cubical dilatation is

$$\Delta = -\Sigma A_i \cos i[pr\cos(\theta - a_i) + t - \beta_i] \,.\, \exp(-iz\sqrt{p^2 - 1/\Omega^2}).$$

The symbols r, θ here denote the cylindrical polars of § 244; the student will find it a good exercise to prove by actual substitution that the displacements

$$u = \frac{\partial\phi}{\partial r}, \quad v = \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \quad w = \frac{\partial\phi}{\partial z} + \mathrm{w}$$

satisfy equations (88) and (89) of that Article, when

$$\Xi = \mathrm{H} = \mathrm{Z} = \Xi' = \mathrm{H}' = \mathrm{Z}' = 0, \text{ and } \Phi = z.$$

9. Plane sound waves travelling through an isotropic elastic medium (m_1, n_1, ρ_1, Ω_1) impinge obliquely on the plane surface separating this from a second medium (m_2, n_2, ρ_2, Ω_2). Prove that the disturbance is partly "reflected" into the first medium, and partly "refracted" into the second; and that if the directions of displacement in the incident, reflected and refracted waves make angles ψ, ψ', ψ'', respectively with the normal to the dividing surface, then

$$\psi' = \pi - \psi; \quad \psi'' = \sin^{-1}[(\Omega_2 \sin\psi)/\Omega_1].$$

Investigate also the distribution of energy between the reflected and refracted waves.

The surface conditions in this problem reduce to those necessary for permanent contact between the two media; and these are that the normal components of displacement and of stress in the two media be equal at every point of the surface.

Take the dividing surface for plane of xy, Oz being directed into the second medium, and assume

$$\phi_1 = A\sin\frac{i}{\Omega_1}(x\cos\psi + y\sin\psi + \Omega_1 t) + B\sin\frac{i}{\Omega_1}(x\cos\psi' + y\sin\psi' + \Omega_1 t),$$

$$\phi_2 = C\sin\frac{i}{\Omega_2}(x\cos\psi'' + y\sin\psi'' + \Omega_2 t).$$

The potential ϕ_1 will then represent the propagation of the incident and reflected waves through the first medium, and ϕ_2 that of the refracted waves through the second.

Equilibrium.

10. Lamé observes, in discussing the results of § 290, that if

$$\Pi > 2\Pi' + \mathbf{T},$$

it will be impossible, *by any expenditure of material*, to make the tube strong enough to resist the stress produced. Expose the inconsistency of this reasoning.

11. Assuming that the rivetted seams of a boiler are its weakest parts, compare the strengths of two cylindrical boilers (§§ 289-291) which are alike in all respects except that in one the seams are parallel and perpendicular to the axis, while in the other they are everywhere inclined to it at angles of 45°.

12. A solid sphere is subjected to a normal pressure $C\cos\theta$ over its whole surface; required the strain produced. A very elegant solution has been obtained for this problem, but it is unfortunately disqualified by an inherent impossibility. What is this?

13. A solid sphere is subjected to a normal surface traction $C\cos\theta$ over the hemisphere from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, and to a normal traction $-C\cos\theta$ over the other hemisphere. Prove that equilibrium will be maintained, and determine the strained form of the sphere (*i.*) when C is positive, (*ii.*) when C is negative.

14. A spherical shell (internal and external radii A and B) is subjected to uniform normal pressures Π, Π' over its surfaces: show that the radial displacement is given by

$$u = \frac{(A^3\Pi - B^3\Pi')r}{3k(B^3 - A^3)} + \frac{A^3B^3(\Pi - \Pi')}{4n(B^3 - A^3)r^2}$$

Adopt the notation of Article 243, and assume the strain to be symmetrical about the centre.

15. If $\Pi = 0$ in the case of the last Example, determine the value of Π' at which the limit of stable elastic resistance will be reached.

16. A sphere of "incompressible" material (*i.e.*, an imaginary substance for which the ratio k/n is infinitely great) is subjected to a surface traction whose components are the harmonics $F = \mathbf{H}_i$, $G = \mathbf{H}_i'$, $H = \mathbf{H}_i''$; prove that the radial displacement is

$$\frac{1}{nA^{i-1}}\left\{\frac{(i-1)A^2\mathrm{X}_{i-1}}{2(2i^2+1)} - \frac{(i-1)(2i+3)r^2\mathrm{X}_{i-1}}{2(2i+1)(2i^2+1)} - \frac{\Phi_{i+1}}{2i(2i+1)}\right\},$$

where A is the radius of the sphere, and X, Φ are solid harmonics, such that

$$(x\mathbf{H}_i + y\mathbf{H}_i' + z\mathbf{H}_i'')r^i = \frac{r^2\mathrm{X}_{i-1} - \Phi_{i+1}}{2i+1}.$$

17. An isotropic cylinder of elliptic section is slightly deformed in such a way that the section of its bounding surface (which remains cylindrical) becomes a confocal ellipse. Determine the displacement throughout the solid.

Use the elliptic cylindrics of Article 246, or a system analogous to the spheroidals of Article 251. The surface condition is that a shall be independent of η all over the surface.

18. A solid sphere is subjected to tangential surface traction, everywhere parallel to the plane of xy and of magnitude $\Sigma C_i \mathbf{P}_i/\sin\theta$, where θ has the same meaning as in § 243, and $\mathbf{P}_i$ is Legendre's coefficient of order i. Show that the system is in equilibrium, and that the point (r, θ) will be displaced parallel to the plane of xy through an arc

$$\frac{r}{n\sin\theta}\sum\frac{B_i}{i}\left(\frac{r}{A}\right)(\mathbf{P}_i - \mathbf{P}_{i+2})$$

where A is the radius of the sphere, and

$$B_i = C_i + C_{i-2} + C_{i-4} + \ldots\ldots$$

If the surface traction be $C(P_2 - P_4)/\sin\theta$, discuss its distribution over the surface, and draw the curve into which any superficial meridian is deformed.

$$[\mathbf{P}_2 = \tfrac{1}{2}(3\cos^2\theta - 1,) \quad \mathbf{P}_4 = \tfrac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3)].$$

19. Investigate the system of forces and tractions required to produce in a solid sphere the distribution of displacement

$$\left.\begin{aligned} u &= \epsilon x + \gamma y - \beta z + \lambda(x^2 - y^2 - z^2) + 2\mu xy + 2\nu zx \\ v &= \epsilon y + a z - \gamma x + 2\lambda xy + \mu(y^2 - z^2 - x^2) + 2\nu yz \\ w &= \epsilon z + \beta x - a y + 2\lambda zx + 2\mu yz + \nu(z^2 - x^2 - y^2) \end{aligned}\right\},$$

where all the coefficients are constants.

20. If any body bounded only by a sphere, or by two concentric spheres, be submitted to any conservative system of impressed forces, the action on the body as a whole reduces to a single resultant force.

21. In the case of § 306, the conditions of equilibrium are

$$\left.\begin{aligned} &\Psi_1 = -\frac{3}{\rho A}[x\mathbf{H}_0 + y\mathbf{H}_0{}' + z\mathbf{H}_0{}''] \\ &\mathbf{H}_1 = \frac{1}{r}\,\frac{\partial \mathbf{U}_2}{\partial x},\quad \mathbf{H}_1{}' = \frac{1}{r}\,\frac{\partial \mathbf{U}_2}{\partial y},\quad \mathbf{H}_1{}'' = \frac{1}{r}\,\frac{\partial \mathbf{U}_2}{\partial z} \end{aligned}\right\},$$

$\mathbf{U}_2$ being any solid harmonic of degree 2.

22. A vertical cylindrical hole of circular section is cut in a rigid body, and an elastic cylinder of density ρ, which, if freed from the action of gravity, would exactly fit the hole, is placed in it and stands upon the bottom. Prove that the sides of the hole suffer the same hydrostatic pressure as if it were filled with a liquid of density $\rho(m-n)/(m+n)$.

23. A glacier fills a valley which is perfectly symmetrical about a vertical plane, and which narrows as it descends. Assuming that ice at temperatures below the freezing point, and under moderate stresses behaves as an isotropic elastic solid, investigate the general character of the strain produced in the superficial lamina of ice by (*i.*) the weight of the glacier tending down the valley, (*ii.*) the lateral compression as the valley narrows, (*iii.*) the friction against the sides. Show that there will be a tendency to form "crevasses" or cracks extending across the glacier, symmetrical about the middle line and *with their concavities turned down the valley.* [W. Hopkins.]

24. Investigate the strain produced in a solid sphere by the mutual gravitation of its parts. Show that if κ represent the mutual attraction of two unit masses concentrated at points separated by the unit distance, a uniform normal surface traction $+ m\kappa\rho^2A^2/15(m+n)$ will preserve the *volume* of the sphere unaltered: the cubical dilatation at the surface being in this case $4\kappa\rho^2A^2/15(m+n)$, and the cubical compression at the centre $2\kappa\rho^2A^2/5(m+n)$.

25. Substituting from equations (187) in (3) and thence in (1), and making use of equations (F, G) of § 307 *bis*, we have in all cases of plane stress under gravity, such as those of §§ 308 and 309,

$$\left.\begin{aligned}
\frac{\partial}{\partial x}\left[qu + (1+\sigma)\frac{\partial\xi}{\partial x}\right] = \frac{\partial}{\partial y}\left[qv + (1+\sigma)\frac{\partial\xi}{\partial y}\right] &= (1-\sigma)[\alpha x + (\beta - \mathbf{g}\rho)y + \gamma] \\
\frac{\partial}{\partial x}\left[qv + (1+\sigma)\frac{\partial\xi}{\partial y}\right] + \frac{\partial}{\partial y}\left[qu + (1+\sigma)\frac{\partial\xi}{\partial x}\right] &= -2(1+\sigma)(\beta x + \alpha y) \\
q\frac{\partial w}{\partial z} &= -2\sigma[\alpha x + (\beta - \mathbf{g}\rho)y + \gamma] \\
\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} &= 0.
\end{aligned}\right\}$$

Hence prove that the component displacements are given by

$$\left.\begin{aligned}
qu &= (1-\sigma)[\tfrac{1}{2}\alpha(x^2-y^2) + (\beta-\mathbf{g}\rho)xy + \gamma x] - (1+\sigma)\left[\alpha y^2 + \frac{\partial\xi}{\partial x}\right] + \sigma\alpha z^2 \\
qv &= (1-\sigma)[\alpha xy - \tfrac{1}{2}(\beta-\mathbf{g}\rho)(x^2-y^2) + \gamma y] - (1+\sigma)\left[\beta x^2 + \frac{\partial\xi}{\partial x}\right] + \sigma(\beta-\mathbf{g}\rho)z^2 \\
qw &= -2\sigma z[\alpha x + (\beta-\mathbf{g}\rho)y + \gamma]
\end{aligned}\right\}$$

26. Verify from the above values of u, v, w that

$$\left.\begin{aligned} P &= \alpha x + (\beta - g\rho)y + \gamma - \frac{\partial^2 \xi}{\partial x^2} \\ Q &= \alpha x + (\beta - g\rho)y + \gamma - \frac{\partial^2 \xi}{\partial y^2} \\ U &= -\beta x - \alpha y - \frac{\partial^2 \xi}{\partial x \partial y} \\ R &= S = T = 0 \end{aligned}\right\};$$

and hence deduce that there is a system of lines of zero stress parallel to Oz, and that the principal normal stresses in any plane perpendicular to Oz are

$$\alpha x + (\beta - g\rho)y + \gamma \pm \sqrt{\left(\frac{\partial^2 \xi}{\partial x^2}\right)^2 + \left(\beta x + \alpha y + \frac{\partial^2 \xi}{\partial x \partial y}\right)^2}.$$

The differential equations of the Lines of Stress along which these act are

$$\left[\frac{\partial^2 \xi}{\partial x^2} \pm \sqrt{\left(\frac{\partial^2 \xi}{\partial x^2}\right)^2 + \left(\beta x + \alpha y + \frac{\partial^2 \xi}{\partial x \partial y}\right)^2}\right] dx + \left[\beta x + \alpha y + \frac{\partial^2 \xi}{\partial x \partial y}\right] dy = 0.$$

27. Apply the above solution to the example considered in § 308 (Figures 36, 37).

28. Solve the case considered in § 309 (Figure 38).

29. A beam without load is supported by vertical forces, without couples, at its ends (Figure 39, Plate III.).

30. A beam without load is supported by vertical forces, together with couples of given magnitude, at its ends (Figure 40, Plate III).

31. A beam without load is supported by vertical forces at its ends, and a couple of known magnitude is applied at one end only (Figure 41, Plate III.).

32. Integrating twice the first three of equations (C), § 307 *bis*, we obtain

$$\left.\begin{aligned} &\frac{\partial}{\partial y}\left\{\int [\Phi - (1+\sigma)\nabla^2 \chi_3] dz + \phi_1(y)\right\} \\ &\qquad + \frac{\partial}{\partial z}\left\{\int [\Phi - (1+\sigma)\nabla^2 \chi_2] dy + \psi_1(z)\right\} = 0 \\ &\frac{\partial}{\partial z}\left\{\int [\Phi - (1+\sigma)\nabla^2 \chi_1] dx + \phi_2(z)\right\} \\ &\qquad + \frac{\partial}{\partial x}\left\{\int [\Phi - (1+\sigma)\nabla^2 \chi_3] dz + \psi_2(x)\right\} = 0 \\ &\frac{\partial}{\partial x}\left\{\int [\Phi - (1+\sigma)\nabla^2 \chi_2] dy + \phi_3(x)\right\} \\ &\qquad + \frac{\partial}{\partial y}\left\{\int [\Phi - (1+\sigma)\nabla^2 \chi_1] dx + \psi_3(y)\right\} = 0 \end{aligned}\right\}$$

where ϕ and ψ denote arbitrary functions.

Hence show, by substituting in (A) the values of the strain components, and integrating, that

$$\left.\begin{aligned} q\theta_1 + (1+\sigma)\frac{\partial^2(\chi_2-\chi_3)}{\partial y\partial z} &= \frac{\partial}{\partial y}\left\{\int[\Phi-(1+\sigma)\nabla^2\chi_3]dz + \phi_1(y)\right\} \\ q\theta_2 + (1+\sigma)\frac{\partial^2(\chi_3-\chi_1)}{\partial z\partial x} &= \frac{\partial}{\partial z}\left\{\int[\Phi-(1+\sigma)\nabla^2\chi_1]dx + \phi_2(z)\right\} \\ q\theta_3 + (1+\sigma)\frac{\partial^2(\chi_1-\chi_2)}{\partial x\partial y} &= \frac{\partial}{\partial x}\left\{\int[\Phi-(1+\sigma)\nabla^2\chi_2]dy + \phi_3(x)\right\} \end{aligned}\right\}$$

33. Deduce from the last example, by substitution in the identical equations

$$\frac{\partial u}{\partial x} = e, \quad \frac{\partial u}{\partial y} = \tfrac{1}{2}c - \theta_3, \quad \frac{\partial u}{\partial z} = \tfrac{1}{2}b + \theta_3; \text{ etc.},$$

that the general solution for strain in three dimensions obtained by Airy's method is

$$\left.\begin{aligned} qu &= \int[\Phi-(1+\sigma)\nabla^2\chi_1]dx - (1+\sigma)\frac{\partial}{\partial x}(\chi_2+\chi_3-\chi_1) + \psi_3(y) + \phi_2(z) \\ qv &= \int[\Phi-(1+\sigma)\nabla^2\chi_2]dy - (1+\sigma)\frac{\partial}{\partial y}(\chi_3+\chi_1-\chi_2) + \psi_1(z) + \phi_3(x) \\ qw &= \int[\Phi-(1+\sigma)\nabla^2\chi_3]dz - (1+\sigma)\frac{\partial}{\partial z}(\chi_1+\chi_2-\chi_3) + \psi_2(x) + \phi_1(y) \end{aligned}\right\}$$

In applying these general formulæ to the case of Plane Stress, worked out independently in Example 25, put $\chi_1 = \chi_2 = 0$, $\chi_3 = \chi$. We must also write

$$\Phi = \frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} - (1-\sigma)\rho\Psi,$$

as will at once appear on forming the equations analogous to (C) of § 307 *bis*. The ϕ and ψ functions are quite determinate, the arbitrary terms which appear on integration representing bodily translations and rotations.

34. Obtain a solution analogous to that of § 307, when a plane stress is caused by the Applied Forces

$$X = \frac{\partial\Psi}{\partial y}, \quad Y = \frac{\partial\Psi}{\partial x},$$

Ψ being any function of x and y which satisfies $\partial^2\Psi/\partial x\partial y = 0$, and the surface tractions being such as to admit of Plane Stress (§ 307): and determine, by the method of § 307 *bis*, the necessary limitation to the form of the function in terms of which the stress components are expressed.

35. A free charge E of electricity distributes itself over a plane disc bounded by the ellipse

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

with surface density

$$\frac{E}{4\pi AB\sqrt{1-x^2/A^2-y^2/B^2}}$$

on either side of the disc: the potential produced being

$$\frac{E}{2}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda(A^2+\lambda)(B^2+\lambda)}}$$

at points within the disc, and

$$\frac{E}{2}\int_\xi^\infty \frac{d\lambda}{\sqrt{\lambda(A^2+\lambda)(B^2+\lambda)}}$$

at points without it; ξ being the greatest root of the cubic

$$\frac{x^2}{A^2+\xi}+\frac{y^2}{B^2+\xi}+\frac{z^2}{\xi}=1.$$

Hence deduce, as in § 321, that a rigid right cylinder of weight **W** whose normal section is of the same form as the disc will, if placed upright upon the ground, descend vertically through a distance

$$w_0=\frac{(m+n)\mathbf{W}}{4\pi mnA}\,.\,\mathrm{F}(\mathrm{e},\ \tfrac{1}{2}\pi),$$

where e is the eccentricity of the elliptic section, and F denotes the elliptic integral of the first kind; and determine the distribution of displacement throughout the earth.

[Take $\psi=-\mathbf{W}/2\pi AB\sqrt{1-x^2/A^2-y^2/B^2}$].

36.] A cylindrical vessel is filled with liquid to a height D, in vacuo. The vessel and its contents are then weighed under an atmospheric pressure Π, and at the same temperature as before. The mean density of the liquid in the vessel being thus found to be ρ', show that its true natural density may be deduced by the formula

$$\rho=\rho'\left\{1-\frac{\Pi}{k}+\frac{\mathrm{g}\rho' D}{2k}\right\},$$

where k is its compressibility for the given temperature.

CHAPTER VII.

BEAMS AND WIRES.

Introductory.

322.] **Definitions.** The terms **Beam**, **Wire**, and **Hoop**, in the most extended sense in which they will be employed in the present Chapter, denote bodies which have the following characteristic property in common:—

Each is so related to a certain straight or continuously curved line, called its **Central Axis**, that *the centre of gravity of every section by a plane perpendicular to the Central Axis lies in that Axis.*

The Central Axis itself may be situated wholly or partly within or without the substance of the body.

The Central Axis of a *Beam* is a *straight line,* and—unless the contrary be expressly stated—the beam is to be supposed cylindrical or prismatic in form, the generators of the *lateral surface* being parallel to the Axis, and the *plane ends* of the beam being perpendicular to it and of dimensions comparable with its length.

The Central Axis of a *Hoop* is any *closed curve* of continuous curvature, and the form of the Hoop is defined by that of its Central Axis and by those of its normal sections. We shall only deal with *uniform circular hoops,* in which the Axis is a circle, while all sections in planes perpendicular to the Axis are equal and similar figures, similarly situated with regard to its polar line (*i.e.*, the straight line drawn through its centre perpendicular to its plane.)

A beam, or hoop of any form, the dimensions of whose transverse sections are all *very small* in comparison with the length of the central axis (but yet finite) will be called a **Wire**. For *purely geometrical purposes,* a wire may be regarded as coincident with its Axis.

We shall confine ourselves to the consideration of wires of naturally *uniform transverse section,* but no restriction will be placed upon the natural form of the Axis.

323.] **The class of Strains to be investigated. Exclusion of Lateral Surface Tractions.** The main object of this Chapter is to obtain reliable data for the employment of beams in structures and mechanism, where their function is to transmit from one body to another forces or couples, the straining effect of which upon themselves is in general very great in comparison with that of their weight.

The distinctive character of all the strains discussed will therefore be *the absence of all stress across the lateral surfaces of the beams, wires, or hoops.*

The straining of beams will be considered as due to forces and couples applied by means of surface tractions *acting over their ends alone.* These may be supplemented, in the case of terminated wires, by impressed forces.

Since closed hoops have no ends, they will be regarded as under the influence of impressed forces only.

St. Venant's Problem: Straining of a naturally Cylindrical Beam, free from Impressed Forces, by Surface Tractions applied to its ends alone.

324.] **Anticipation of the General Character of the Strain. Geometrical conditions imposed.** It is sufficiently obvious, from a superficial view of the conditions of equilibrium of the beam as a whole (§ 146), that the most general form of small strain which external action of the supposed kind will tend to produce must be compounded of the three comparatively simple types—

(*i.*) *Longitudinal Extension* of the beam, accompanied by lateral contraction: due to equilibrating forces parallel to the Axis.

(*ii.*) *Torsion*, or twisting of the beam about some straight line parallel to its Central Axis, with or without warping of the transverse sections and distortion of the lateral surfaces; due to equilibrating couples in planes perpendicular to the Axis.

(*iii.*) *Flexion* of the beam, of such a kind that the Central Axis assumes the form of a plane curve: due to equilibrating couples in planes parallel to the Axis.

We shall find, on analysing the general equations of strain obtained in § 327 below, that this anticipation is fully borne out.

To simplify the geometrical conditions of the problem, we shall suppose the centre of gravity of the area of one end of the beam (henceforth referred to as the *Base*) to be an absolutely fixed point, which we shall take for origin. The Central Axis of the beam will be our axis of z, and the principal Axes of Inertia of the area of the base our axes of x and y.

Thus we shall have,

$$\iint x\,dx\,dy = \iint y\,dx\,dy = \iint xy\,dx\,dy = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

identically, where the integrals are taken over the whole area of any normal section of the beam. Also, if $\mathfrak{A}$ denote the area of the transverse section, $\mathfrak{I}_1$, $\mathfrak{I}_2$ its moments of inertia about the principal axes through its centre of gravity parallel to Ox, Oy, and $\mathfrak{I}_3$ its moment of inertia about the Central Axis of the beam,

$$\left.\begin{aligned} \iint dx\,dy &= \mathfrak{A} \\ \iint y^2\,dx\,dy &= \mathfrak{I}_1 \\ \iint x^2\,dx\,dy &= \mathfrak{I}_2 \\ \iint (x^2+y^2)\,dx\,dy &= \mathfrak{I}_1 + \mathfrak{I}_2 = \mathfrak{I}_3 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

These quantities are of course constants which depend only on the natural form and dimensions of the beam, and not at all on its material.

We shall further suppose that the element of the base immediately surrounding the origin always retains its initial plane, and that an elementary line in that plane—for simplicity, say the initial element of Ox—retains its natural direction.* The geometrical conditions to be satisfied at the origin are therefore

$$u = v = w = 0,\quad \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = \frac{\partial v}{\partial x} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

325.] Conditions of Equilibrium. Besides the general equations of equilibrium (103) or (104) of § 285, the stress components must satisfy identically six relations imposed upon them by the peculiar circumstances of the strain. Since the sides of the beam are free from stress, it follows that any length of it, bounded by normal sections, is held in equilibrium solely by the total stresses across its ends. Hence the component forces and couples across those ends must be equal, and it follows from equations (6) and (7) of § 146 that the six integrals

$$\left.\begin{matrix} \iint R\,dx\,dy, & \iint S\,dx\,dy, & \iint T\,dx\,dy, \\ \iint (yR - zS)\,dx\,dy, & \iint (zT - xR)\,dx\,dy, & \iint (xS - yT)\,dx\,dy \end{matrix}\right\}\ldots\ldots(4)$$

taken over any transverse section, must be absolutely independent of z.

* It will appear later that if *any* line in the fixed plane element of the base be constrained to retain its initial direction, *every* line in that element will do the same.

326.] Statement of St. Venant's Problem. Since the lateral surfaces, over which the stress components are everywhere zero, are parallel to the Axis, the boundary conditions reduce to

$$\lambda P+\mu U=\lambda U+\mu Q=\lambda T+\mu S=0.$$

The first two of these will be satisfied identically if we assume * that

$$P=Q=U=0 \quad \ldots\ldots (5)$$

throughout the body; and in this case the only condition to be satisfied by the special values of the stress components at the lateral bounding surface is

$$\lambda T+\mu S=0 \quad \ldots\ldots (6)$$

To obtain a solution of the general equations which will satisfy (4) and (5) throughout the body (6) all over the lateral surface, and (3) at the origin, is the problem justly named by Clebsch "St. Venant's Problem."

The peculiarity of the solution is that $c=0$, and $e=f=-\sigma g$ throughout the body, so that each longitudinal "fibre," or elementary prism parallel to the Axis is extended longitudinally and contracted laterally just as if it were solitary (§ 213), while its transverse sections do not suffer shear.

327.] Solution of the Problem. Substituting from (5) in equations (40) of § 214, we have

$$\left.\begin{aligned} \frac{\partial w}{\partial z}&=\frac{R}{q} \quad &\ldots\ldots (7)\\ \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}&=-\sigma\frac{\partial w}{\partial z} \quad &\ldots\ldots (8)\\ \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}&=0 \quad &\ldots\ldots (9)\end{aligned}\right\}$$

and the general equations (104) of § 285 reduce to

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial z\partial x}&=0 \quad &\ldots\ldots (10)\\ \frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y\partial z}&=0 \quad &\ldots\ldots (11)\\ \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+2\frac{\partial^2 w}{\partial z^2}&=0 \quad &\ldots\ldots (12)\end{aligned}\right\}$$

* This will probably appear to the student a very sweeping assumption to make at such an early stage of the investigation. The solution of the general problem is, however, of a "semi-inverse" character, the conditions of each of the simpler component strains of Article 324 having been fully analysed by St. Venant in his two splendid memoirs—"Sur la Torsion des Prismes," *Mém. des Sav. Étr.:* t. xiv. (1855), "Sur la Flexion des Prismes," *Liouville:* 2° sér. t. i. (1856). It must be remembered that *any* solution which satisfies all the conditions is *the* solution.

while the boundary condition (6) may be written

$$\lambda\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)=0 \text{......................}(13)$$

Differentiating (10), (11), (12) as to x, y, z respectively, and subtracting the first two results from the third,

$$\frac{\partial^2}{\partial z^2}\left(2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)=0,$$

and therefore by (8)

$$\frac{\partial^3 w}{\partial z^3}=0 \text{..................................}(14)$$

Again, differentiating (10) as to y and (11) as to x, and adding the results,

$$2\frac{\partial^3 w}{\partial x\partial y\partial z}+\frac{\partial^2}{\partial z^2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=0,$$

and therefore by (9)

$$\frac{\partial^3 w}{\partial x\partial y\partial z}=0.$$

Lastly, differentiating (12) as to z, and taking account of (14),

$$\frac{\partial^3 w}{\partial x^2\partial z}+\frac{\partial^3 w}{\partial y^2\partial z}=0:$$

ut if we differentiate (10) as to y, and (11) as to x, and subtract he results,

$$\frac{\partial^3 w}{\partial x^2\partial z}-\frac{\partial^3 w}{\partial y^2\partial z}=0.$$

Thus

$$\frac{\partial^3 w}{\partial x^2\partial z}=\frac{\partial^3 w}{\partial y^2\partial z}=0,$$

nd finally

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial z}\right)=\frac{\partial^2}{\partial y^2}\left(\frac{\partial w}{\partial z}\right)=\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial w}{\partial z}\right)=\frac{\partial^2}{\partial z^2}\left(\frac{\partial w}{\partial z}\right)=0.$$

ence it appears that $\partial w/\partial z$ cannot contain any power of x, y or above the first, nor the product xy: and it follows at once from t and from (8) that

$$\left.\begin{aligned}\frac{\partial w}{\partial z}&=\epsilon-\varpi_2 x-\varpi_1 y+z(\beta+\beta_2 x+\beta_1 y)\\ \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}&=-\sigma(\epsilon-\varpi_2 x-\varpi_1 y)-\sigma z(\beta+\beta_2 x+\beta_1 y)\end{aligned}\right\}\text{.........}(15)$$

where all the coefficients are absolute constants.

Equations (10) and (11) may now be written

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial z^2} &= \varpi_2 - \beta_2 z\\ \frac{\partial^2 v}{\partial z^2} &= \varpi_1 - \beta_1 z\end{aligned}\right\}\quad\ldots\ldots(16)$$

and from (15) and (16) we easily deduce that

(*i.*) u contains no higher power of x than x^2, and no higher power of z than z^3;

(*ii.*) v contains no higher power of y than y^2, and no higher power of z than z^3;

(*iii.*) w contains no higher power of z than z^2.

Integrating (15) and (16), and supplying arbitrary functions with due regard to these limitations, to equation (9), and to the conditions (3) which are to hold at the origin, we have finally

$$\left.\begin{aligned}u =& -\sigma[\epsilon x - \tfrac{1}{2}\varpi_2(x^2-y^2) - \varpi_1 xy] - \sigma z[\beta x + \tfrac{1}{2}\beta_2(x^2-y^2) + \beta_1 xy]\\ &+ z\left[\left(\frac{\partial\psi}{\partial x}\right)_0 - \tau y\right] + \tfrac{1}{2}\varpi_2 z^2 - \tfrac{1}{6}\beta_2 z^3\\ v =& -\sigma[\epsilon y - \varpi_2 xy - \tfrac{1}{2}\varpi_1(y^2-x^2)] - \sigma z[\beta y + \beta_2 xy + \tfrac{1}{2}\beta_1(y^2-x^2)]\\ &+ z\left[\left(\frac{\partial\psi}{\partial y}\right)_0 + \tau x\right] + \tfrac{1}{2}\varpi_1 z^2 - \tfrac{1}{6}\beta_1 z^3\\ w =& z[\epsilon - \varpi_2 x - \varpi_1 y] + \tfrac{1}{2}z^2[\beta + \beta_2 x + \beta_1 y] + \psi\\ &- [\tfrac{1}{2}\beta(x^2+y^2) + \beta_2 xy^2 + \beta_1 x^2 y] - \left[x\left(\frac{\partial\psi}{\partial x}\right)_0 + y\left(\frac{\partial\psi}{\partial y}\right)_0\right]\end{aligned}\right\}\ldots(17)$$

where the new coefficients introduced are also arbitrary constants, ψ is any function of x and y satisfying

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0\quad\ldots\ldots(18)$$

and vanishing at the origin, and

$$\left(\frac{\partial\psi}{\partial x}\right)_0,\ \left(\frac{\partial\psi}{\partial y}\right)_0$$

denote the values of its derivatives at the origin.

The stress components are

$$\left.\begin{aligned}R &= q[\epsilon - \varpi_2 x - \varpi_1 y] + qz[\beta + \beta_2 x + \beta_1 y]\\ S &= n\left[\tau x - (1+\sigma)\beta y - (2+\sigma)\beta_2 xy - \beta_1 x^2 + \tfrac{1}{2}\sigma\beta_1(x^2-y^2) + \frac{\partial\psi}{\partial y}\right]\\ T &= n\left[-\tau y - (1+\sigma)\beta x - (2+\sigma)\beta_1 xy - \beta_2 y^2 + \tfrac{1}{2}\sigma\beta_2(y^2-x^2) + \frac{\partial\psi}{\partial x}\right]\end{aligned}\right\}(19)$$

and the boundary condition (13) becomes

$$\lambda\frac{\partial\psi}{\partial x}+\mu\frac{\partial\psi}{\partial y}-\tau(\lambda y-\mu x)-(1+\sigma)\beta(\lambda x+\mu y)$$
$$-\beta_1\{(2+\sigma)\lambda xy+\mu[x^2-\tfrac{1}{2}\sigma(x^2-y^2)]\}$$
$$-\beta_2\{(2+\sigma)\mu xy+\lambda[y^2-\tfrac{1}{2}\sigma(y^2-x^2)]\}=0.\qquad(20)$$

Thus the mechanical conditions only involve ψ in connection with the four constants τ, β, β_1, β_2; but these constants are perfectly arbitrary, and therefore independent, and the terms multiplied by them may be taken to represent independent strains of simpler forms. In order that each of these forms of strain may satisfy the conditions of the problem, it is clear that ψ must be the sum of four functions of x and y, multiplied respectively by τ, β, β_1, β_2, and such that equations (18), (20), and the further equations of condition to be deduced below from (4), are satisfied separately by those terms which involve each coefficient.

Assuming therefore that

$$\psi=\tau\mathrm{w}+\beta\mathrm{w}'+\beta_1\mathrm{w}_1+\beta_2\mathrm{w}_2,$$

and substituting in (18) and (20), we have the general equations

$$\nabla^2\mathrm{w}=\nabla^2\mathrm{w}'=\nabla^2\mathrm{w}_1=\nabla^2\mathrm{w}_2=0,$$

with the boundary conditions

$$\left.\begin{aligned}
\lambda\frac{\partial\mathrm{w}}{\partial x}+\mu\frac{\partial\mathrm{w}}{\partial y}&=\lambda y-\mu x\\
\lambda\frac{\partial\mathrm{w}'}{\partial x}+\mu\frac{\partial\mathrm{w}'}{\partial y}&=(1+\sigma)(\lambda x+\mu y)\\
\lambda\frac{\partial\mathrm{w}_1}{\partial x}+\mu\frac{\partial\mathrm{w}_1}{\partial y}&=(2+\sigma)\lambda xy+\mu[x^2-\tfrac{1}{2}\sigma(x^2-y^2)]\\
\lambda\frac{\partial\mathrm{w}_2}{\partial x}+\mu\frac{\partial\mathrm{w}_2}{\partial y}&=(2+\sigma)\mu xy+\lambda[y^2-\tfrac{1}{2}\sigma(y^2-x^2)]
\end{aligned}\right\}.$$

We may show at once that w′ cannot possibly satisfy both the general equation and the boundary condition, for if we take the integral

$$\iint\left(\frac{\partial^2\mathrm{w}'}{\partial x^2}+\frac{\partial^2\mathrm{w}'}{\partial y^2}\right)dxdy$$

over any normal section, and integrate it by parts, it becomes

$$\int\left(\lambda\frac{\partial\mathrm{w}'}{\partial x}+\mu\frac{\partial\mathrm{w}'}{\partial y}\right)ds$$

where ds is an elementary arc of the periphery of the section. Now, we have by (2)

$$\mathfrak{A} = \iint dxdy$$
$$= \tfrac{1}{2}\int(\lambda x + \mu y)ds,$$

so that the boundary condition satisfied by w′ requires that

$$\iint\left(\frac{\partial^2 \mathrm{w}'}{\partial x^2} + \frac{\partial^2 \mathrm{w}'}{\partial y^2}\right)dxdy = 2(1+\sigma)\mathfrak{A},$$

which is obviously inconsistent with

$$\frac{\partial^2 \mathrm{w}'}{\partial x^2} + \frac{\partial^2 \mathrm{w}'}{\partial y^2} = 0.$$

Hence it follows that β must be zero.

It will be found, on substituting from (19) in (4) and taking account of (1), that the first, second, third and sixth conditions of equilibrium are satisfied identically. In order that the fourth and fifth may be satisfied we must have

$$\left.\begin{aligned}\iint \frac{\partial \psi}{\partial y} dxdy &= \tfrac{1}{2}\beta_1[(2-\sigma)\mathfrak{I}_2 - (4+3\sigma)\mathfrak{I}_1] \\ \iint \frac{\partial \psi}{\partial x} dxdy &= \tfrac{1}{2}\beta_2[(2-\sigma)\mathfrak{I}_1 - (4+3\sigma)\mathfrak{I}_2]\end{aligned}\right\}.$$

Finally then, collecting terms according to the six arbitrary coefficients, the displacements are

$$\left.\begin{aligned}
u = &-\epsilon\sigma x - \tau z\left[y - \left(\frac{\partial \mathrm{w}}{\partial x}\right)_0\right] \\
&+ \varpi_1\sigma xy - \beta_1 z\left[\sigma xy - \left(\frac{\partial \mathrm{w}_1}{\partial x}\right)_0\right] \\
&+ \varpi_2[\tfrac{1}{2}\sigma(x^2-y^2) + \tfrac{1}{2}z^2] - \beta_2 z\left[\tfrac{1}{2}\sigma(x^2-y^2) + \tfrac{1}{6}z^2 - \left(\frac{\partial \mathrm{w}_2}{\partial x}\right)_0\right] \\
v = &-\epsilon\sigma y + \tau z\left[x + \left(\frac{\partial \mathrm{w}}{\partial y}\right)_0\right] \\
&+ \varpi_1[\tfrac{1}{2}\sigma(y^2-x^2) + \tfrac{1}{2}z^2] - \beta_1 z\left[\tfrac{1}{2}\sigma(y^2-x^2) + \tfrac{1}{6}z^2 - \left(\frac{\partial \mathrm{w}_1}{\partial y}\right)_0\right] \\
&+ \varpi_2\sigma xy - \beta_2 z\left[\sigma xy - \left(\frac{\partial \mathrm{w}_2}{\partial y}\right)_0\right] \\
w = &\ \epsilon z + \tau\left[\mathrm{w} - x\left(\frac{\partial \mathrm{w}}{\partial x}\right)_0 - y\left(\frac{\partial \mathrm{w}}{\partial y}\right)_0\right] \\
&- \varpi_1 yz - \beta_1\left[x^2y - \tfrac{1}{2}yz^2 - \mathrm{w}_1 + x\left(\frac{\partial \mathrm{w}_1}{\partial x}\right)_0 + y\left(\frac{\partial \mathrm{w}_1}{\partial y}\right)_0\right] \\
&- \varpi_2 xz - \beta_2\left[xy^2 - \tfrac{1}{2}xz^2 - \mathrm{w}_2 + x\left(\frac{\partial \mathrm{w}_2}{\partial x}\right)_0 + y\left(\frac{\partial \mathrm{w}_2}{\partial y}\right)_0\right]
\end{aligned}\right\} \quad \text{...(21)}$$

and the stress components are

$$\left.\begin{aligned}
R &= \epsilon q - \varpi_1 q y + \beta_1 q y z - \varpi_2 q x + \beta_2 q x z \\
S &= \tau n\left[x + \frac{\partial \mathrm{w}}{\partial y}\right] - \beta_1 n\left[x^2 + \tfrac{1}{2}\sigma(y^2 - x^2) - \frac{\partial \mathrm{w}_1}{\partial y}\right] - \beta_2 n\left[(2+\sigma)xy - \frac{\partial \mathrm{w}_2}{\partial y}\right] \\
T &= -\tau n\left[y - \frac{\partial \mathrm{w}}{\partial x}\right] - \beta_1 n\left[(2+\sigma)xy - \frac{\partial \mathrm{w}_1}{\partial x}\right] - \beta_2 n\left[y^2 + \tfrac{1}{2}\sigma(x^2 - y^2) - \frac{\partial \mathrm{w}_2}{\partial x}\right]
\end{aligned}\right\} \quad (22)$$

where w, w_1, w_2 are functions of x and y, vanishing at the origin, and satisfying the equations

$$\left.\begin{aligned}
\frac{\partial^2 \mathrm{w}}{\partial x^2} + \frac{\partial^2 \mathrm{w}}{\partial y^2} &= 0 \\
\frac{\partial^2 \mathrm{w}_1}{\partial x^2} + \frac{\partial^2 \mathrm{w}_1}{\partial y^2} &= 0 \\
\frac{\partial^2 \mathrm{w}_2}{\partial x^2} + \frac{\partial^2 \mathrm{w}_2}{\partial y^2} &= 0
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(23)$$

throughout the body, and the boundary conditions

$$\lambda\frac{\partial \mathrm{w}}{\partial x} + \mu\frac{\partial \mathrm{w}}{\partial y} = \lambda y - \mu x \quad \ldots\ldots\ldots\ldots(24)$$

$$\lambda\frac{\partial \mathrm{w}_1}{\partial x} + \mu\frac{\partial \mathrm{w}_1}{\partial y} = (2+\sigma)\lambda xy + \mu[x^2 + \tfrac{1}{2}\sigma(y^2 - x^2)] \quad \ldots\ldots\ldots\ldots(25)$$

$$\lambda\frac{\partial \mathrm{w}_2}{\partial x} + \mu\frac{\partial \mathrm{w}_2}{\partial y} = (2+\sigma)\mu xy + \lambda[y^2 + \tfrac{1}{2}\sigma(x^2 - y^2)] \quad \ldots\ldots\ldots\ldots(26)$$

over its lateral surface: and also the conditions of equilibrium

$$\iint \frac{\partial \mathrm{w}}{\partial x} dx dy = \iint \frac{\partial \mathrm{w}}{\partial y} dx dy = 0 \quad \ldots(27)$$

$$\iint \frac{\partial \mathrm{w}_1}{\partial x} dx dy = \iint \frac{\partial \mathrm{w}_1}{\partial y} dx dy - \tfrac{1}{2}[(2-\sigma)\mathfrak{I}_2 - (4+3\sigma)\mathfrak{I}_1] = 0 \quad \ldots(28)$$

$$\iint \frac{\partial \mathrm{w}_2}{\partial y} dx dy = \iint \frac{\partial \mathrm{w}_2}{\partial x} dx dy - \tfrac{1}{2}[(2-\sigma)\mathfrak{I}_1 - (4+3\sigma)\mathfrak{I}_2] = 0 \quad \ldots(29)$$

where the double integrals are taken all over any transverse section.

It should be observed that the stress components S and T are independent of z, and therefore constant along each longitudinal fibre."

328.] **Determinateness of the Solution.** We already know from general principles (§ 255) that the solution is perfectly determinate when the distribution of stress over the ends of the beam is given. We may however show that the solution (21), subject to the boundary conditions (24-26), is perfectly deter-

minate in itself, so that the distribution of stress over the ends, as deduced by means of (22), is not at all arbitrary, but is governed by fixed laws depending only on the form and dimensions of the beam. To prove this, it will be sufficient to show, by a method equally applicable to all,* that any one of the w functions is completely determined by (23) and the appropriate boundary condition (24), (25) or (26).

If possible let these conditions be both satisfied by two different values of w (for example): let ζ be the difference of these two values. Then ζ must satisfy

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}=0$$

throughout the body, and

$$\lambda\frac{\partial\zeta}{\partial x}+\mu\frac{\partial\zeta}{\partial y}=0$$

all over the lateral surface. Now if we integrate the expression

$$\iint\left[\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2\right]dxdy$$

by parts it becomes

$$\iint\zeta\left[\lambda\frac{\partial\zeta}{\partial x}+\mu\frac{\partial\zeta}{\partial y}\right]ds-\iint\zeta\left[\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right]dxdy,$$

and each of these terms is identically zero. But (compare §§ 254, 256) the original integral is the sum of a number of essentially positive quantities, each of which must therefore vanish separately. Consequently

$$\frac{\partial\zeta}{\partial x}=\frac{\partial\zeta}{\partial y}=0$$

throughout the body, and since w is supposed to vanish at the origin, and thus cannot involve a constant term, ζ must be zero throughout. Hence the two values of w are identical, and it is obvious that the same may be proved, in precisely the same words, of w_1 and w_2.

First Component.—Simple Extension.

329.] **Complete Solution.** Making all the arbitrary constants zero, with the exception of ϵ, we have the simple strain

giving
$$\left.\begin{array}{c} u=-\sigma\epsilon x, \quad v=-\sigma\epsilon y, \quad w=\epsilon z; \\ R=q\epsilon, \quad S=T=0. \end{array}\right\} \qquad (30)$$

* This is, in effect, a special proof of Green's general theorem, adapted to the case in which the solution of Laplace's equation is independent of z, while the surfaces bounding the region within which that equation is satisfied are either parallel or perpendicular to Oz.

This is the case already fully discussed in § 213. A longitudinal tension $\mathbf{E}$ is applied to the beam by means of a uniform traction $\mathbf{E}/\mathfrak{A}$ over each end, and produces a uniform extension $\epsilon = \mathbf{E}/\mathfrak{A}q$ throughout the beam, accompanied by a uniform contraction $\sigma\epsilon$ in every transverse direction. The ratio $\mathfrak{e} = \mathfrak{A}q$ of the tension to the consequent elongation is called the *Coefficient of Longitudinal Extension* of the beam, and sometimes *Hooke's modulus;* but it must be remembered that it depends upon the dimensions of the body, as well as on the properties of the material, so that it is not a true specific modulus, in the sense in which we have always employed the term. For a beam of given material it is proportional to the sectional area, and for a beam of given section to the Young's modulus of the material.

If L be the length of the beam, equation (41) of § 214 gives for the total potential energy due to extension

$$W = R^2 L\mathfrak{A}/2q = L\mathbf{E}^2/2\mathfrak{e} = \tfrac{1}{2}L\mathfrak{e}\epsilon^2 \quad \text{.....................(31)}$$

If $\mathfrak{e}'$ be the coefficient of extension of a second beam of the same material, of the same length L but of section $\mathfrak{A}'$, then $\mathfrak{e} = \mathfrak{A}'q$, and

$$\mathfrak{e}' : \mathfrak{e} :: \mathfrak{A}' : \mathfrak{A};$$

but the masses of the beams are also in the ratio

$$M' : M :: \mathfrak{A}' : \mathfrak{A},$$

and therefore

$$\mathfrak{e}'/M' = \mathfrak{e}/M.$$

Hence we deduce that *the resistance to tension or thrust, proportionally to its mass, of a beam of given length is precisely the same whatever its transverse section.* [Compare the results of §§ 336 and 338.]

Second Component.—Torsion.

330.] **Equations of Strain.** Annulling all the arbitrary constants but τ, we have

$$\left.\begin{aligned} u &= -\tau z\left[y - \left(\frac{\partial \text{w}}{\partial x}\right)_0\right] \\ v &= \tau z\left[x + \left(\frac{\partial \text{w}}{\partial y}\right)_0\right] \\ w &= \tau\left[\text{w} - x\left(\frac{\partial \text{w}}{\partial x}\right)_0 - y\left(\frac{\partial \text{w}}{\partial y}\right)_0\right] \end{aligned}\right\} \quad \text{..................(32)}$$

and

$$\left.\begin{aligned} R &= 0 \\ S &= n\tau\left[x + \frac{\partial \mathrm{w}}{\partial y}\right] \\ T &= -n\tau\left[y - \frac{\partial \mathrm{w}}{\partial x}\right] \end{aligned}\right\} \qquad (33)$$

where w may be any function of x and y which vanishes at the origin, and satisfies

$$\frac{\partial^2 \mathrm{w}}{\partial x^2} + \frac{\partial^2 \mathrm{w}}{\partial y^2} = 0 \qquad (34)$$

throughout the body,

$$\lambda\left(\frac{\partial \mathrm{w}}{\partial x} - y\right) + \mu\left(\frac{\partial \mathrm{w}}{\partial y} + x\right) = 0 \qquad (35)$$

over the lateral surface, and the conditions of equilibrium

$$\iint \frac{\partial \mathrm{w}}{\partial x} dx dy = \iint \frac{\partial \mathrm{w}}{\partial y} dx dy = 0 \qquad (36)$$

331.] **Geometrical character of the Strain.** The simplest way of ascertaining this is to investigate (*i.*) the curve assumed by any longitudinal fibre of the beam, and (*ii.*) the surface into which any (initially plane) normal section is warped.

Let the point initially at (x, y, z) be displaced by the strain to (x', y', z'), so that $x' = x + u$, $y' = y + v$, $z' = z + w$.

(*i.*) Along any longitudinal fibre of the beam the initial coördinates x, y are constants. Thus (see § 68) the form assumed by any such fibre is represented by the equations

$$\left.\begin{aligned} x' &= x - \tau z'\left[y - \left(\frac{\partial \mathrm{w}}{\partial x}\right)_0\right] \\ y' &= y + \tau z'\left[x + \left(\frac{\partial \mathrm{w}}{\partial y}\right)_0\right] \end{aligned}\right\}$$

or

$$-\frac{x' - x}{y - \left(\frac{\partial \mathrm{w}}{\partial x}\right)_0} = \frac{y' - y}{x + \left(\frac{\partial \mathrm{w}}{\partial y}\right)_0} = \tau z'.$$

Thus, when the strain is very small, each fibre remains a straight line, and the *foot* of each fibre (the point in which it cuts the plane of xy) retains its initial position. In general each fibre is inclined to the Axis of the beam at a small angle

$$\tan^{-1}\left\{\tau\sqrt{\left[x + \left(\frac{\partial \mathrm{w}}{\partial y}\right)_0\right]^2 + \left[y - \left(\frac{\partial \mathrm{w}}{\partial x}\right)_0\right]^2}\right\},$$

but the particular fibre for which

$$x = -\left(\frac{\partial \mathrm{w}}{\partial y}\right)_0, \quad y = \left(\frac{\partial \mathrm{w}}{\partial x}\right)_0$$

is altogether unaffected by the strain. This fibre is called the **Axis of Torsion**, and the strain is said to be a Torsion about this axis. Again, each strained fibre lies in a plane

$$(x'-x)\left[x + \left(\frac{\partial \mathrm{w}}{\partial y}\right)_0\right] + (y'-y)\left[y - \left(\frac{\partial \mathrm{w}}{\partial x}\right)_0\right] = 0$$

which is perpendicular to the straight line joining its foot to that of the Axis of Torsion. Thus the generators of any circular cylinder

$$\left[x + \left(\frac{\partial \mathrm{w}}{\partial y}\right)_0\right]^2 + \left[y - \left(\frac{\partial \mathrm{w}}{\partial x}\right)_0\right]^2 = C^2,$$

described about the Axis of Torsion in the unstrained beam, become one set of generators of the one-sheet hyperboloid of revolution

$$(x'-x)^2 + (y'-y)^2 = C^2\tau^2 z'^2.$$

This surface is represented, on an exaggerated scale of torsion, in Figure 44.

The strained fibres may however, to the same degree of approximation, be regarded as helices of pitch

$$\frac{\pi}{2} - \tau\sqrt{\left[x + \left(\frac{\partial \mathrm{w}}{\partial y}\right)_0\right]^2 + \left[y - \left(\frac{\partial \mathrm{w}}{\partial x}\right)_0\right]^2}$$

described on circular cylinders about the Axis of Torsion, and this is the form they actually take under torsion of finite amount.

(*ii.*) Over any naturally plane normal section of the beam, the initial coördinate z is constant, and it appears by applying § 68 to the third of equations (32) that every such section is warped into the (generally) curved surface

$$z' = z + \tau\left[\mathrm{w}' - x'\left(\frac{\partial \mathrm{w}}{\partial x}\right)_0 - y'\left(\frac{\partial \mathrm{w}}{\partial y}\right)_0\right],$$

where w' denotes the same function of x' and y' that w does of x and y.

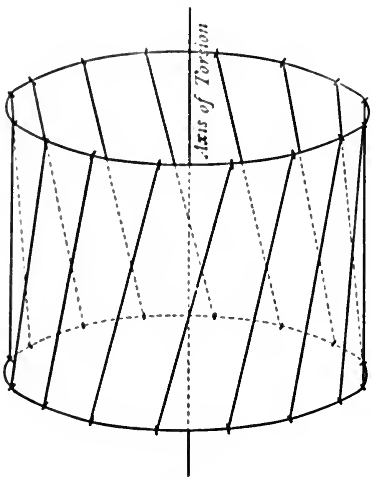


FIG. 44.

Torsion about the Central Axis of the Beam. St. Venant's Solution for a certain class of Beams.

332.] **Equations of the Strain.** When the Axis of Torsion coincides with the Central Axis of the beam, we have

$$\left(\frac{\partial \mathrm{w}}{\partial x}\right)_0 = \left(\frac{\partial \mathrm{w}}{\partial y}\right)_0 = (\mathrm{w})_0 = 0 \qquad (37)$$

In this case, equations (32) reduce to

$$u = -\tau yz, \quad v = \tau xz, \quad w = \tau \mathrm{w}; \qquad (38)$$

the other equations of § 330 being unaffected.

The strain now obviously consists of the bodily rotation of each normal section through an angle τz about the axis, together with a general warping of these sections by longitudinal displacement. The quantity τ is called the *Twist per unit length* of beam, or the *Amount of Torsion.*

333.] **St. Venant's Solution.** The problem can now be readily solved for a large and important class of beams, as follows. Let us suppose that the equation of the cylindrical surface (or of the closed curve bounding the base) can be put into the form

$$\phi + \tfrac{1}{2}(x^2 + y^2) = C, \qquad (39)$$

where ϕ is *any* solution of

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad (40)$$

and C is a constant.

Then

$$\lambda : \mu :: \frac{\partial \phi}{\partial x} + x : \frac{\partial \phi}{\partial y} + y,$$

and the boundary condition (35) becomes

$$\frac{\partial \mathrm{w}}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial \mathrm{w}}{\partial y}\frac{\partial \phi}{\partial y} + x\left(\frac{\partial \mathrm{w}}{\partial x} + \frac{\partial \phi}{\partial y}\right) + y\left(\frac{\partial \mathrm{w}}{\partial y} - \frac{\partial \phi}{\partial x}\right) = 0;$$

thus (34) and (35) will in all such cases be satisfied if we suppose

$$\frac{\partial \mathrm{w}}{\partial y} - \frac{\partial \phi}{\partial x} = \frac{\partial \mathrm{w}}{\partial x} + \frac{\partial \phi}{\partial y} = 0, \qquad (41)$$

that is if we choose w so that ϕ and w may be *Conjugate Functions* * of x and y. This is St. Venant's celebrated solution which is developed so skilfully and with such beautiful results in his Memoir on the Torsion of Prisms, already referred to.

* See Article 245, and Examples 1-4 on Chapter V.

The conditions of equilibrium (36) will also be satisfied identically upon this assumption, for they may now be written

$$\iint \frac{\partial \phi}{\partial y} dxdy = \iint \frac{\partial \phi}{\partial x} dxdy = 0,$$

or

$$\int \phi dx = \int \phi dy = 0,$$

the single integrals being taken round the perimeter of the section. But by (1)

$$\iint x dxdy = \iint y dxdy = 0,$$

and therefore ϕ must be such that

$$\int x^2 dy = \int y^2 dx = 0\,;$$

also, since (39) must be supposed to represent a *closed curve*,

$$\int y^2 dy = \int x^2 dx = 0.$$

and

$$\int dy = \int dx = 0.$$

Since $\phi + \frac{1}{2}(x^2 + y^2) = C$ all round the periphery, these last equations give us

$$\int [\phi + \tfrac{1}{2}(x^2 + y^2)] dx = \int [\phi + \tfrac{1}{2}(x^2 + y^2)] dy = 0,$$

and consequently

$$\int \phi dx = \int \phi dy = 0$$

identically.

In order that (37) may be satisfied w must not contain a linear function of x and y: nor therefore must ϕ. But this condition is necessarily satisfied; for if ϕ did involve any such terms, the integrals $\int \phi dx$ and $\int \phi dy$ would involve terms of the form $\int x dy$ and $\int y dx$, which are proportional to the area $\mathfrak{A}$ of the transverse section, and consequently cannot vanish.

Hence all the conditions of the problem are satisfied by any value of ϕ which satisfies (40) and makes the boundary (39) of the base a closed curve, provided that the included area has the origin for its centre of gravity and Ox and Oy for its principal axes of inertia.

334.] **The Torsion-Couple, Coefficient of Torsion and Potential Energy of the Strain.** It follows from (1) and (36) that the distribution of stress (33) over any transverse section of the beam (which is the same for all such sections) reduces to a couple in the plane of the section.

If **T** be the magnitude of the couple applied to either end of the beam

$$\mathbf{T} = \iint (xS - yT)\,dxdy$$
$$= n\tau \iint \left[x^2 + y^2 + x\frac{\partial \mathrm{w}}{\partial y} - y\frac{\partial \mathrm{w}}{\partial x}\right] dxdy$$
$$= n\tau \left[\mathfrak{I}_3 + \iint \left(x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y}\right) dxdy\right]$$

by (2) and (41). Thus if

$$\mathfrak{t} = n\left[\mathfrak{I}_3 + \iint \left(x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y}\right) dxdy\right] \quad \text{.................(42)}$$

the torsion-couple required to produce an amount of torsion τ is given by

$$\mathbf{T} = \mathfrak{t}\tau, \quad \text{.................................(43)}$$

and $\mathfrak{t}$ may be called the *Coefficient of Torsion* * of the beam. For a beam of given material it depends only on the form and dimensions of the transverse section, and for a beam of given dimensions it is proportional to the rigidity n of the material.

Again, if L be the length of the beam, the total potential energy of the strain is by (41) of § 214.

$$W = \frac{L}{2n}\iint (S^2 + T^2)\,dxdy$$
$$= \frac{n\tau^2 L}{2}\iint \left[\left(x + \frac{\partial \mathrm{w}}{\partial y}\right)^2 + \left(y - \frac{\partial \mathrm{w}}{\partial x}\right)^2\right] dxdy$$
$$= \frac{n\tau^2 L}{2}\iint \left[x^2 + y^2 + x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y}\right.$$
$$\left. + \frac{\partial \phi}{\partial x}\left(x + \frac{\partial \phi}{\partial x}\right) + \frac{\partial \phi}{\partial y}\left(y + \frac{\partial \phi}{\partial y}\right)\right] dxdy$$
$$= \frac{n\tau^2 L}{2}\left\{\frac{\mathfrak{t}}{n} + \iint \left[\frac{\partial \phi}{\partial x}\left(x + \frac{\partial \phi}{\partial x}\right) + \frac{\partial \phi}{\partial y}\left(y + \frac{\partial \phi}{\partial y}\right)\right] dxdy\right\}.$$

Now

$$\iint \left[\frac{\partial \phi}{\partial x}\left(x + \frac{\partial \phi}{\partial x}\right) + \frac{\partial \phi}{\partial y}\left(y + \frac{\partial \phi}{\partial y}\right)\right] dxdy$$
$$= \iint \left[\frac{\partial \phi}{\partial x}\frac{\partial}{\partial x}\left(\phi + \frac{x^2 + y^2}{2}\right) + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial y}\left(\phi + \frac{x^2 + y^2}{2}\right)\right] dxdy$$
$$= \int \left(\phi + \frac{x^2 + y^2}{2}\right)\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy\right)$$
$$- \iint \left(\phi + \frac{x^2 + y^2}{2}\right)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) dxdy$$
$$= C\int \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy\right), \text{ by (39) and (40),}$$
$$= 0, \text{ because the periphery is a closed curve.}$$

* Also known as the *Torsional Rigidity*.

Thus finally

$$W = \tfrac{1}{2}L\mathfrak{t}\tau^2 = L\mathrm{T}^2/2\mathfrak{t}\,; \qquad\qquad (44)$$

this formula should be compared with (31) of § 329.

335.] **Circular Cylinder.** If the base of the beam be a circle its centre must be at the origin, and we must therefore put $\phi = 0$ in (39) and take $\frac{1}{2}A^2$ for the constant term. A will then be the radius of the base.

It is evident that w vanishes with ϕ, and therefore

$$u = -\tau yz, \quad v = \tau xz, \quad w = 0\,; \qquad\qquad (45)$$

so that each normal section is simply *rotated bodily in its own plane* about the Central Axis, through an Angle proportional to its distance from the fixed base, without warping or distortion of any kind. The Coefficient of Torsion for a beam of circular section is by (42)

$$\mathfrak{t} = n\mathfrak{J}_3 = n\mathfrak{A}^2/2\pi = \tfrac{1}{2}\pi n A^4 \qquad\qquad (46)$$

The formula

$$\mathrm{T} = n\mathfrak{J}_3\tau, \qquad\qquad (47)$$

for the torsion of a circular cylinder, was first obtained by Coulomb in his researches into the theory of the Torsion Balance, and is usually alluded to as *Coulomb's Formula.*

It is interesting to note that, since by equations (33) the tangential stresses depend upon the rigidity n, they will vanish identically, and the boundary conditions will in consequence be satisfied, for all forms of w, if the beam be formed of a *viscous liquid* instead of an elastic solid material. The formulæ of this Article apply to such a beam, whatever be the form of its section, as may easily be shown by twisting very gradually and evenly a square stick of fine sealing wax. Boundaries of transverse sections, scratched on the wax beforehand, will be found to remain truly plane curves. Compare Article 340, below.

336.] **Hollow Circular Cylinder.** Let the beam, instead of being solid, have a coaxial cylindrical cavity of radius κA. Then ϕ is zero for both surfaces, and w is zero throughout the body, as before. The coefficient of torsion will be in this case

$$\mathfrak{t}' = n\mathfrak{J}_3' = \tfrac{1}{2}\pi n A^4(1 - \kappa^4)\,;$$

and if we compare this with the coefficient of the solid circular beam, as given by (46), we find

$$\mathfrak{t}' : \mathfrak{t} :: 1 - \kappa^4 : 1.$$

But the masses of the two beams, if their lengths are equal, are in the ratio

$$M' : M :: 1 - \kappa^2 : 1,$$

so that

$$\mathfrak{t}'/M' : \mathfrak{t}/M :: 1 + \kappa^2 : 1;$$

whence we deduce that *the resistance to torsion, proportionally to its mass, of a circular cylindrical beam of given length and external radius is increased by making it hollow.*

This principle is of great importance in the economy of structural materials, and will be referred to again later.

337.] **Elliptic Cylinder.** If in equation (39) we make

$$\phi = \tfrac{1}{2}a(y^2 - x^2),$$

it becomes

$$(1-a)x^2 + (1+a)y^2 = 2C, \qquad \text{(48)}$$

and if C is positive, and a is positive and less than unity, this represents an ellipse having its major and minor axes along Ox and Oy.

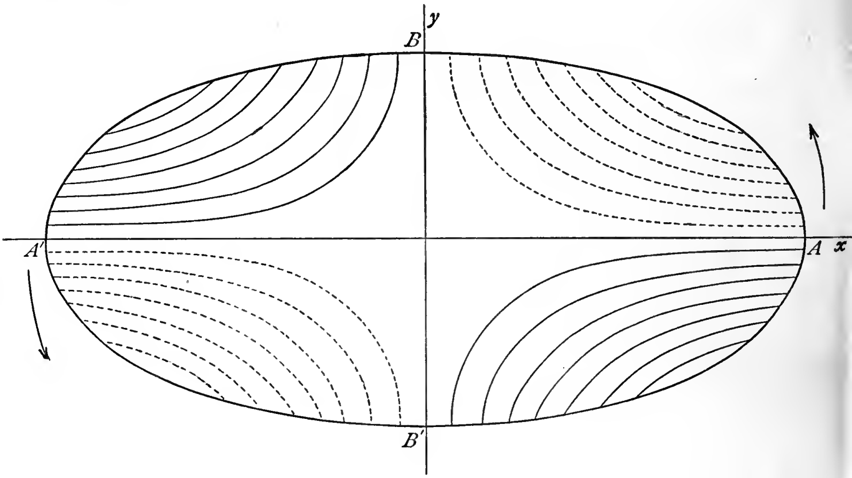


Fig. 45.

If A and B be the semi-axes of this ellipse

$$\frac{a}{A^2 - B^2} = \frac{C}{A^2B^2} = \frac{1}{A^2 + B^2},$$

and we have

$$\text{w} = -axy = -\frac{A^2 - B^2}{A^2 + B^2}xy;$$

thus

$$u = -\tau yz, \quad v = \tau xz, \quad w = -\frac{A^2 - B^2}{A^2 + B^2}\tau xy \qquad \text{(49)}$$

Also

$$\begin{aligned}\mathfrak{t} &= n[\mathfrak{J}_3 + a(\mathfrak{J}_2 - \mathfrak{J}_1)] \\ &= \tfrac{1}{4}\pi n AB[A^2 + B^2 + a(A^2 - B^2)] \\ &= \frac{\pi n}{2}\,\frac{AB(A^4 + B^4)}{A^2 + B^2} \qquad \text{.................................(50)}\end{aligned}$$

The transverse sections are in this case warped into hyperbolic paraboloids, any one of which is cut by planes perpendicular to the Central Axis in a series of hyperbolas having their asymptotes coincident with the principal axes of the unstrained elliptic section. It is evident from the form and sign of w that these surfaces are *concave* towards the positive direction of Oz in the $(+x, +y)$ and $(-x, -y)$ quadrants, and *convex* in the remaining quadrants.

Figure 45 represents the "contour lines" (*coupes topographiques*) in which the warped section is cut by a series of planes perpendicular to the axis. The principal axes AA', BB' of the unstrained section are unaltered by the strain, being merely twisted bodily through an angle τz about the axis of torsion; they are therefore the contour lines for the *original level* of the section. The dotted hyperbolas in the quadrants AB, $A'B'$ are *below* the original level (as looked at from the free end of the beam), and those in the remaining quadrants are *above* it. Figure 46 shows very clearly the warping of the sections, as it may be realised in practice on a greatly exaggerated scale, by twisting an indiarubber band of elliptic section.

FIG.46.

338.] **Hollow Beam, bounded by cylindrical surfaces of similar Elliptic sections.** If the beam is hollow and bounded internally by the similar and coaxial elliptic cylinder

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = \kappa^2,$$

ϕ is of the same form for both surfaces, and w and w have the same form as before.

If $\mathfrak{t}'$ be the coefficient of torsion of such a beam

$$\begin{aligned}\mathfrak{t}' &= n[\mathfrak{J}_3 + a(\mathfrak{J}_2 - \mathfrak{J}_1)] \\ &= \frac{\pi n}{2}\cdot\frac{AB(A^4 + B^4)}{A^2 + B^2}(1 - \kappa^4).\end{aligned}$$

Thus the resistance to torsion of an elliptic cylindrical beam of given material and given length, per unit mass of the beam, is increased in the ratio $1+\kappa^2:1$ by hollowing it. [Compare the corresponding result for beams of circular section in § 336.]

It is easy to show from equation (42) that this result applies to beams of any section, provided that the internal surface is similar to and similarly situated with the external surface.

339.] **Beam of Equilateral Triangular Section.** If in equation (39) we write for the constant term $C^2/18$, and put

$$\phi = (3xy^2 - x^3)/C\sqrt{3},$$

it becomes

$$-\frac{x^3-3xy^2}{C\sqrt{3}} + \frac{x^2+y^2}{2} - \frac{C^2}{18} = 0,$$

or

$$6\sqrt{3}(x^3-3xy^2) - 9C(x^2+y^2) + C^3 = 0.$$

The expression on the left hand side splits into three linear factors

$$(2x\sqrt{3}+C)(x\sqrt{3}+3y-C)(x\sqrt{3}-3y-C),$$

and it is easily verified that the boundary represented by the above equation is an equilateral triangle, having its centre of gravity at the origin and one vertex on Ox, the length of each side being C.

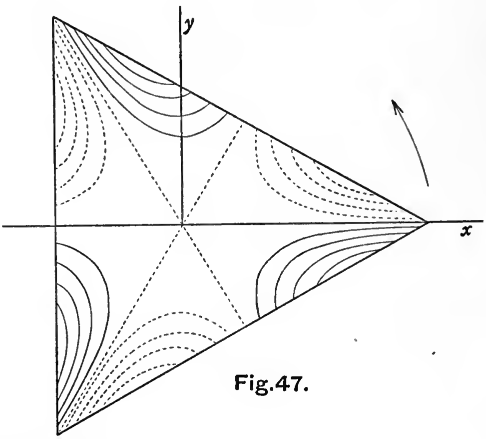


Fig.47.

Thus if we write

$$w = (y^3 - 3x^2y)/C\sqrt{3}$$

we obtain at once the solution for an equilateral triangular beam: the contour lines, given by

$$y^3 - 3x^2y = constant,$$

are represented in Figure 47.

340.] **Beam of Square Section.** Let the transverse section of the beam be a square, with its sides (of length C) parallel to Ox and Oy. The problem of determining the appropriate form of ϕ in this case will be much simplified by writing

$$\phi = \psi + \tfrac{1}{2}(y^2 - x^2):$$

it is easily seen that ψ also must satisfy (40), while the equation (39) of the bounding curve may now (by writing $\frac{1}{4}C^2$ instead of C) be put in the form

$$\psi = \tfrac{1}{4}C^2 - y^2.$$

Our function ψ must consequently be such a solution of (40) that it vanishes identically when $y = \pm\frac{1}{2}C$ for all values of x between $+\frac{1}{2}C$ and $-\frac{1}{2}C$, and is equal to $\frac{1}{4}C^2 - y^2$ when $x = \pm\frac{1}{2}C$ for all values of y between $+\frac{1}{2}C$ and $-\frac{1}{2}C$.

Now constants and even powers of y may be expanded by Fourier's Theorem in series of cosines of multiples of y, and on

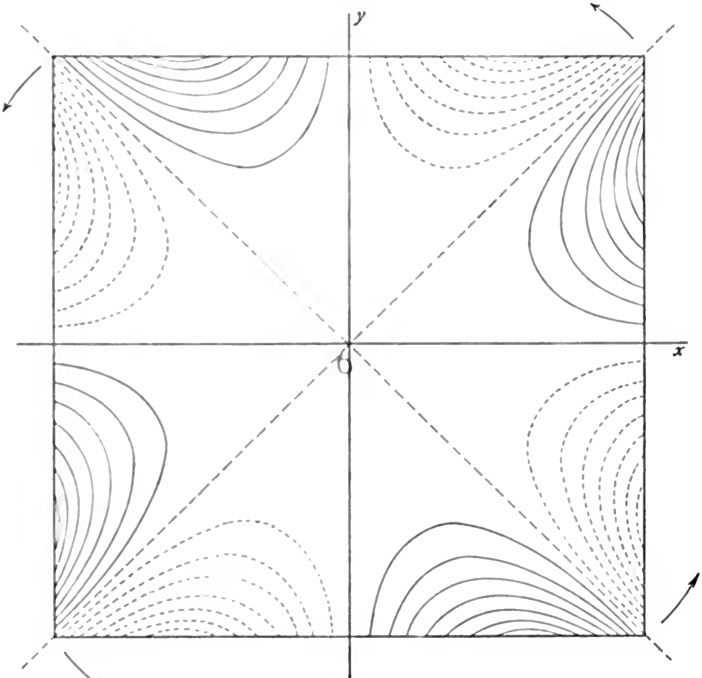


Fig. 48.

referring to Example 4 (*iii.*), page 258, it is at once apparent that the appropriate form of solution is

$$\psi = \sum_{p=0}^{p=\infty} (A_p e^{px} + B_p e^{-px}) \cos py.$$

The first of the required conditions will be satisfied identically if we suppose all the values of p included in this series to be of the form $(2i+1)\pi/C$, where i is any integer, or zero: and the solution will then be fully determined by the remaining conditions

$$\tfrac{1}{4}C^2 - y^2 = \sum_{i=0}^{i=\infty}\left[A_i e^{(2i+1)\pi/2} + B_i e^{-(2i+1)\pi/2}\right]\cos(2i+1)\pi y/C$$
$$= \sum_{i=0}^{i=\infty}\left[A_i e^{-(2i+1)\pi/2} + B_i e^{(2i+1)\pi/2}\right]\cos(2i+1)\pi y/C,$$

from which it is at once evident that $A_i = B_i$.

By Fourier's Theorem *

$$\tfrac{1}{4}C^2 - y^2 = \frac{4}{C}\sum_{i=0}^{i=\infty}\cos\frac{(2i+1)\pi y}{C}\int_0^{\frac{1}{2}C}(\tfrac{1}{4}C^2 - y^2)\cos\frac{(2i+1)\pi y}{C}dy,$$

and consequently

$$A_i = B_i = \frac{2}{C\cosh\dfrac{(2i+1)\pi}{2}}\int_0^{\frac{1}{2}C}(\tfrac{1}{4}C^2 - y^2)\cos\frac{(2i+1)\pi y}{C}dy$$
$$= \frac{(-1)^i \,.\, 4C^2}{(2i+1)^3\pi^3\cosh\dfrac{(2i+1)\pi}{2}}.$$

Thus finally

$$\phi = \tfrac{1}{2}(y^2 - x^2) + \frac{8C^2}{\pi^3}\sum_{i=0}^{i=\infty}\frac{(-1)^i\cosh\dfrac{(2i+1)\pi x}{C}}{(2i+1)^3\cosh\dfrac{(2i+1)\pi}{2}}\cos\frac{(2i+1)\pi y}{C}$$

and †

$$w = -xy + \frac{8C^2}{\pi^3}\sum_{i=0}^{i=\infty}\frac{(-1)^i\sinh\dfrac{(2i+1)\pi x}{C}}{(2i+1)^3\cosh\dfrac{(2i+1)\pi}{C}}\sin\frac{(2i+1)\pi y}{C}.$$

The contour lines are represented as before in Figure 48, and Figure 49 gives a view of the warped sections for comparison with those of the Elliptic Beam.

341.] **Character of the Stress.** By equations (5) and (33) the only existing stress components are R and S; thus equations (21) and (22) of § 163 reduce to

$$\left.\begin{aligned} &N(N^2 - S^2 - T^2) = 0 \\ &\frac{T\nu}{\lambda} = \frac{S\nu}{\mu} = \frac{T\lambda + S\mu}{\nu} = N \end{aligned}\right\} \quad\text{..........................(51)}$$

* Todhunter's *Integral Calculus*, Article 326, formula 5.
† Example 4 (*iv.*), page 258.

One of the principal stresses therefore vanishes at every point, and since the directions of the lines of zero stress are given by

$$\nu = 0, \quad T\lambda + S\mu = 0,$$

they are plane curves in planes perpendicular to the Central Axis and cutting the lateral surface at right angles.

The remaining principal stresses are equal in magnitude and of opposite sign, so that the stress at every point is a simple shearing stress, of magnitude

$$\sqrt{S^2 + T^2},$$

in a plane parallel to Oz the direction cosines of which are given by

$$\frac{\lambda}{S} = -\frac{\mu}{T} = \frac{1}{\sqrt{S^2 + T^2}}.$$

The system of Principal Surfaces enveloping these planes has for its differential equation

$$S dx - T dy = 0,$$

the integral of which is, by (33) and (41)

$$\phi + \tfrac{1}{2}(x^2 + y^2) = \text{constant}.$$

This system therefore includes the lateral surface of the beam.

It may also be deduced from (51) that the directions of the principal traction and pressure at every point are inclined at angles of 45° to the Central Axis.

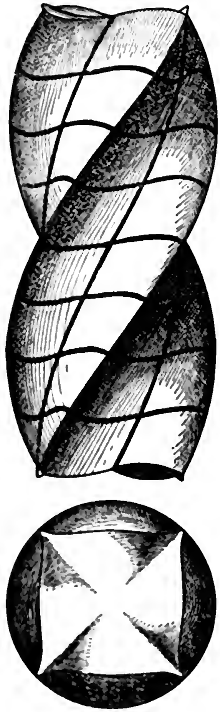


FIG. 49.

The magnitude of the resultant shearing stress is

$$\mathbf{S} = n\tau \sqrt{\left(x + \frac{\partial \phi}{\partial x}\right)^2 + \left(y + \frac{\partial \phi}{\partial y}\right)^2}.$$

Let

$$\Phi = \phi + \tfrac{1}{2}(x^2 + y^2),$$

so that the surfaces $\Phi = \text{constant}$ are those Principal Surfaces across which there is no stress, but which envelope the planes of shearing stress; then

$$\mathbf{S} = n\tau \sqrt{\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2} \quad \text{..........................(52)}$$

Since w, $\partial \mathrm{w}/\partial x$, $\partial \mathrm{w}/\partial y$ all vanish at the origin, so also do ϕ, $\partial \phi/\partial x$, $\partial \phi/\partial y$, and Φ, $\partial \Phi/\partial x$ and $\partial \Phi/\partial y$. Φ therefore increases

continuously in numerical value from 0 (along the Central Axis) to C (over the lateral surface). Similarly, $\mathbf{S}$ is zero along the Central Axis, and for corresponding points * on the different Φ surfaces $\mathbf{S}$ increases continuously with the numerical value of Φ until we reach the surface. To determine therefore the points of maximum stress ("points dangereux") we have only to determine those points on the lateral surface of the body at which the expression (52) for $\mathbf{S}$ becomes a maximum.

It follows at once from (52) that $\mathbf{S}$ is *zero* at any *projecting* angle (such as the edges of the square and triangular beams) and *infinite* at any *reëntrant* angle. *Angular* grooves † are therefore fatal to beams intended to sustain torsion, and the slightest crack in the surface will tend to spread indefinitely until the beam is destroyed. On the other hand, *angular* ridges add nothing to the torsional strength of the beam.

St. Venant has, however, proved a more general, and perhaps more striking property of torsion-shear. This is that *the stress at the surface is always a maximum at those points nearest to the axis, and a minimum at those points farthest from it.*

We can prove this property without difficulty for the cases which we have solved.

(*i.*) *Circular Beam*:

Here $$\Phi = \tfrac{1}{2}(x^2 + y^2),$$

$$\mathbf{S} = n\tau\sqrt{x^2+y^2} = n\tau A.$$

Thus $\mathbf{S}$ is constant all over the surface.

(*ii.*) *Elliptic Beam*:

Here $$\Phi = (A^2y^2 + B^2x^2)/(A^2+B^2),$$

and

$$\mathbf{S} = 2n\tau\sqrt{A^4y^2+B^4x^2}/(A^2+B^2) = 2n\tau B[A^4 - (A^2-B^2)x^2]^{\frac{1}{2}}/(A^2+B^2).$$

Thus $\mathbf{S}$ has its maximum value $2n\tau A^2B/(A^2+B^2)$ when $x=0$, *i.e.*, at the extremities of the *minor* axis, and its minimum

* Corresponding points on any family of curves, involving one variable parameter, are those points in which the family are cut by any one of the orthogonal system. With the notation of Chapter V., any function taken along the curve $\eta = const.$, $\zeta = const.$, can only vary with ξ.

† It seems possible that the curious twisting of old poplar trees, growing in situations where they are exposed to prevalent winds in a fairly definite direction, may be due in part to the presence of deep and sharply cut longitudinal grooves in the trunk. The unsymmetrical growth of the boughs affords a leverage to the wind, which thus exerts a powerful torsion couple. This tendency is of course greatly increased when the trees form an avenue, for they are then much more exposed on one side than on any other.

The Cambridge student will find excellent examples of the action here referred to in the old poplar avenue at Newnham Croft, near the University Swimming Club's sheds.

$2n\tau AB^2/(A^2+B^2)$ when $x=\pm A$, *i.e.*, at the extremities of the *major* axis. There are consequently two lines of minimum stress (AA, $A'A'$ in Figure 46), and two lines of maximum stress (BB, $B'B'$) along the whole length of the beam.

(*iii.*) *Equilateral Triangular Beam.*

Here $$\Phi = (3xy^2 - x^3)/C\sqrt{3} + \tfrac{1}{2}(x^2+y^2),$$
and $$\mathbf{S} = n\tau\sqrt{[(y^2-x^2)\sqrt{3}+Cx]^2 + y^2(2x\sqrt{3}+C)^2}/C.$$
The sides of the beam are represented by
$$2x\sqrt{3}+C=0,\quad x\sqrt{3}+3y-C=0,\quad x\sqrt{3}-3y-C=0.$$
Thus over the first side
$$\mathbf{S} = n\tau\sqrt{3}(\tfrac{1}{4}C^2-y^2)/C,$$
and this expression vanishes when $y=\pm\frac{1}{2}C$ (*i.e.*, along the edges which bound the side), and has its maximum value $\frac{1}{4}n\tau C\sqrt{3}$ when $y=0$ (*i.e.*, along the straight line drawn parallel to the Axis to bisect the face). Similarly for the other two sides.

(*iv.*) *Square Beam.*

Here
$$\Phi = y^2 + \frac{8C^2}{\pi 3}\sum_{i=0}^{i=\infty}\frac{(-1)^i\cosh\dfrac{(2i+1)\pi x}{C}}{(2i+1)^3\cosh\dfrac{(2i+1)\pi}{2}}\cos\frac{(2i+1)\pi y}{C};$$
the component stresses can easily be deduced by differentiation, and calculated numerically, and the resultant deduced.

St. Venant gives tables of the results (*Mémoire sur la Torsion des Prismes*, pp. 393, 394), which show conclusively that $\mathbf{S}$ is a maximum when $x=0$, $y=\pm\frac{1}{2}C$, and when $x=\pm\frac{1}{2}C$, $y=0$, the four corresponding values being equal, and that $\mathbf{S}$ vanishes at the angles.

This latter property may very easily be proved directly; for when $\pm x=\pm y=C$,
$$\partial\Phi/\partial x = 0,$$
$$\frac{\partial\Phi}{\partial y} = \pm C\left\{1-\frac{8}{\pi^2}\sum_{i=0}^{i=\infty}\frac{1}{(2i+1)^2}\right\}=0,$$
(Todhunter's *Plane Trigonometry*, Ch. xxiii., Ex. 3.)

Thus it appears that very little strength is gained by making beams, intended to resist torsion *only*, with projecting longitudinal ridges, or flanges. We shall however presently see that such flanges are of the greatest possible value in resisting *flexion*, if properly disposed.

342.] Erroneous Extension of Coulomb's Formula. It was assumed by engineers,* before St. Venant had obtained the complete solution of the problem, that all beams—of whatever section—behaved under torsion like circular cylinders: *i.e.*, that their normal sections rotated without distortion in their own planes. Thus the formula $\mathrm{T} = n\mathfrak{J}_3\tau$ was supposed to be universally applicable, whereas we know from formula (42) that it is a unique property of the circular cylinder.

The true value of $\mathfrak{t}$ as calculated from (42) for a beam of any other form, is found always to be less than that given by the application of Coulomb's formula, and also (as we might have expected from the last Article) less than that of a circular cylinder of the same sectional area. Figure 50 shows the results of St. Venant's comparison: the first line of numbers giving the ratios of the values of $\mathfrak{t}$ for beams of the sections represented to

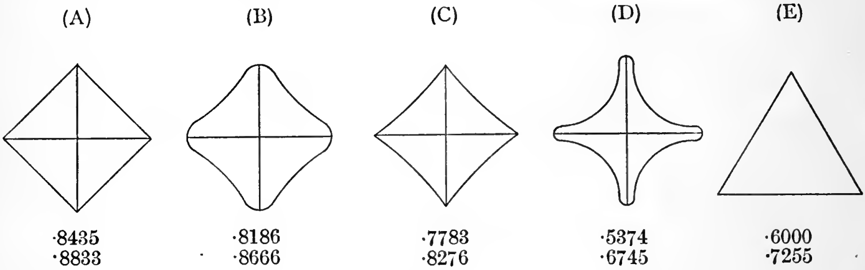


Fig.50.

those deduced from the fallacious theory just referred to, and the second line their ratios to the value of $\mathfrak{t}$ for a circular cylinder of the same sectional area. The waste of material in forming projecting ridges is very conspicuous in case (D).

Third Component.—Flexion.

343.] Equations of Strain. Retaining only the terms in the second lines of equations (21)—*i.e.*, annulling all the arbitrary constants but ϖ_1 and β_1—the displacements take the form

$$\left.\begin{aligned} u &= \varpi_1\sigma xy - \beta_1 z\left[\sigma xy - \left(\frac{\partial \mathrm{w}_1}{\partial x}\right)_0\right] \\ v &= \varpi_1\left[\tfrac{1}{2}\sigma(y^2 - x^2) + \tfrac{1}{2}z^2\right] - \beta_1 z\left[\tfrac{1}{2}\sigma(y^2 - x^2) + \tfrac{1}{6}z^2 - \left(\frac{\partial w_1}{\partial y}\right)_0\right] \\ w &= -\varpi_1 yz - \beta_1\left[x^2y - \tfrac{1}{2}yz^2 - \mathrm{w}_1 + x\left(\frac{\partial \mathrm{w}_1}{\partial x}\right)_0 + y\left(\frac{\partial \mathrm{w}_1}{\partial y}\right)_0\right] \end{aligned}\right\}\ldots(53)$$

* This statement is made by St. Venant, and quoted by Thomson and Tait. Neither authority gives any references, and I have not been able to verify it personally.

while the stress components are

$$\left.\begin{aligned} R &= -\varpi_1 \eta y + \beta_1 \eta y z \\ S &= -\beta_1 n\left[x^2 + \tfrac{1}{2}\sigma(y^2 - x^2) - \frac{\partial \mathrm{w}_1}{\partial y}\right] \\ T &= -\beta_1 n\left[(2+\sigma)xy - \frac{\partial \mathrm{w}_1}{\partial x}\right] \end{aligned}\right\} \qquad (54)$$

The function w_1 must satisfy the conditions

$$\frac{\partial^2 \mathrm{w}_1}{\partial x^2} + \frac{\partial^2 \mathrm{w}_1}{\partial y^2} = 0 \qquad (55)$$

throughout the body,

$$\lambda\frac{\partial \mathrm{w}_1}{\partial x} + \mu\frac{\partial \mathrm{w}_1}{\partial y} = (2+\sigma)\lambda xy + \mu[x^2 + \tfrac{1}{2}\sigma(x^2 - y^2)] \qquad (56)$$

over the lateral surface, and finally

$$\iint \frac{\partial \mathrm{w}_1}{\partial x}dxdy = \iint \frac{\partial \mathrm{w}_1}{\partial y}dxdy - \tfrac{1}{2}[(2-\sigma)\mathfrak{I}_2 - (4+3\sigma)\mathfrak{I}_1] = 0 \qquad (57)$$

for the preservation of equilibrium.

344.] **Geometrical Character of the Strain.** Any longitudinal fibre (x, y) of the beam is strained into the *plane* curve

$$\left.\begin{aligned} x' &= x(1 + \varpi_1 \sigma y) - \beta_1 z'\left[\sigma xy - \left(\frac{\partial \mathrm{w}_1}{\partial x}\right)_0\right] \\ y' &= y + \tfrac{1}{2}\varpi_1\sigma(y^2 - x^2) - \beta_1 z'\left[\tfrac{1}{2}\sigma(y^2 - x^2) - \left(\frac{\partial \mathrm{w}_1}{\partial y}\right)_0\right] + \tfrac{1}{2}\varpi_1 z'^2 - \tfrac{1}{6}\beta_1 z'^3 \end{aligned}\right\} (58)$$

The plane of each strained fibre is parallel to Oy, and the curve assumed by it is a parabola of the third degree, which however—the strain being very small—does not differ much from a parabola of the second degree, nor from a circle of large radius.

Every set of fibres which before the strain formed the generators of a rectangular hyperbolic cylinder

$$\sigma xy = C,$$

having the planes of yz and zx for its asymptotic planes, are strained into curves lying in the system of *parallel* planes

$$x' = x + \varpi_1 C - \beta_1 z'\left[C - \left(\frac{\partial \mathrm{w}_1}{\partial x}\right)_0\right];$$

and, in particular, the generators of the cylinder

$$\sigma xy = \left(\frac{\partial \mathrm{w}_1}{\partial x}\right)_0$$

become curves in planes parallel to yz.

The Central Axis itself lies when strained in the plane through Oy

$$x' = \beta_1 z' \left(\frac{\partial \mathrm{w}_1}{\partial x}\right)_0,$$

its curvature at a distance z from the fixed base being

$$\varpi_1 - \beta_1 z.$$

The elongation of the longitudinal fibres of the beam is given by

$$g = -y(\varpi_1 - \beta_1 z),$$

so that all fibres initially in the plane of zx—and in particular the Central Axis—retain their natural lengths unaltered. We have thus a *Plane of Zero Extension* dividing the beam longitudinally into two portions. If ϖ_1/β_1 be numerically greater than the length L of the beam, all fibres on one side of this plane will be elongated, and all fibres on the other side of it contracted, throughout their whole length. If however ϖ_1/β_1 be numerically less than L, the elongation of every fibre not in the plane of zero extension changes sign at a point initially distant ϖ_1/β_1 from its foot. The curvature of the Central Axis, and indeed of every fibre, changes sign at the same distance from the base, so that in this case each strained fibre has a point of inflexion.

The plane transverse sections are deformed into the surfaces

$$z' = z - \beta_1 x' \left(\frac{\partial \mathrm{w}_1}{\partial x}\right)_0 - y' \left[\varpi_1 z - \tfrac{1}{2}\beta_1 z^2 + \beta_1 \left(\frac{\partial \mathrm{w}_1}{\partial y}\right)_0\right] + \beta_1 \mathrm{w}_1', \ldots\ldots (59)$$

where, as in § 331, w_1' bears the same relation to x', y' as w_1 to x, y.

The tangent plane to any such surface at the point where it is cut by the Central Axis is found on expanding w_1' by MacLaurin's Theorem to be

$$z' = z - y'(\varpi_1 z - \tfrac{1}{2}\beta_1 z^2);$$

it is therefore parallel to Ox.

345.] **The Second Flexion Component.** The terms involving ϖ_2, β_2 and w_2 in equations (21), (22), (23), (26), (29) may be deduced from those discussed in the last two Articles by interchanging the suffixes 1 and 2, and the coördinates x and y. The strain represented by them is therefore a flexion of precisely the same character, only in converse relation to the principal planes of the beam.

Plane Circular Flexion in a Principal Plane.

346.] **Reduction of the Strain.** If we now annul β_1, and with it all the terms involving w_1 (see § 327), and retain only those which have ϖ_1 for coefficient, the character of the strain is greatly simplified.

The displacements become

$$\left.\begin{aligned} u &= \varpi_1 \sigma x y \\ v &= \tfrac{1}{2}\varpi_1[\sigma(y^2 - x^2) + z^2] \\ w &= -\varpi_1 y z \end{aligned}\right\} \qquad (60)$$

and the stress components

$$R = -\varpi_1 q y, \quad P = Q = S = T = U = 0 \qquad (61)$$

347.] **Geometrical Character of the Strain.** The easiest way to realise the effects of this strain is to resolve the displacements (60) into the three simple component systems

$$\left.\begin{aligned} u &= 0 \\ v &= \tfrac{1}{2}\varpi_1 z^2 \\ w &= -\varpi_1 y z \end{aligned}\right\}(i.) \qquad \left.\begin{aligned} u &= \varpi_1 \sigma x y \\ v &= -\tfrac{1}{2}\varpi_1 \sigma x^2 \\ w &= 0 \end{aligned}\right\}(ii.) \qquad \left.\begin{aligned} u &= 0 \\ v &= \tfrac{1}{2}\varpi_1 \sigma y^2 \\ w &= 0 \end{aligned}\right\}(iii.)$$

(*i.*) First, we have

$$y' - y = \tfrac{1}{2}\varpi_1 z^2, \quad z' - z = -\varpi_1 y z;$$

thus

$$(y' - y)(2 - 2\varpi_1 y) = \varpi_1 z z',$$

and to our order of approximation

$$(y' - y)(2 - \varpi_1 y - \varpi_1 y') = \varpi_1 z'^2,$$

or

$$y'^2 - \frac{2y'}{\varpi_1} + z'^2 = y^2 - \frac{2y}{\varpi_1},$$

or

$$\left(\frac{1}{\varpi_1} - y'\right)^2 + z'^2 = \left(\frac{1}{\varpi_1} - y\right)^2.$$

Hence every fibre parallel to the Central Axis is strained into an arc of a circle, lying in the plane drawn through its original direction parallel to the plane of yz, and having its centre in a straight line drawn parallel to Ox to cut Oy at a distance $1/\varpi_1$ from the origin in the positive direction. All fibres initially in the plane of zx retain their natural length unaltered.

Secondly, since

$$y' - \tfrac{1}{2}\varpi_1 z^2 = \frac{z - z'}{\varpi_1 z} = y,$$

it follows that

$$z' - z + \varpi_1 z y' = 0,$$

or

$$\varpi_1 y' + \frac{z'}{z} = 1.$$

Thus every line in the beam initially parallel to Oy is strained into a straight line parallel to the plane of yz, and meeting the line

of centres of the circular fibres. In fact, each such straight line becomes a radius of all those circular fibres which lie in the same plane parallel to yz.

Thirdly, every straight line in the beam parallel to Ox remains a straight line, and is shifted bodily parallel to itself. See Figure 51.

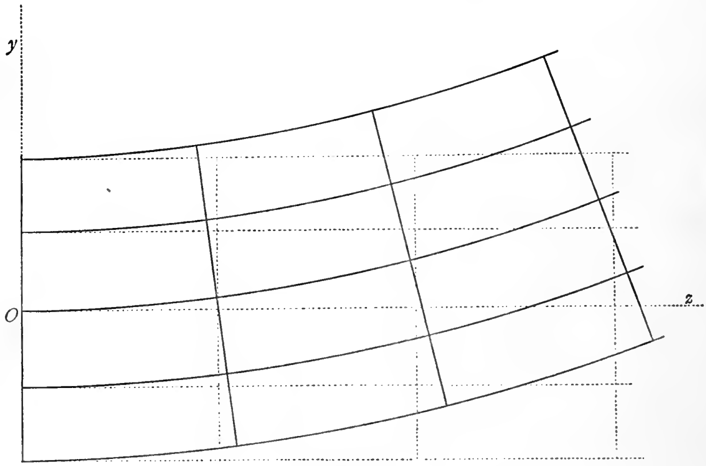


Fig. 51

(*ii.*) This component is obviously similar to the first, x, z, u, w, $-\sigma\varpi_1$ being substituted for z, x, w, u, ϖ_1.

Every longitudinal fibre remains straight, and is shifted bodily parallel to itself.

Every line in the body parallel to Ox becomes a circular arc in the plane drawn through its initial direction parallel to xy, and having its centre in a straight line drawn parallel to Oz to cut Oy at a distance $1/\sigma\varpi_1$ from the origin in the *negative* direction.

Every line in the body parallel to Oy remains a straight line in the same plane parallel to xy, and becomes a radius of all the circular arcs in that plane. See Figure 52.

(*iii.*) represents a displacement of every point in the body perpendicular to the plane of xz, and in the positive direction of Oy (*i.e.*, that towards which the beam is bent), the amount of which depends only on the initial distance of the point from this plane. Thus every straight line of the three principal systems remains straight, and parallel to its initial direction.

Superposing these results we see that

(*i.*) Every longitudinal fibre of the beam is strained into a circular arc of radius $1/\varpi_1 - y$ in a plane making a small angle $\sigma\varpi_1 x$ with yz.

(*ii.*) Every straight line parallel to Ox is strained into a circular arc of radius $1/\sigma\varpi_1 + y$ in a plane making an angle $\varpi_1 z$ with xy.

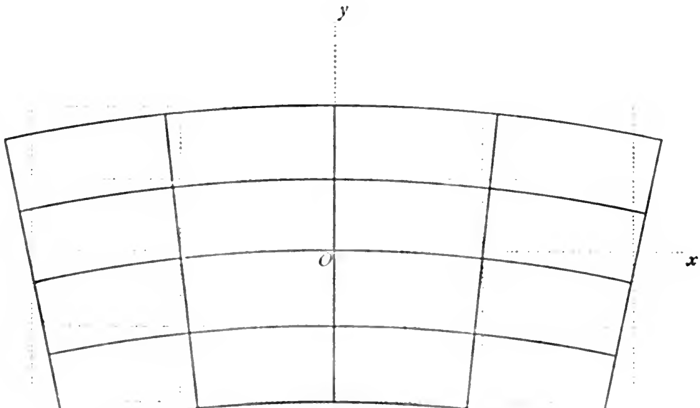


Fig. 52

(*iii.*) Every straight line parallel to Oy remains a straight line, the equations of which are

$$\frac{x' - x}{\sigma x} = \varpi y' = \frac{z - z'}{z},$$

and which is a radius of all the circular arcs of either system which intersect it.

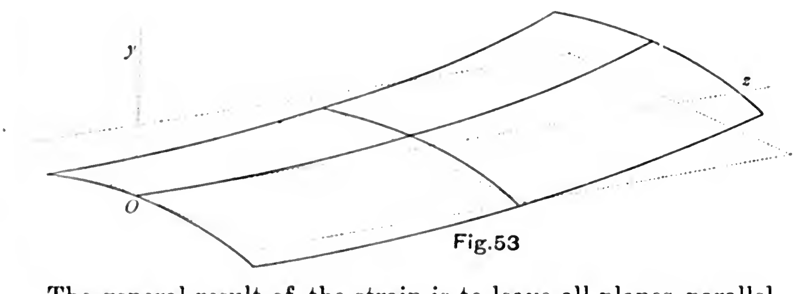


Fig. 53

The general result of the strain is to leave all planes parallel to xy and yz in the form of planes, and to warp all planes parallel to zx into surfaces of anticlastic curvature. The strained form of the plane of zx is shewn in Figure 53.

Since $$e = f = \sigma\varpi_1 y, \quad g = -\varpi_1 y,$$

this plane is a *Plane of Zero Strain*, that is to say, it is warped in such a manner that any figures drawn in it preserve (to the first order of small quantities) the dimensions and proportions of their elementary parts unaltered, although they cease to be plane figures. This plane is known as the *Neutral Plane*.

All fibres on the side of this plane towards which flexion takes place are uniformly shortened and dilated, and all fibres on the other side of it are uniformly lengthened and compressed, the loss or gain in length being proportional to the distance of the fibre from the Neutral Plane. This proportionality ensures that all transverse sections—and in particular the ends of the beam—remain plane.

All fibres initially in the plane of yz become circular arcs in that plane, and a strain of the kind that we have been investigating is called a *Circular Flexion* in the principal plane of yz, or about the principal axis Ox. The *amount of flexion* is measured by the curvature ϖ_1 of the Central Axis. The plane in which the Central Axis is flexed is called the *Plane of Flexion*.

348.] **Character of the Stress.** All the stress components vanish except the longitudinal traction, and by (61)

$$R = -q\varpi_1 y.$$

The *tension* across any transverse section of the rod is

$$\iint R\,dxdy = -q\varpi_1 \iint y\,dxdy = 0$$

by (1). The action across every transverse section consequently reduces to a couple; and since the component couple in the plane of zx is

$$\iint xR\,dxdy = -q\varpi_1 \iint xy\,dxdy = 0$$

by (1), this couple is wholly in the plane of flexion.

349.] **The Flexion Couple, Coefficient of Flexion, and Potential Energy.** The couple in the plane of flexion, applied to either end of the rod, is the same as that acting across each transverse section throughout its length, its amount being

$$\mathbf{P}_1 = -\iint Ry\,dxdy = q\varpi_1 \iint y^2 dxdy = q\varpi_1 \mathfrak{J}_1 .\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,(62)$$

This is called the Flexion Couple, and its ratio to the amount of flexion produced, namely

$$\mathfrak{p}_1 = \mathbf{P}_1/\varpi_1 = q\mathfrak{J}_1, .\,(63)$$

is called the *Coefficient of Flexion* in the principal plane yz.

If L be the length of the beam, the total potential energy of the strain is

$$W = \tfrac{1}{2}L\iint Rg\,dx\,dy = \tfrac{1}{2}q\varpi_1^2\iint y^2\,dx\,dy$$
$$= \tfrac{1}{2}L\mathfrak{p}_1\varpi_1^2 = L\mathbf{P}_1^2/2\mathfrak{p}_1 \qquad (64)$$

The Second Principal Component of Circular Flexion.

350.] **Displacements, Stress, Coefficient of Flexion, etc.** In exactly the same way we may show that, if in equations (21) we annul all terms but those which have ϖ_2 for coefficient, they will represent a circular flexion of amount ϖ_2 in the principal plane zx, the plane of yz being now the plane of zero strain.

The displacements are

$$\left.\begin{aligned} u &= \tfrac{1}{2}\varpi_2[\sigma(x^2-y^2)+z^2] \\ v &= \varpi_2\sigma xy \\ w &= -\varpi_2 xz \end{aligned}\right\} \qquad (65)$$

and the longitudinal traction is

$$R = -q\varpi_2 x.$$

The flexion couple is

$$\mathbf{P}_2 = q\varpi_2\mathfrak{I}_2, \qquad (66)$$

he coefficient of flexion

$$\mathfrak{p}_2 = q\mathfrak{I}_2, \qquad (67)$$

. nd the potential energy

$$W = \tfrac{1}{2}L\mathfrak{p}_2\varpi_2^2 = L\mathbf{P}_2^2/2\mathfrak{p}_2 \qquad (68)$$

Circular Flexion in any Plane.

351.] **Equations of Displacement.** Let the beam be fixed in such a way that the Central Axis takes the form of a circular arc of curvature ϖ in a plane inclined at an angle a to the principal plane of zx.

Then, by a simple application of Meunier's Theorem the component curvatures of the Axis in the two principal planes are

$$\varpi_1 = \varpi\cos a, \quad \varpi_2 = \varpi\sin a \qquad (69)$$

Making these substitutions in equations (21) and (22), we have for the displacements

$$\left.\begin{aligned} u &= \varpi\{\sigma xy\cos a + \tfrac{1}{2}[\sigma(x^2-y^2)+z^2]\sin a\} \\ v &= \varpi\{\sigma xy\sin a + \tfrac{1}{2}[\sigma(y^2-x^2)+z^2]\cos a\} \\ w &= -\varpi\{yz\cos a + xz\sin a\} \end{aligned}\right\} \qquad (70)$$

and for the longitudinal traction

$$R = -q\varpi(x \sin \alpha + y \cos \alpha), \quad \text{.......................(71)}$$

the other stress components vanishing, as before.

352.] **Flexion Couple, etc.** The total action across any transverse section still reduces to a couple, but this couple is no longer in the Plane of Flexion. The component couple in this plane, which may still be called the *Flexion Couple proper*, is

$$\mathbf{P} = -\cos \alpha \iint yRdxdy - \sin \alpha \iint xRdxdy$$
$$= q\varpi(\mathfrak{J}_1 \cos^2\alpha + \mathfrak{J}_2 \sin^2\alpha), \quad \text{.............................(72)}$$

so that the Coefficient of Flexion

$$\mathfrak{p} = q(\mathfrak{J}_1 \cos^2\alpha + \mathfrak{J}_2 \sin^2\alpha) \quad \text{..........................(73)}$$

is still, as in the simpler cases [(63) and (67)] of flexion in a principal plane, equal to *the product of Young's Modulus into the moment of inertia of the transverse section about an axis in its own plane, through its centre of gravity, perpendicular to the plane of flexion.*

It will be observed that

$$\mathfrak{p} = \mathfrak{p}_1 \cos^2\alpha + \mathfrak{p}_2 \sin^2\alpha, \quad \text{..............................(74)}$$

where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are the *principal* coefficients of flexion.

The component couple *perpendicular* to the plane of flexion is

$$q\varpi(\mathfrak{J}_1 - \mathfrak{J}_2) \sin \alpha \cos \alpha = (\mathfrak{p}_1 - \mathfrak{p}_2) \sin \alpha \cos \alpha, \quad \text{...............(75)}$$

the sign being taken so that it tends to bend the Axis towards the plane of yz, in which the coefficient of flexion is $\mathfrak{p}_1$. In other words, this couple is necessary to prevent the beam from acquiring the given amount of flexion in the easiest possible way, *i.e.*, in that principal plane in which the coefficient of flexion is least.

If the plane of the *resultant couple* make an angle ψ with zx, the component couple perpendicular to this plane vanishes, so that

$$\sin \psi \iint Rydxdy - \cos \psi \iint Rxdxdy = 0,$$

or

$$\tan \psi = \frac{\mathfrak{J}_2}{\mathfrak{J}_1} \tan \alpha = \frac{\mathfrak{p}_2}{\mathfrak{p}_1} \tan \alpha \quad \text{........................(76)}$$

Hence, when the *form* of the transverse section of the beam is given, and either the plane in which the couple is to be applied, or the plane in which flexion is to be produced, the second plane can be found by the following geometrical constructions:—

(*i.*) Describe any *momental ellipse* *

$$\mathfrak{J}_1 x^2 + \mathfrak{J}_2 y^2 = \textit{constant}$$

of the transverse section, and a central radius of this ellipse to

* Routh's *Rigid Dynamics:* Volume i., Article 19 (4th edition).

represent the trace on the plane of the section of the plane of flexion. The perpendicular from the centre on the tangent to the ellipse at the extremity of this radius will be the trace of the plane of the resultant couple.

(*ii.*) Describe the *ellipse of gyration* *

$$x^2/\mathfrak{I}_1 + y^2/\mathfrak{I}_2 = 1/\mathfrak{A}$$

of the transverse section, and a central radius representing the trace of the plane of the resultant couple. The perpendicular from the centre on the tangent to the ellipse at the extremity of this radius will be the trace of the plane of flexion.

353.] **Beams of Equal Flexibility in all directions.** It is obvious from these constructions, or directly from (75) and (76), that the plane of the resultant couple does not in general coincide with the plane of flexion unless the latter is a principal plane.

In all cases, however, in which the area of the transverse section is *kinetically symmetrical* about its centre of gravity, $\mathfrak{I}_1 = \mathfrak{I}_2$, the ellipses become circles, and $\mathfrak{p} = \mathfrak{p}_1 = \mathfrak{p}_2$. The beam is then said to be *equally flexible in all directions*, and flexion takes place accurately in the plane of the applied couple.

354.] **The Potential Energy of Flexion.** By equation (20) of § 199

$$\begin{aligned} W &= \tfrac{1}{2}L\iint Ry\,dx\,dy \\ &= \tfrac{1}{2}Lq\varpi^2\iint(y\cos\alpha + x\sin\alpha)^2 dx\,dy \\ &= \tfrac{1}{2}L\varpi^2(\mathfrak{I}_1\cos^2\alpha + \mathfrak{I}_2\sin^2\alpha) \\ &= \tfrac{1}{2}L\mathfrak{p}\varpi^2 = L\mathfrak{P}^2/2\mathfrak{p}. \qquad\qquad (77)\end{aligned}$$

as before.

355.] **The Stress. Economy of Material in Beams designed to resist Flexion only. The I beam.** It follows at once from (71) that—whatever be the form of the transverse section, the longitudinal traction is zero throughout the Neutral Plane, drawn through the Central Axis perpendicular to the Plane of Flexion. The traction also has its maximum positive and negative values along those generators of the beam which are farthest from the Neutral Plane, on either side.

Since the coefficients of flexion, like that of torsion, depend upon the moments of inertia of the cross section, a precisely similar economy of material, or increase of strength per unit mass, is effected by hollowing out that portion of the beam which surrounds the Central Axis.

* *Ibid*, Article 26.

When the plane of the straining couple is determinate—in actual structures this is usually the vertical plane through the Central Axis—a still greater economy of material is possible, because our only object is then to make the coefficient of flexion in one given plane as great as possible, while that in the perpendicular plane may theoretically be reduced to any extent. We shall therefore gain by concentrating the substance of the beam as near as possible to the plane of flexion, and as far as possible from the neutral plane. In practice we have to take into account possible small flexions in other planes, as well as accidental torsions, so that the reduction of material in the central portion of the beam must not be carried too far. The best practical compromise is found in the "I beam," in which the section consists of two *flanges* connected by a *web*, the whole being symmetrical about the plane of flexion.

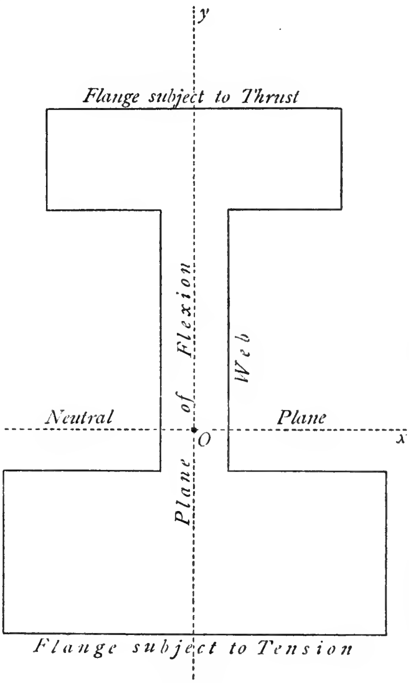


Fig.54.

In the case of wrought iron, and the other more perfectly elastic materials in which the working strengths under tension and compression are approximately equal [see Table (D), page 203], the Neutral Plane should be equidistant from the two extremes of the section. In cast iron, however, the working strength under compression is nearly three times that under tension, so that the greatest economy of strength will be gained by making the distances of the Neutral Plane from the extreme surfaces of the flanges in the same ratio. Since the centre of gravity of the entire section must always lie in the neutral plane, this consideration of course requires that the sectional area of the *stretched* flange should be considerably greater than that of the *compressed* flange.

The Coefficient of Flexion for an I beam of given dimensions is easily calculated.* Let "depth" denote dimensions parallel

* In practice the inward angles are rounded off, to guard against accidental *torsion*, and other shearing actions (Art. 341).

to the plane of flexion (yz), and "breadth" dimensions perpendicular to this plane. Let B_1, F_1 be the breadth and depth of the flange on the side *towards* which flexion takes place, and which is therefore subject to longitudinal *thrust* (§ 348), and let B_2, F_2 be the dimensions of the flange subject to *tension*. Let B_3 be the breadth of the web, and Y_1, Y_2 the distances from the neutral plane (xz) of the extreme surfaces of the contracted and extended flanges. Then

(*i.*) Since the neutral plane contains the centre of gravity of the section

$$\tfrac{1}{2}B_3(Y_1 - F_1 + Y_2 - F_2)(Y_1 - F_1 - Y_2 + F_2)$$
$$= B_2F_2(Y_2 - \tfrac{1}{2}F_2) - B_1F_1(Y_1 - \tfrac{1}{2}F_1) \text{..............} (78)$$

(*ii.*) Since the extreme stresses, and therefore also by (61) the extreme ordinates, are to be in the ratio of the working strength under thrust (**C**) to that under tension (**T**),

$$Y_1 : Y_2 :: \mathbf{C} : \mathbf{T} \text{..............................} (79)$$

(*iii.*) By (63) of § 349

$$\mathfrak{p}/q = B_1F_1[\tfrac{1}{12}F_1^2 + (Y_1 - \tfrac{1}{2}F_1)^2] + B_2F_2[\tfrac{1}{12}F_2^2 + (Y_2 - \tfrac{1}{2}F_2)^2]$$
$$+ B_3(Y_1 - F_1 + Y_2 - F_2)[\tfrac{1}{12}(Y_1 - F_1 + Y_2 - F_2)^2 + \tfrac{1}{4}(Y_1 - F_1 - Y_2 + F_2)^2] \text{...} (80)$$

By means of (78) and (79) $\mathfrak{p}$ may easily be found in terms of the dimensions of the flanges, and of the total depth ($Y_1 + Y_2$) of the beam. The dimensions adopted in practice are such that the strength of the beam is about six times that of a simple rectangular beam of the same sectional area (Cotterill).

The Small Strain compounded of Uniform Extension, Uniform Torsion about the Central Axis, and Plane Circular Flexion.

356.] **The Displacements and Stress Components.** Compounding the equations of §§ 329, 332 and 350, we obtain for the resultant displacements

$$\left.\begin{aligned} u &= -\sigma\epsilon x - \tau yz + \varpi\{\sigma xy\cos\alpha + \tfrac{1}{2}[\sigma(x^2 - y^2) + z^2]\sin\alpha\} \\ v &= -\sigma\epsilon y + \tau xz + \varpi\{\sigma xy\sin\alpha + \tfrac{1}{2}[\sigma(y^2 - x^2) + z^2]\cos\alpha\} \\ w &= \epsilon z + \tau \mathrm{w} - \varpi z(x\sin\alpha + y\cos\alpha) \end{aligned}\right\} \text{......} (81)$$

and for the stress components

$$\left.\begin{aligned} R &= q[\epsilon - \varpi(x\sin\alpha + y\cos\alpha)] \\ S &= n\tau\left(x + \frac{\partial \mathrm{w}}{\partial y}\right) \\ T &= -n\tau\left(y - \frac{\partial \mathrm{w}}{\partial x}\right) \end{aligned}\right\} \text{........................} (82)$$

357.] **The External Forces and Couples to which this Strain is due. Independence of their effects.** Over either end of the beam $T=F$, $S=G$, $R=H$, and on substituting from formulæ (82) in the surface integrals (6) and (7) of § 146, and integrating over the area of the transverse section, we find for the system of external forces and couples which must be applied to the ends of the beam to produce the strain represented by (81)

(*i.*) A force, parallel to the Axis, of amount

$$\mathbf{E} = q\mathfrak{A}\epsilon = \mathfrak{e}\epsilon, \qquad (83)$$

(*ii.*) A couple, in the principal plane of yz, of amount *

$$-\mathbf{P}_1 = -q\mathfrak{J}_1\varpi\cos\alpha = -\mathfrak{p}_1\varpi\cos\alpha \qquad (84)$$

(*iii.*) A couple, in the principal plane of zx, of amount

$$\mathbf{P}_2 = q\mathfrak{J}_2\varpi\sin\alpha = \mathfrak{p}_2\varpi\sin\alpha \qquad (85)$$

(*iv.*) A couple, in the plane of xy perpendicular to the Central Axis, of amount

$$\mathbf{T} = n\tau\left\{\mathfrak{J}_3 + \iint\left(x\frac{\partial \mathrm{w}}{\partial y} - y\frac{\partial \mathrm{w}}{\partial x}\right)dxdy\right\} = \mathfrak{t}\tau \qquad (86)$$

Thus each of the component distortions ϵ, $\varpi_1(=\varpi\cos\alpha)$, $\varpi_2(=\varpi\sin\alpha)$, τ, is related to the corresponding external action as if it existed alone. Consequently, if the external action on the ends of the beam is distributed according to the laws of equation (82), the effects of the longitudinal force and the component couples will be entirely independent.

Let $\mathbf{E}$ be the longitudinal tension, and $\mathbf{C}$ the resultant couple applied in the plane having λ, μ, ν for its direction cosines: then by equations (83-86)

$$\left.\begin{aligned} \epsilon &= \mathbf{E}/\mathfrak{e} \\ \varpi &= \mathbf{C}\sqrt{(\lambda/\mathfrak{p}_1)^2 + (\mu/\mathfrak{p}_2)^2} \\ \alpha &= \tan^{-1}(-\mu\mathfrak{p}_1/\lambda\mathfrak{p}_2) \\ \tau &= \nu\mathbf{C}/\mathfrak{t} \end{aligned}\right\} \qquad (87)$$

It should also be observed that, even in the most general form of strain, the force and couple across any transverse section of the beam are transmitted, unaltered in magnitude, from one end to the other.

358.] **The Total Potential Energy.** By equation (20) of § 199, we have

$$W = \tfrac{1}{2}L\iint\{Rg + Sa + Tb\}dxdy,$$

* The couples are here taken in the standard directions of Appendix I. If the plane of flexion lies between the positive directions of Ox and Oy, the effective couple in the plane of yz must be negative.

and it is easily deduced, on integrating and using where necessary the results of the previous Articles, that

$$\left.\begin{aligned} W &= \tfrac{1}{2}L[\mathfrak{e}\epsilon^2 + \mathfrak{p}_1\varpi_1^2 + \mathfrak{p}_2\varpi_2^2 + \mathfrak{t}\tau^2] \\ &= \tfrac{1}{2}L[\mathfrak{e}\epsilon^2 + \mathfrak{p}\varpi^2 + \mathfrak{t}\tau^2] \end{aligned}\right\} \quad \text{.....................(88)}$$

Thus the potential energy is a homogeneous quadratic function of the extension, torsion and flexions, each term appearing in exactly the same form as if the corresponding kind of strain existed alone.

The Equilibrium and Motion of Naturally Straight Wires.

359.] **General considerations applicable to bodies of infinitely small transverse dimensions.** The first point to be observed in connection with such bodies is, that *the infinite smallness of relative displacement of points situated at elementary distances from one another* (the necessary and sufficient condition for the infinite smallness of strain, assumed by our theory) *may be consistent with finite (and even very great) relative displacement of points separated by finite distances.*

This apparently obscure statement may be illustrated by a simple example: let us take the case of plain circular flexion of a wire. From equations (60) it appears that the strain components are of the dimensions ϖy, where ϖ is the curvature of the Central Axis and y the distance of any point from that Axis, measured parallel to the plane of flexion. Thus, if D be what we have called the "depth" of the wire (§ 355), *i.e.*, its maximum dimension in the plane of flexion, the product ϖD may be taken to represent the dimensions of the strain. Now, suppose a wire of finite length L, naturally straight, to be bent into a circle. Supposing the law of flexional strain to hold for such a case, the Central Axis will be unaltered in length, and its curvature will therefore be $\varpi = 2\pi/L$. Thus the strain will be of dimensions D/L, and if the transverse dimensions of the wire be infinitely small in comparison with its finite length, the strain at every point will remain infinitely small although the relative displacements of transverse sections separated by finite distances will obviously be finite. The case of torsion may be treated in a precisely similar way.

These results may be generalised as follows: *If one or more of the dimensions of a body be infinitely small in comparison with the remainder, points in the body separated by finite distances, measured parallel to a finite dimension, may suffer finite relative displacements, parallel to an infinitely small dimension, without producing other than infinitely small strains and stresses at any point of the body.*

On the other hand, by considering the longitudinal extension of an infinitely fine wire, we may easily show that *the relative displacements of all points in such a body, parallel to a finite dimension, must be infinitely small, if the strain and stress are to be infinitely small.*

Thus, if an infinitely fine wire have one point fixed, the necessary and sufficient conditions that the strain and stress may be infinitely small throughout are that the longitudinal displacements of all points be infinitely small, and that the transverse displacements of points at finite distances from the fixed point be either finite or infinitely small. And, in such a case, the distribution of small strain and stress throughout the wire may be assumed to be of the same form as in a beam of finite section under the same mechanical conditions.

Secondly, it is evident that the impressed forces (if any) on an *elementary* length of the wire are always negligible in comparison with the tensions or couples acting across its ends: the factors expressing the impressed force per unit mass and the stress per unit area being supposed finite (compare the method of § 144). The *form* in which stress is transmitted along the wire may therefore be assumed to be independent of the existence of impressed forces, the only effect of the latter being to cause the *magnitude* of the stress to vary from element to element. The same reasoning is of course applicable to the *effective forces*, if the wire be in motion. It should be observed that a transverse force, impressed or effective, of finite magnitude per unit mass, acting on a finite length of wire, will in general require the application of equilibrating transverse forces to its ends, in the form of tangential stress of finite magnitude per unit area. These will give rise to tangential or shearing stresses between consecutive elements of the wire, which may be taken into account by simple superposition.*

Thirdly, although the forces and couples found in § 357 to be transmitted along a beam of finite section depend upon a certain definite distribution of surface traction over the ends of the beam, yet, as the section is indefinitely diminished, we may take it for granted that the exact distribution of stress over it becomes of less and less importance, until finally we may assume that *any* equilibrating forces and couples, of the general type described in that article, applied to the ends of an infinitely fine wire, must distribute themselves over those ends in such a way as to transmit throughout the length of the wire forces and

* An elementary length of wire is a body all of whose dimensions are of the same order of magnitude. *All* the relative displacements of points included in such an element must be infinitely small, and the principle of superposition may therefore be safely employed.

couples of the form we have found necessary for equilibrium in the case of a beam of finite section.

Finally, the curvature of the infinitely small transverse sections, due to the strain, may be ignored, and they may be regarded as plane elements, everywhere cut perpendicularly in their centres of gravity by the Central Axis.

360.] **Approximate application of the foregoing considerations to Wires whose transverse dimensions, though finite, are very small in comparison with their length.** The considerations of the last article are rigorously true only of wires of infinitely small section, but they also apply in a lesser degree to the Wires defined in § 322, and the close approximation of the theoretical formulæ deduced from them with the results of experiment amply justify the application.

We now propose to consider the equilibrium and vibrations of such wires (which we shall in general assume, for simplicity, to be of equal flexibility in all directions), under forces and couples applied to its ends, together with any system of impressed forces throughout its mass. We shall assume

(*i.*) that the transverse sections are approximately small plane surfaces, cut perpendicularly in their centres of gravity by the Central Axis.

(*ii.*) that the normal component of the tension or thrust across any section is equal to $\mathfrak{e}\epsilon$ (§ 329), where ϵ is the elongation of the Central Axis at the point where it cuts the section, and $\mathfrak{e}$ the coefficient of extension at the point.

(*iii.*) that the couple between the two elements separated by any transverse section is in a plane perpendicular to the principal normal to the Central Axis at the corresponding point.

(*iv.*) that the component of this couple in the plane of the section, or about the tangent to the Central Axis, is $\mathfrak{t}\tau$ (§ 334), where τ is the rate of twist of the wire about the Axis, and $\mathfrak{t}$ the coefficient of torsion at the point.

(*v.*) that the component of this couple in the osculating plane of the Central Axis (*i.e.*, the *plane of flexion* at the point), or about its binormal, is $\mathfrak{p}\varpi$ (§§ 349, 353), where ϖ is the curvature of the Central Axis and $\mathfrak{p}$ the coefficient of flexion at the point.

If the wire be of uniform section throughout, the coefficients $\mathfrak{e}$, $\mathfrak{t}$, $\mathfrak{p}$ are of course constants.

Let the origin O be any fixed point of the Central Axis, and let s denote the length of an arc of the strained Axis, reaching from O to the point (x', y', z') initially at (x, y, z). Then, if $\mathfrak{A}$ be the small sectional area of the wire at the point (x', y', z'), the element of volume may be taken as $d\mathfrak{A}ds$.

To express the impressed and effective forces and couples as functions of s, we may proceed as follows:—

Let $\mathrm{x}' \equiv x' + \xi'$, $\mathrm{y}' \equiv y' + \eta'$, $\mathrm{z}' \equiv z' + \zeta'$ be the coördinates of any point in the section $\mathfrak{A}$ terminating the length s of the wire, so that ξ', η', ζ' are the coördinates of any point of the section, in its strained position, referred to axes through its centre of gravity, parallel to Ox, Oy, Oz.

If X, Y, Z be the components of the impressed force per unit mass at (x, y, z), and if

$$\iint X d\mathfrak{A} = \mathfrak{A}\mathfrak{X}, \text{ etc.,} \qquad (89)$$

then $\rho\mathfrak{A}\mathfrak{X}$, $\rho\mathfrak{A}\mathfrak{Y}$, $\rho\mathfrak{A}\mathfrak{Z}$ are the components of the impressed force per unit length of wire.

The components of the impressed couple per unit length about the axes of reference are

$$\iint \rho(\mathrm{y}'Z - \mathrm{z}'Y)d\mathfrak{A}, \text{ etc.,}$$

or

$$\iint \rho[(y' + \eta')Z - (z' + \zeta')Y]d\mathfrak{A}, \text{ etc.,}$$

or

$$\rho\mathfrak{A}[y'\mathfrak{Z} - z'\mathfrak{Y}] + \rho\iint(\eta'Z - \zeta'Y)d\mathfrak{A}, \text{ etc.,}$$

which we shall write $\rho\mathfrak{A}(y'\mathfrak{Z} - z'\mathfrak{Y} + \mathfrak{L})$, $\rho\mathfrak{A}(z'\mathfrak{X} - x'\mathfrak{Z} + \mathfrak{M})$, $\rho\mathfrak{A}(x'\mathfrak{Y} - y'\mathfrak{X} + \mathfrak{N})$, where obviously

$$\left.\begin{aligned} \iint(\eta'Z - \zeta'Y)d\mathfrak{A} &= \mathfrak{A}\mathfrak{L} \\ \iint(\zeta'X - \xi'Z)d\mathfrak{A} &= \mathfrak{A}\mathfrak{M} \\ \iint(\xi'Y - \eta'X)d\mathfrak{A} &= \mathfrak{A}\mathfrak{N} \end{aligned}\right\} \qquad (90)$$

The components of the effective force per unit length are

$$\iint \rho\ddot{\mathrm{x}}'d\mathfrak{A}, \text{ etc.} = \iint \rho(\ddot{x}' + \ddot{\xi}')d\mathfrak{A}, \text{ etc.,}$$

and these obviously reduce to $\rho\mathfrak{A}\ddot{x}'$, $\rho\mathfrak{A}\ddot{y}'$, $\rho\mathfrak{A}\ddot{z}'$.

Lastly, the effective couple about Ox,

$$\iint \rho(\mathrm{y}'\ddot{\mathrm{z}}' - \mathrm{z}'\ddot{\mathrm{y}}')d\mathfrak{A}$$

or

$$\iint \rho[(y' + \eta')(\ddot{z}' + \ddot{\zeta}') - (z' + \zeta')(\ddot{y}' + \ddot{\zeta}')]d\mathfrak{A}$$

reduces to

$$\rho\mathfrak{A}(y'\ddot{z}' - z'\ddot{y}') + \rho\iint(\eta'\ddot{\zeta}' - \zeta'\ddot{\eta}')d\mathfrak{A};$$

so that the component effective couples may be written $\rho\mathfrak{A}(y'\ddot{z}' - z'\ddot{y}' + \mathfrak{l})$, $\rho\mathfrak{A}(z'\ddot{x}' - x'\ddot{z}' + \mathfrak{m})$, $\rho\mathfrak{A}(x'\ddot{y}' - y'\ddot{x}' + \mathfrak{n})$, where

$$\left.\begin{aligned} \iint(\eta'\ddot{\zeta}' - \zeta'\ddot{\eta}')d\mathfrak{A} &= \mathfrak{l} \\ \iint(\zeta'\ddot{\xi}' - \xi'\ddot{\zeta}')d\mathfrak{A} &= \mathfrak{m} \\ \iint(\xi'\ddot{\eta}' - \eta'\ddot{\xi}')d\mathfrak{A} &= \mathfrak{n} \end{aligned}\right\} \qquad (91)$$

Now let us consider the equations of motion of the length s of the wire, extending from $(0, 0, 0)$ to (x', y', z'). Let A, B, C denote the components, parallel to the arbitrarily directed axes of reference, of the tension across the further end, and let the suffix 0 distinguish the values assumed by functions of s when $s=0$. Then, on resolving parallel to Ox, Oy, Oz, we have

$$A - A_0 + \rho\int_0^s \mathfrak{A}_1 \mathfrak{X}_1 ds_1 = \rho\int_0^s \mathfrak{A}_1 \ddot{x}_1' ds_1, \text{ etc.,}$$

or

$$\left.\begin{aligned} A + \rho\int_0^s \mathfrak{A}_1(\mathfrak{X}_1 - \ddot{x}_1')ds_1 &= A_0 \\ B + \rho\int_0^s \mathfrak{A}_1(\mathfrak{Y}_1 - \ddot{y}_1')ds_1 &= B_0 \\ C + \rho\int_0^s \mathfrak{A}_1(\mathfrak{Z}_1 - \ddot{z}_1')ds_1 &= C_0 \end{aligned}\right\} \quad\text{.....................} (92)$$

Again, the direction cosines of the tangent to the Central Axis (which is the axis of the torsion couple) are $\partial x'/\partial s$, $\partial y'/\partial s$, $\partial z'/\partial s$, so that the components of this couple about the axes of reference are

$$\mathfrak{t}\tau\frac{\partial x'}{\partial s},\ \mathfrak{t}\tau\frac{\partial y'}{\partial s},\ \mathfrak{t}\tau\frac{\partial z'}{\partial s}.$$

Similarly, the direction cosines of the binormal to the Central Axis (which is the axis of the flexion couple) being

$$\frac{1}{\varpi}\left(\frac{\partial y'}{\partial s}\frac{\partial^2 z'}{\partial s^2} - \frac{\partial z'}{\partial s}\frac{\partial^2 y'}{\partial s^2}\right), \text{ etc.,}$$

the components of this couple are

$$\mathfrak{p}\left(\frac{\partial y'}{\partial s}\frac{\partial^2 z'}{\partial s^2} - \frac{\partial z'}{\partial s}\frac{\partial^2 y'}{\partial s^2}\right), \text{ etc.}$$

Thus, taking moments about Ox, we have

$$\mathfrak{t}\tau\frac{\partial x'}{\partial s} + \mathfrak{p}\left(\frac{\partial y'}{\partial s}\frac{\partial^2 z'}{\partial s^2} - \frac{\partial z'}{\partial s}\frac{\partial^2 y'}{\partial s^2}\right) - \left[\mathfrak{t}\tau\frac{\partial x'}{\partial s}\right]_0 - \left[\mathfrak{p}\left(\frac{\partial y'}{\partial s}\frac{\partial^2 z'}{\partial s^2} - \frac{\partial z'}{\partial s}\frac{\partial^2 y'}{\partial s^2}\right)\right]_0$$

$$+ y'C - z'B + \rho\int_0^s \mathfrak{A}_1(y_1'\mathfrak{Z}_1 - z_1'\mathfrak{Y}_1 + \mathfrak{L}_1)ds_1 = \rho\int_0^s \mathfrak{A}_1(y_1'\ddot{z}_1' - z_1'\ddot{y}_1' + \mathfrak{l}_1)ds_1$$

or

$$\begin{aligned} &\mathfrak{t}\tau\frac{\partial x'}{\partial s} + \mathfrak{p}\left(\frac{\partial y'}{\partial s}\frac{\partial^2 z'}{\partial s^2} - \frac{\partial z'}{\partial s}\frac{\partial^2 y'}{\partial s^2}\right) + y'C - z'B \\ &+ \rho\int_0^s \mathfrak{A}_1[y_1'(\mathfrak{Z}_1 - \ddot{z}_1') - z_1'(\mathfrak{Y}_1 - \ddot{y}_1') + \mathfrak{L}_1 - \mathfrak{l}_1]ds_1 \\ &= \left[\mathfrak{t}\tau\frac{\partial x'}{\partial s} + \mathfrak{p}\left(\frac{\partial y'}{\partial s}\frac{\partial^2 z'}{\partial s^2} - \frac{\partial z'}{\partial s}\frac{\partial^2 y'}{\partial s^2}\right)\right]_0 \quad\text{.....................} (93) \end{aligned}$$

Now, if we multiply the second of equations (92) by z', and the third by y', and subtract one from the other, we obtain

$$y'C - z'B = C_0 y' - B_0 z' - \rho\int_0^s \mathfrak{A}_1[y'(\mathfrak{Z}_1 - \ddot{z}_1') - z'(\mathfrak{Y}_1 - \ddot{y}_1')]ds_1;$$

and on substituting* this result in (93) it becomes

$$\mathfrak{t}\tau\frac{\partial x'}{\partial s}+\mathfrak{p}\left(\frac{\partial y'}{\partial s}\,\frac{\partial^2 z'}{\partial s^2}-\frac{\partial z'}{\partial s}\,\frac{\partial^2 y'}{\partial s^2}\right)+C_0 y'-B_0 z'+\rho\int^s\mathfrak{A}_1(\mathfrak{L}_1-\mathfrak{l}_1)ds_1$$

$$+\rho\int_0^s\mathfrak{A}_1[(y_1'-y')(\mathfrak{Z}_1-\ddot{z}_1')-(z_1'-z')(\mathfrak{Y}_1-\ddot{y}_1')]ds_1$$

$$=\left[\mathfrak{t}\tau\frac{\partial x'}{\partial s}+\mathfrak{p}\left(\frac{\partial y'}{\partial s}\,\frac{\partial^2 z'}{\partial s^2}-\frac{\partial z'}{\partial s}\,\frac{\partial^2 y'}{\partial s^2}\right)\right]_0; \qquad \text{(94)}$$

and two similar equations may be obtained by taking moments in the remaining coördinate planes.

For practical purposes equations (92) and (94) are more readily available after being differentiated as to s; so that finally we write

$$\left.\begin{aligned}\frac{\partial A}{\partial s}+\rho\mathfrak{A}(\mathfrak{X}-\ddot{x}')=0\\ \frac{\partial B}{\partial s}+\rho\mathfrak{A}(\mathfrak{Y}-\ddot{y}')=0\\ \frac{\partial C}{\partial s}+\rho\mathfrak{A}(\mathfrak{Z}-\ddot{z}')=0\end{aligned}\right\}; \qquad \text{(95)}$$

and [on elimination of A_0, B_0, C_0 by means of (92) *after* differentiation of (94)]

$$\left.\begin{aligned}\frac{\partial}{\partial s}\left[\mathfrak{t}\tau\frac{\partial x'}{\partial s}+\mathfrak{p}\left(\frac{\partial y'}{\partial s}\,\frac{\partial^2 z'}{\partial s^2}-\frac{\partial z'}{\partial s}\,\frac{\partial^2 y'}{\partial s^2}\right)\right]+C\frac{\partial y'}{\partial s}-B\frac{\partial z'}{\partial s}\\ +\rho\mathfrak{A}(\mathfrak{L}-\mathfrak{l})=0\\ \frac{\partial}{\partial s}\left[\mathfrak{t}\tau\frac{\partial y'}{\partial s}+\mathfrak{p}\left(\frac{\partial z'}{\partial s}\,\frac{\partial^2 x'}{\partial s^2}-\frac{\partial x'}{\partial s}\,\frac{\partial^2 z'}{\partial s^2}\right)\right]+A\frac{\partial z'}{\partial s}-C\frac{\partial x'}{\partial s}\\ +\rho\mathfrak{A}(\mathfrak{M}-\mathfrak{m})=0\\ \frac{\partial}{\partial s}\left[\mathfrak{t}\tau\frac{\partial z'}{\partial s}+\mathfrak{p}\left(\frac{\partial x'}{\partial s}\,\frac{\partial^2 y'}{\partial s^2}-\frac{\partial y'}{\partial s}\,\frac{\partial^2 x'}{\partial s^2}\right)\right]+B\frac{\partial x'}{\partial s}-A\frac{\partial y'}{\partial s}\\ +\rho\mathfrak{A}(\mathfrak{N}-\mathfrak{n})=0\end{aligned}\right\} \qquad \text{(96)}$$

We also have, as the analytical expression of assumption (*ii.*) above,

$$A\frac{\partial x'}{\partial s}+B\frac{\partial y'}{\partial s}+C\frac{\partial z'}{\partial s}=\mathfrak{e}\epsilon \qquad \text{(97)}$$

The geometrical equations, expressing τ, $\mathfrak{l}$, $\mathfrak{m}$, $\mathfrak{n}$ as functions of x', y', z', s, are troublesome to obtain in the general case of finite curvature and twist; and as we shall only apply the dynamical equations involving them to cases in which the transverse displacements are small, we shall investigate later on the simple forms they assume under those circumstances.

* This transformation of course only amounts to shifting the point about which moments are taken from the origin to the farther end of the arc s.

The terminal conditions, to be satisfied at each end of the wire, are as follows:—

(*i.*) A, B, C must be equal to the components of the applied tension.

(*ii.*) $\mathfrak{t}\tau$ must be equal to the applied couple *in* the plane of the terminal section.

(*iii.*) $\mathfrak{p}\varpi$ must be equal to the applied couple in the plane *perpendicular to* the terminal section.

(*iv.*) the terminal values of

$$\frac{1}{\varpi}\left(\frac{\partial y'}{\partial s}\frac{\partial^2 z'}{\partial s^2}-\frac{\partial z'}{\partial s}\frac{\partial^2 y'}{\partial s^2}\right),\ \text{etc.},$$

must equal the direction cosines of the plane in which the couple (*iii.*) is applied.

(*v.*) $\mathfrak{e}\epsilon$ must equal the normal component of the applied tension.

It is evident that, if there be no torsion, and if the flexion be in one plane throughout, the equations of motion and terminal conditions will be equally applicable to a wire of any form, such that the principal axes of its sections lie throughout in two planes, one of which coincides with the plane of flexion. In such a case, we shall merely have to write $\mathfrak{p}_1$ or $\mathfrak{p}_2$ for $\mathfrak{p}$ (§§ 349, 350).

361.] **The "Linea Elastica" of James Bernoulli.** If we suppose a wire of uniform section to be held in equilibrium by equal opposing tensions applied to its ends, either directly, or by means of rigid arms in one plane with the unstrained axis and the lines in which the tensions act, it is evident that there will be no torsion, and that the flexion will be wholly in that plane.

Taking the line of action of the tensions as axis of z, and the plane of flexion as plane of zx, the equations of equilibrium reduce to

$$\left.\begin{aligned}&\frac{dA}{ds}=\frac{dB}{ds}=\frac{dC}{ds}=0\\&\frac{d}{ds}\left[\mathfrak{p}\left(\frac{dz'}{ds}\frac{d^2x'}{ds^2}-\frac{dx'}{ds}\frac{d^2z'}{ds^2}\right)+Az'-Cx'\right]=0\end{aligned}\right\}$$

with the terminal conditions $A=B=0$, $C=$ the tension C_0 applied to either end, and

$$\mathfrak{p}\varpi\equiv\mathfrak{p}\left(\frac{dz'}{ds}\frac{d^2x'}{ds^2}-\frac{dx'}{ds}\frac{d^2z'}{ds^2}\right)$$

$=$ the couple (if any) applied to either end by means of the rigid arms.

Thus $A=B=0$, and $C=C_0$, throughout the length of the wire, and the remaining equation of equilibrium is

$$\frac{d}{ds}(\mathfrak{p}\varpi - C_0 x') = 0.$$

If a be the length of either arm, we have $\mathfrak{p}\varpi = C_0 a$ when $x'=a$, so the constant of integration is zero, and

$$\mathfrak{p}\varpi = C_0 x',$$

or *the curvature at every point is numerically proportional to the distance from the line of tension.* Transforming the independent variable from s to z', this equation may be written

$$\frac{d^2x'}{dz'^2}\left[1+\left(\frac{dx'}{dz'}\right)^2\right]^{-\frac{3}{2}} = \frac{C_0 x'}{\mathfrak{p}},$$

and, on multiplying by $2dx'/dz'$ and integrating,

$$\left[1+\left(\frac{dx'}{dz'}\right)^2\right]^{-\frac{1}{2}} = \frac{C_0}{2\mathfrak{p}}(D^2 - x'^2),$$

or

$$z' = \int \frac{C_0(D^2 - x'^2)dx'}{\sqrt{4\mathfrak{p}^2 - C_0^2(D^2 - x'^2)^2}},$$

where D is an arbitrary parameter. This is then the equation of the curve into which the wire is strained; it is known as the *Linea Elastica.*

When $dx'/dz'=0$, $x' = \pm\sqrt{D^2 \pm 2\mathfrak{p}/C_0}$, so that if C_0 be taken to represent the *numerical magnitude** of the tension, the maximum distance of the curve from the line of tension is $\sqrt{D^2+2\mathfrak{p}/C_0}$, and the minimum distance (if any true minimum exists) is $\sqrt{D^2-2\mathfrak{p}/C_0}$. In the cases of Figures 55-58 D^2 must be therefore taken *less* than $2\mathfrak{p}/C_0$, and in the case of Figure 60 *greater*. Figure 59 represents the intermediate case, in which $D^2 = 2\mathfrak{p}/C_0$, where the equation of the curve reduces to the integrable from

$$z' = \int\left[\frac{2\mathfrak{p}}{x'\sqrt{4\mathfrak{p} - C_0 x'^2}} - \frac{C_0 x'}{\sqrt{4\mathfrak{p} - C_0 x'^2}}\right]\frac{dx'}{\sqrt{C_0}},$$

or, C_0 being necessarily positive in this case,

$$z' = \sqrt{\frac{4\mathfrak{p} - C_0 x'^2}{C_0}} - \sqrt{\frac{\mathfrak{p}}{C_0}}\log\frac{2\sqrt{\mathfrak{p}} + \sqrt{4\mathfrak{p} - C_0 x'^2}}{x'\sqrt{C_0}}.$$

Figures 55-60 are taken from Thomson and Tait's *Natural Philosophy:* they are copied from the actual forms assumed by flat springs of such small breadth that no appreciable tortuosity (and consequent torsion) was introduced by the crossing of the different branches.

* This is *positive* if the ends are pulled apart as in Figures 58, 59, 60, and *negative* if they are pulled towards one another as in Figures 55, 56, 57.

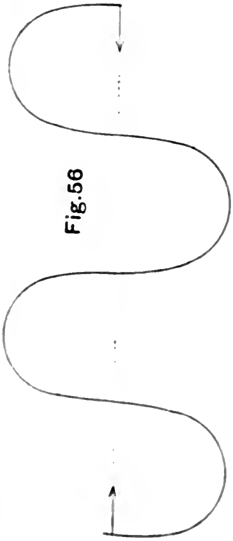


Fig.56

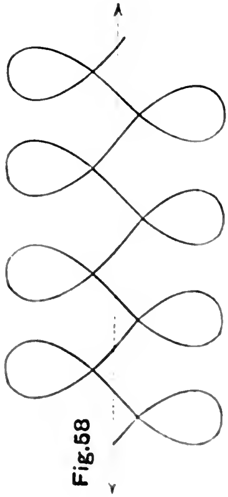


Fig.58

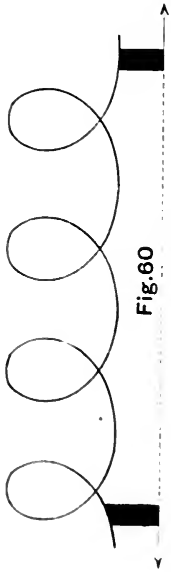


Fig.60

Fig.55

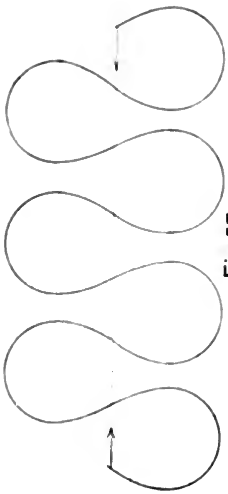


Fig.57

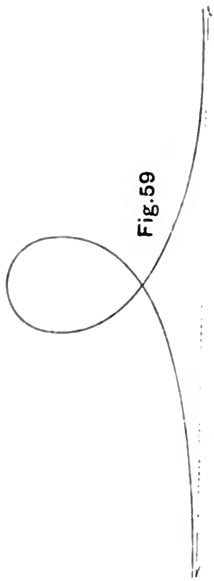


Fig.59

362.] **The Helix of Equilibrium of a Uniform Wire under no Impressed Force or Couple.** Writing in the general equations of equilibrium ϖ for the resultant curvature, (λ, μ, ν) for the direction cosines of the tangent, and (λ', μ', ν') for those of the binormal to the curve assumed by the Central Axis, under no impressed force or couple, they become

$$\left.\begin{aligned}
&\frac{dA}{ds} = \frac{dB}{ds} = \frac{dC}{ds} = 0\\
&\frac{d}{ds}[\mathfrak{t}\tau\lambda + \mathfrak{p}\varpi\lambda'] + C\mu - B\nu = 0\\
&\frac{d}{ds}[\mathfrak{t}\tau\mu + \mathfrak{p}\varpi\mu'] + A\nu - C\lambda = 0\\
&\frac{d}{ds}[\mathfrak{t}\tau\nu + \mathfrak{p}\varpi\nu'] + B\lambda - A\mu = 0\\
&\lambda A + \mu B + \nu C = \mathfrak{e}\epsilon
\end{aligned}\right\}.$$

Thus $A = A_0$, $B = B_0$, $C = C_0$ throughout the wire, or the tension is constant in magnitude and direction. Let $\mathbf{E}$ be its magnitude, and let us choose the arbitrary axes of reference so that Oz may be parallel to its direction. Then $C = \mathbf{E}$, $A = B = 0$, and the equations of equilibrium will be satisfied by the assumptions that ϵ, τ, ϖ are also constant, and that

$$\left.\begin{aligned}
&\nu = \mathfrak{e}\epsilon/\mathbf{E}\\
&\mathfrak{t}\tau\frac{d\lambda}{ds} + \mathfrak{p}\varpi\frac{d\lambda'}{ds} + \mathbf{E}\mu = 0\\
&\mathfrak{t}\tau\frac{d\mu}{ds} + \mathfrak{p}\varpi\frac{d\mu'}{ds} - \mathbf{E}\lambda = 0\\
&\frac{d\nu'}{ds} = 0
\end{aligned}\right\} \qquad (98)$$

Thus ν, ν' are constant and $\lambda\mu' - \lambda'\mu = 0$, or the tangent and binormal are inclined at constant angles to Oz, and the principal normal is everywhere perpendicular to Oz. The curvature being constant, it is obvious that the Central Axis of the wire assumes the form of a regular Helix, described upon a right circular cylinder having Oz for a generator. If r be the radius of this cylinder, and a the pitch of the helix, $\mathrm{r} = \cos^2 a/\varpi$, $\epsilon = \mathbf{E}\sin a/\mathfrak{e}$.

If we now transform the origin to a point in the axis of the cylinder, and choose Ox so that it shall pass through the end $z' = 0$ of the wire, and if ϕ denote the angle through which the arc s of the wire turns about Oz, we shall have

$$x' = \mathrm{r}\cos\phi, \quad y' = \mathrm{r}\sin\phi, \quad z' = \mathrm{r}\phi\tan a, \quad s = \mathrm{r}\phi\sec a,$$

and the second and third of equations (98) both reduce to

$$\mathfrak{p}\varpi\sin a - \mathfrak{t}\tau\cos a + \mathrm{r}\mathbf{E} = 0 \qquad (99)$$

When the magnitude of the couple applied to either end and the inclination of its axis to Oz are given, ϖ and τ can be determined, and (99) will then serve to determine α; whence finally r and ϵ can be found.

363.] **Equilibrium under Terminal Couples only.** Writing $\mathbf{E}=0$, we have $\epsilon=0$, and if $\mathbf{P}$, $\mathbf{T}$ be the flexion and torsion couples (99) reduces to $\mathbf{P}\sin\alpha-\mathbf{T}\cos\alpha=0$, while $\mathrm{r}=\mathfrak{p}\cos^2\alpha/\mathbf{P}$. Let the magnitude of the couple be $\mathbf{C}$, and let its axis in any possible position of equilibrium make an angle ψ with Oz, on the same side of it as the tangent to the curve. Then $\mathbf{P}=\mathbf{C}\cos(\alpha+\psi)$, $\mathbf{T}=\mathbf{C}\sin(\alpha+\psi)$, and by (99) $\sin\psi=0$. Consequently the axis of the helix is perpendicular to the planes of the terminal couples.

Thus if equal opposing couples in parallel planes be applied to the ends of a fine wire, it will take the form of a uniform helix described upon a cylinder with its axis perpendicular to the planes of the couples, the length l and radius r of the cylinder, the pitch α of the helix (or angle at which it cuts the planes of the couples) and the angle ϕ through which it turns about the axis, being connected by the equations

$$\mathrm{r}=\mathfrak{p}\cos\alpha/\mathbf{C},\quad \mathrm{l}=L\sin\alpha,\quad \phi=L\mathbf{C}/\mathfrak{p}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(100)$$

If only the magnitude $\mathbf{C}$ of the couples be given, there are an infinite number of possible positions of equilibrium, but if in addition one of the geometrical quantities l, r, or α be known, the solution is completely determinate. The curvature of the wire will be $\mathbf{C}\cos\alpha/\mathfrak{p}$ and its torsion $\mathbf{C}\sin\alpha/\mathfrak{p}$ throughout.

364.] **Simplified form of the equations, when the maximum curvature of the Central Axis is very small.** In this case the transverse displacements of all points of the Axis are small, and if Oz be taken to coincide with its unstrained direction, x' or u and y' or v may be regarded as small quantities, as well as $z'-z$ or w (§ 359). To the first order of small quantities, s is equal to z', and the operator $\partial/\partial s$ is equivalent to $\partial/\partial z$. Thus equations (93) may be written

$$\left.\begin{aligned}\frac{\partial A}{\partial z}+\rho\mathfrak{A}(\mathfrak{X}-\ddot{u})&=0\\ \frac{\partial B}{\partial z}+\rho\mathfrak{A}(\mathfrak{Y}-\ddot{v})&=0\\ \frac{\partial C}{\partial z}+\rho\mathfrak{A}(\mathfrak{Z}-\ddot{w})&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(101)$$

The elongation of the wire at any point will now of course be given by

$$\epsilon=\frac{\partial w}{\partial z},\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(102)$$

and the approximate values of τ, $\mathfrak{l}$, $\mathfrak{m}$, $\mathfrak{n}$ may be found as follows. The curvature being very small, the displacement of any point in the transverse section $\mathfrak{A}$, relative to its centre of gravity, will be very approximately perpendicular to Oz, and due entirely to twist about the Central Axis. If θ be the angular rotation of this section about the Central Axis due to torsion (§ 332), which we must in general suppose to vary from one section to another (§ 359) under the influence of impressed couple, and which is not necessarily very small, the amount of torsion, or rate of twist per unit length of wire, is evidently

$$\tau = \frac{\partial \theta}{\partial z} \qquad (103)$$

Let (ξ, η, z) be the initial coördinates of that point of the section $\mathfrak{A}$ whose strained coördinates we have denoted in § 360 by $(x'+\xi', y'+\eta', z'+\zeta')$; then we shall have very approximately

$$\xi' = \xi - \eta\theta, \quad \eta' = \eta + \xi\theta, \quad \zeta' = 0;$$
$$\ddot{\xi}' = -\xi\dot{\theta}^2 - \eta\ddot{\theta}, \quad \ddot{\eta}' = -\eta\dot{\theta}^2 + \xi\ddot{\theta}, \quad \ddot{\zeta}' = 0.$$

Substituting these values in (91), and remembering that every diameter of the section through its centre of gravity (§ 353) is a principal axis of inertia, we find (on the assumption that the angular velocity $\dot{\theta}$ of rotation of transverse sections about the Central Axis is small)

$$\mathfrak{l} = \mathfrak{m} = 0; \quad \mathfrak{A}\mathfrak{n} = \mathfrak{J}_3\ddot{\theta}. \qquad (104)$$

Equations (96) and (97) now reduce to

$$\left.\begin{aligned}
&\frac{\partial}{\partial z}\left[\mathfrak{t}\frac{\partial \theta}{\partial z}\frac{\partial u}{\partial z} - \mathfrak{p}\frac{\partial^2 v}{\partial z^2}\right] + C\frac{\partial v}{\partial z} - B\left(1 + \frac{\partial w}{\partial z}\right) + \rho\mathfrak{A}\mathfrak{L} = 0 \\
&\frac{\partial}{\partial z}\left[\mathfrak{t}\frac{\partial \theta}{\partial z}\frac{\partial v}{\partial z} + \mathfrak{p}\frac{\partial^2 u}{\partial z^2}\right] + A\left(1 + \frac{\partial w}{\partial z}\right) - C\frac{\partial u}{\partial z} + \rho\mathfrak{A}\mathfrak{M} = 0 \\
&\frac{\partial}{\partial z}\left[\mathfrak{t}\frac{\partial \theta}{\partial z}\left(1 + \frac{\partial w}{\partial z}\right)\right] + B\frac{\partial u}{\partial z} - A\frac{\partial v}{\partial z} + \rho(\mathfrak{A}\mathfrak{N} - \mathfrak{J}_3\ddot{\theta}) = 0
\end{aligned}\right\} \qquad (105)$$

and

$$A\frac{\partial u}{\partial z} + B\frac{\partial v}{\partial z} + C\left(1 + \frac{\partial w}{\partial z}\right) = \mathfrak{c}\frac{\partial w}{\partial z}. \qquad (106)$$

In applying the terminal conditions it is to be observed that the components of the flexion couple in the coördinate planes are

$$-\mathfrak{p}\frac{\partial^2 v}{\partial z^2}, \quad +\mathfrak{p}\frac{\partial^2 u}{\partial z^2}, \quad 0. \qquad (107)$$

Of course, if the wire be of uniform section, $\mathfrak{t}$, $\mathfrak{p}$, $\mathfrak{c}$, $\mathfrak{A}$ and $\mathfrak{J}_3$ are absolute constants.

Examples of Small Vibrations of Wires of Uniform Section under no Impressed Forces or Couples.

365.] **Free Longitudinal Vibrations.** Annulling all the impressed forces and couples, and all the displacements but w, we have

$$\left.\begin{aligned} &\frac{\partial A}{\partial z} = 0, \quad \frac{\partial B}{\partial z} = 0, \quad \frac{\partial C}{\partial z} - \rho \mathfrak{A} \ddot{w} = 0, \\ B\left(1 + \frac{\partial w}{\partial z}\right) = 0, \quad &A\left(1 + \frac{\partial w}{\partial z}\right) = 0, \quad C\left(1 + \frac{\partial w}{\partial z}\right) = \mathfrak{c}\frac{\partial w}{\partial z} \end{aligned}\right\}.$$

These equations are obviously satisfied (to the first order of small quantities) by $A = B = 0$, and

$$C = \mathfrak{c}\frac{\partial w}{\partial z}, \quad \frac{\partial^2 w}{\partial z^2} - \frac{\rho \mathfrak{A} \ddot{w}}{\mathfrak{c}} = 0.$$

The latter is the equation of vibration to be satisfied throughout the wire, and it is evident that the sole remaining conditions that the motion may be entirely free are that

$$\frac{\partial w}{\partial z} = 0, \text{ when } z = 0 \text{ and when } z = L.$$

Writing for w the form (34) of § 260, we find that w_i and w_i' satisfy the equation

$$\frac{d^2 w_i}{dz^2} + \frac{i^2 \rho \mathfrak{A}}{\mathfrak{c}} w_i = 0,$$

or (§ 329)

$$\frac{d^2 w_i}{dz^2} + \frac{i^2 \rho}{q} w_i = 0.$$

Applying the terminal conditions to the general solution of this equation we have finally

$$w = \Sigma M_i \cos\frac{i\pi z}{L} \sin\frac{i\pi}{L}\sqrt{\frac{q}{\rho}}(t - a_i)$$

where i is any integer, and M_i, a_i are arbitrary constants.

The velocity of transmission of sound along a wire is therefore $\sqrt{q/\rho}$. This result should be compared with the velocity $\sqrt{(m+n)/\rho}$, obtained in § 268, for transmission through an infinite medium. We shall write Ω_1 for $\sqrt{q/\rho}$.

366.] **Lateral Vibrations in a Fixed Plane.** The wire being equally flexible in all directions, we may take any plane

through the Central Axis, *e.g.* the plane of zx, as plane of vibration. Annulling therefore v, w, and θ, we have

$$\left.\begin{aligned} B=0,\quad \frac{\partial C}{\partial z}=0,\quad A\frac{\partial u}{\partial z}+C&=0\\ \frac{\partial A}{\partial z}-\rho\mathfrak{A}\ddot{u}&=0\\ \mathfrak{p}\frac{\partial^3 u}{\partial z^3}+A-C\frac{\partial u}{\partial z}&=0 \end{aligned}\right\}.$$

Eliminating C between the third and fifth of these equations, and neglecting the square of $\partial u/\partial z$, we obtain

$$\mathfrak{p}\frac{\partial^3 u}{\partial z^3}+A=0\,;$$

and, on differentiating as to z and eliminating A, the equation of motion reduces to

$$\mathfrak{p}\frac{\partial^4 u}{\partial z^4}+\rho\mathfrak{A}\ddot{u}=0.$$

The tension is, to our order of approximation, entirely transverse (*i.e.* due to shearing stress only), and its value is

$$A=-\frac{\partial^3 u}{\partial z^3},\qquad\qquad(108)$$

the flexion couple being

$$\mathfrak{p}\varpi=+\mathfrak{p}\frac{\partial^2 u}{\partial z^2}\qquad\qquad(109)$$

The equation satisfied by the amplitude u_i (§ 260) is

$$\mathfrak{p}\frac{d^4 u_i}{dz^4}-i^2\rho\mathfrak{A}u_i=0,$$

or, if we write

$$i=\frac{\mathrm{i}^2}{L^2}\sqrt{\frac{\mathfrak{p}}{\rho\mathfrak{A}}},$$

then

$$\frac{d^4 u_i}{dz^4}=\frac{\mathrm{i}^4}{L^4}u_i.\qquad\qquad(110)$$

The general solution of this equation is

$$u_i=M_i\sin\frac{\mathrm{i}z}{L}+M_i'\cos\frac{\mathrm{i}z}{L}+N_i\sinh\frac{\mathrm{i}z}{L}+N_i'\cosh\frac{\mathrm{i}z}{L}.\qquad(111)$$

where the four coefficients are arbitrary constants, to be determined by means of the terminal conditions at the two ends. These are as follows:—

(*i.*) at an end that is absolutely free, there can be neither force nor flexion couple, so that at such an end

$$\frac{\partial^2 u}{\partial z^2}=0,\quad \frac{\partial^3 u}{\partial z^3}=0 \qquad \text{.........................}(112)$$

(*ii.*) at an end that is fixed in position, but so that the terminal portion of the wire is free to change its direction (*e.g.* a hinge), the displacement is zero, and so is the flexion couple, and at such an end

$$u=0,\quad \frac{\partial^2 u}{\partial z^2}=0 \qquad \text{.........................}(113)$$

(*iii.*) at an end that is *clamped* so that the terminal portion of the wire is fixed *in direction,* the tangent to the Axis at its termination must coincide with its initial direction: at such an end therefore

$$u=0,\quad \frac{\partial u}{\partial z}=0 \qquad \text{.........................}(114)$$

(*iv.*) at an end carrying a rigid mass **M**, but otherwise free, the couple vanishes, while the transverse force must obviously be equal to the effective force on **M**. Thus at such an end

$$\frac{\partial^2 u}{\partial z^2}=0,\quad \mathfrak{p}\frac{\partial^3 u}{\partial z^3}=-\mathbf{M}\ddot{u} \qquad \text{.........................}(115)$$

For the applications of these conditions, and the different forms of the general solution to which they give rise, the student is referred to Lord Rayleigh's "Theory of Sound," Chapter VII., and to Donkin's "Acoustics," Chapter IX.

Deflection of Uniform Rods from the Horizontal, under Gravity.

367.] **General Equation of Equilibrium.** Let a thin rod or wire rest upon any number of rigid supports in one horizontal straight line. It is required to determine the small deflections of the rod from the horizontal, between the supports, caused by its own weight.

Take any point in the line of supports for origin, and that line for axis of z, and let Ox be directed vertically downwards. It is evident that the deflection will be entirely in the vertical plane of zx, and we have $X=\mathbf{g}$, $Y=Z=0$, so that $\mathfrak{X}=\mathbf{g}$, $\mathfrak{Y}=\mathfrak{Z}=0$, $\mathfrak{L}=\mathfrak{M}=\mathfrak{N}=0$. Thus the equations of equilibrium reduce to

$$\left.\begin{aligned} \frac{dA}{dz}+\rho\mathbf{g}\mathfrak{A}=0 \\ \mathfrak{p}\frac{d^3u}{dz^3}+A=0 \end{aligned}\right\},$$

and, on elimination of A,

$$\mathfrak{p}\frac{d^4u}{dz^4} = \rho g \mathfrak{A}.$$

Integrating this equation four times, we see that the curve assumed by each portion of the rod terminated by consecutive supports, or by a free end and a support, is represented by an equation of the form

$$u = \kappa_0 + \kappa_1 z + \kappa_2 z^2 + \kappa_3 z^3 + \rho g \mathfrak{A} z^4/24\mathfrak{p}, \qquad (116)$$

where κ_0, κ_1, κ_2, κ_3 are constants, different in general for each such portion.

To show that the solution is completely determinate, we will take the general case in which there are p supports at given distances apart, and the ends are either free or clamped. The rod will be divided into $p+1$ curves, each represented by an equation of the form (116), and there will consequently be $4p+4$ constants to be determined. Now,

(*i.*) the line of supports being the axis of z, the values of u, as deduced from the equation of any portion of the rod, must vanish at each of the supports which bound that portion ($2p$ equations),

(*ii.*) the curvature of the rod being necessarily continuous, the values of du/dz and d^2u/dz^2 at each support, as deduced from the equations of the curves on either side of it, are necessarily equal ($2p$ equations).

(*iii*) at either end, whether free or clamped (§ 366) two conditions must be satisfied (4 equations).

Thus, on the whole, we have exactly $4p+4$ equations of condition to determine the $4p+4$ constants involved.

The thrust on any support is equal to the difference in the values of A, immediately on either side of it.

368.] **Rod supported by one end only, that end being clamped in a horizontal position.** In this case, u and du/dz must vanish at the clamped end ($z=0$), while d^2u/dz^2 and d^3u/dz^3 vanish at the free end ($z=L$) (§ 366). Thus

$$\kappa_0 = \kappa_1 = 2\kappa_2 + 6\kappa_3 L + \rho g \mathfrak{A} L^2/2\mathfrak{p} = 6\kappa_3 + \rho g \mathfrak{A} L/\mathfrak{p} = 0,$$

and the curve assumed by the Axis of the rod is

$$u = \frac{\rho g \mathfrak{A} z^2}{24\mathfrak{p}}(z^2 - 4Lz + 6L^2), \qquad (117)$$

giving an extreme depression at the free end of $\rho g \mathfrak{A} L^4/8\mathfrak{p}$.

369.] **Rod freely supported at its middle point.** This case may be at once deduced from the last, for it is obvious from symmetry that the tangent to the Axis of the rod at its middle point will be horizontal, so that the geometrical conditions are the same as if that point were clamped. Taking the middle point for origin, and writing $\frac{1}{2}L$ for L in (117), we have for the equation of that half of the rod for which z is positive

$$u = \frac{\rho g \mathfrak{A} z^2}{48\mathfrak{p}}(2z^2 - 4Lz + 3L^2) \qquad \text{(118)}$$

The extreme depression of either free end is therefore $\rho g \mathfrak{A} L^4/128\mathfrak{p}$.

370.] **Rod supported (but not clamped) at its two ends.** Taking one end as origin, u and d^2u/dz^2 vanish when $z=0$ and when $z=L$, so that

$$\kappa_0 = \kappa_2 = \kappa_1 + \kappa_3 L^2 + \rho g \mathfrak{A} L^3/24\mathfrak{p} = \kappa_3 + \rho g \mathfrak{A} L/12\mathfrak{p} = 0.$$

Thus

$$u = \frac{\rho g \mathfrak{A} z}{24\mathfrak{p}}(z^3 - 2Lz^2 + L^3) \qquad \text{(119)}$$

and the extreme depression at the middle point is $5\rho g \mathfrak{A} L^4/384\mathfrak{p}$.

The ends of a rod supported by its middle point suffer therefore only $\frac{3}{5}$ of the depression experienced by the middle point of a rod supported by its ends.

Greenhill's Problems on Stability.

371.] **Method of Investigation.** In the three following problems we have to determine the conditions under which the natural straight form of a beam, in given circumstances, becomes unstable, In each case it is obvious that this instability will be caused by the undue increase of a geometrical quantity involved (in §§ 372 and 374 the length of the beam, and in § 373 the angular velocity of rotation). Professor Greenhill's method consists in supposing the limit to be passed, and a small deflection of the Central Axis from the straight line to have taken place, and in determining the *least* value of the critical quantity for which this deflection can be maintained. This, being the limiting value which divides the two states, is obviously also the *greatest* value consistent with stability in the original form.

372.] **The maximum height of a vertical pole, consistent with stability under gravity.*** Let a pole of length L, in the form of any solid of revolution about a vertical axis,

* Mathematical Tripos, 1879, and *Proceedings* of Camb. Phil. Society, vol. iv, page 65.

with its base rigidly fixed in a vertical direction, be supposed slightly deflected from its naturally straight form. Taking the highest point of the axis for origin, we have $X = Y = 0$, $Z = \mathrm{g}$, and consequently $\mathfrak{X} = \mathfrak{Y} = 0$, $\mathfrak{Z} = \mathrm{g}$, $\mathfrak{L} = \mathfrak{M} = \mathfrak{N} = 0$. If the plane of xz coincide with the plane of flexion, equations (101) and (105) reduce to

$$\left.\begin{aligned} \frac{dC}{dz} + \rho\mathrm{g}\mathfrak{A} = 0 \\ \frac{d}{dz}\left(\mathfrak{p}\frac{d^2u}{dz^2}\right) - C\frac{du}{dz} = 0 \end{aligned}\right\},$$

and, when C has been eliminated, the equation of equilibrium becomes

$$\frac{d}{dz}\left(\mathfrak{p}\frac{d^2u}{dz^2}\right) + \rho\mathrm{g}\frac{du}{dz}\int_0^z \mathfrak{A}_1 dz_1 = 0.$$

If r denote the radius of the section $\mathfrak{A}$ at distance z from the summit, $\mathfrak{A} = \pi r^2$, and $\mathfrak{p} = \frac{1}{4}\pi q r^4$ (§§ 349, 353), so that

$$\frac{d}{dz}\left(r^4\frac{d^2u}{dz^2}\right) + \frac{4\rho\mathrm{g}}{q}\frac{du}{dz}\int_0^z r^2 dz = 0. \qquad \text{(120)}$$

When r varies as any given power of z, the solution may be obtained in terms of Bessel's functions, for on putting $r = z^p/D^{p-1}$ our equation reduces to

$$z^2\frac{d^2}{dz^2}\left(\frac{du}{dz}\right) + 4pz\frac{d}{dz}\left(\frac{du}{dz}\right) + \frac{4\rho\mathrm{g}D^{2p-2}}{(2p+1)qz^{2p-3}}\frac{du}{dz} = 0,$$

the solution of which—satisfying the terminal condition that the curvature d^2u/dz^2 vanishes at the free end $z = 0$—is

$$\frac{du}{dz} = \kappa z^{\frac{1}{2}-2p} \mathrm{J}_{\frac{4p-1}{3-2p}}\left[\frac{4D^{p-1}}{(2p-3)z^{p-\frac{3}{2}}}\sqrt{\frac{\rho\mathrm{g}}{(2p+1)q}}\right],$$

where κ is an arbitrary constant.

Since the base of the pole is rigidly fixed, we must have $du/dz = 0$, when $z = L$, and consequently

$$\mathrm{J}_{\frac{4p-1}{3-2p}}\left[\frac{4D^{p-1}}{(2p-3)L^{(2p-3)/2}}\sqrt{\frac{\rho\mathrm{g}}{(2p+1)q}}\right] = 0.$$

Thus, if i be the *least* positive root of the equation

$$\mathrm{J}_{\frac{4p-1}{3-2p}}(\mathrm{i}) = 0, \qquad \text{(121)}$$

and if

$$L_0 = \left[\frac{16\rho\mathrm{g}D^{2p-2}}{(2p+1)(2p-3)^2 q\mathrm{i}^2}\right]^{\frac{1}{2p-3}}, \qquad \text{(122)}$$

then if $L < L_0$ it is impossible for the pole to maintain a slightly curved form as a stable form of equilibrium, and consequently the original straight form is stable. L_0 is therefore the critical height required.

Example.—If the pole be cylindrical, $p=0$, and D is the radius of the base. In this case the critical height is given by

$$L_0 = \left(\frac{9qD^2\mathrm{i}^2}{16\rho g}\right)^{\frac{1}{3}},$$

where i is the least positive root of

$$J_{-\frac{1}{3}}(\mathrm{i}) = 0.$$

373.] **A cylindrical shaft, of equal flexibility in all directions, rotates without torsion between fixed bearings (clamps); required the greatest angular velocity of rotation consistent with the stability of the natural staight form.*** Let us suppose that, when the angular velocity is ω, the shaft rotates steadily with the axis slightly curved. Then $\ddot{u} = -\omega^2 u$, $\ddot{v} = -\omega^2 v$, and the equations of motion become

$$\left.\begin{aligned} &\frac{dA}{dz} + \rho\mathfrak{A}\omega^2 u = \frac{dB}{dz} + \rho\mathfrak{A}\omega^2 v = 0 \\ &\mathfrak{p}\frac{d^3u}{dz^3} + A = \mathfrak{p}\frac{d^3v}{dz^3} + B = A\frac{dv}{dz} - B\frac{du}{dz} = 0 \end{aligned}\right\}.$$

First of all, by eliminating A and B between the last three equations, we obtain

$$\frac{du}{dz}\frac{d^3v}{dz^3} - \frac{dv}{dz}\frac{d^3u}{dz^3} = 0,$$

or

$$\frac{du}{dz}\frac{d^2v}{dz^2} - \frac{dv}{dz}\frac{d^2u}{dz^2} = \text{constant}:$$

and, since du/dz and dv/dz both vanish at the bearings, this constant is zero.

Thus the component curvature of the Axis in the plane perpendicular to Oz is everywhere zero, or the form assumed by the strained Axis is a plane curve. Writing then $r=(u^2+v^2)^{\frac{1}{2}}$ for the radial displacement, we have

$$\mathfrak{p}\frac{d^4r}{dz^4} - \rho\mathfrak{A}\omega^2 r = 0,$$

the solution of which (compare § 366) is

$$r = M\cosh\frac{\mathrm{i}z}{L} + M'\sinh\frac{\mathrm{i}z}{L} + N\cos\frac{\mathrm{i}z}{L} + N'\sin\frac{\mathrm{i}z}{L},$$

where

$$\mathrm{i}^4 = \rho\mathfrak{A}L^4\omega^2/\mathfrak{p}.$$

* Mathematical Tripos, 1878.

Since r and dr/dz vanish at both bearings ($z=0$ and $z=L$), the coefficients must satisfy the relations

$$M+N=M'+N'=M\cosh \mathrm{i}+M'\sinh \mathrm{i}+N\cos \mathrm{i}+N'\sin \mathrm{i}$$
$$=M\sinh \mathrm{i}+M'\cosh \mathrm{i}-N\sin \mathrm{i}+N'\cos \mathrm{i}=0,$$

and consequently on elimination of M, M', N, N',

$$\cos \mathrm{i} \,.\, \cosh \mathrm{i}=1 \qquad \text{.....(123)}$$

If therefore i be the *least* positive root of (123), the critical angular velocity is given by

$$\omega_0=\frac{\mathrm{i}^2}{L^2}\sqrt{\frac{\mathfrak{p}}{\rho\mathfrak{A}}}\,; \qquad \text{.....(124)}$$

for if $\omega<\omega_0$ steady motion is impossible with the Central Axis curved, and the straight form is consequently stable.

374.] **A cylindrical shaft, of equal flexibility in all directions, rotates between bearings under twisting couple and longitudinal thrust; required the greatest length of the shaft consistent with stable motion with the Central Axis straight.*** Let $\mathbf{T}$ be the twisting couple, $-\mathbf{E}$ the longitudinal thrust, and ω the angular velocity of rotation about the line of bearings. Assuming the Central Axis to be slightly deflected from its naturally straight form, the equations of motion are when the motion is steady

$$\left.\begin{aligned}
\frac{dA}{dz}+\rho\mathfrak{A}\omega^2u=\frac{dB}{dz}+\rho\mathfrak{A}\omega^2v=\frac{dC}{dz}=0\,;\\
\frac{d}{dz}\left[\mathfrak{t}\frac{d\theta}{dz}\frac{du}{dz}-\mathfrak{p}\frac{d^2v}{dz^2}\right]+C\frac{dv}{dz}-B\left(1+\frac{dw}{dz}\right)=0,\\
\frac{d}{dz}\left[\mathfrak{t}\frac{d\theta}{dz}\frac{dv}{dz}+\mathfrak{p}\frac{d^2u}{dz^2}\right]+A\left(1+\frac{dw}{dz}\right)-C\frac{du}{dz}=0,\\
\frac{d}{dz}\left[\mathfrak{t}\frac{d\theta}{dz}\left(1+\frac{dw}{dz}\right)\right]+B\frac{du}{dz}-A\frac{dv}{dz}=0,\\
A\frac{du}{dz}+B\frac{dv}{dz}+C\left(1+\frac{dw}{dz}\right)=\mathfrak{e}\frac{dw}{dz}.
\end{aligned}\right\},$$

while at either end

$$\mathfrak{t}\frac{d\theta}{dz}=\mathbf{T},\quad C=\mathbf{E}.$$

These equations are satisfied by assuming that

$$\frac{d\theta}{dz}=\frac{\mathbf{T}}{\mathfrak{t}},\quad C=\mathbf{E}=\mathfrak{e}\frac{dw}{dz} \qquad \text{.....(125)}$$

* Mathematical Tripos, 1881, and *Proceedings* of the Institution of Mechanical Engineers, April, 1883.

throughout the shaft, if

$$\left.\begin{array}{l} \mathfrak{P}\dfrac{d^4u}{dz^4} + \mathrm{T}\dfrac{d^3v}{dz^3} - \mathrm{E}\dfrac{d^2u}{dz^2} - \rho\mathfrak{A}\omega^2 u = 0 \\ \mathfrak{P}\dfrac{d^4v}{dz^4} - \mathrm{T}\dfrac{d^3u}{dz^3} - \mathrm{E}\dfrac{d^2v}{dz^2} - \rho\mathfrak{A}\omega^2 v = 0 \end{array}\right\}.$$

A possible solution is

$$\left.\begin{array}{l} u = \kappa_1 \cos iz + \kappa_2 \cos jz + \kappa_3 \cos \alpha z \cosh \beta z + \kappa_4 \sin \alpha z \sinh \beta z \\ v = -\kappa_1 \sin iz - \kappa_2 \sin jz - \kappa_3 \sin \alpha z \cosh \beta z - \kappa_4 \cos \alpha z \sinh \beta z \end{array}\right\} \quad \ldots(126)$$

where i, j are the real roots, and $\alpha \pm \iota\beta$ are the imaginary roots of the quartic equation

$$\mathfrak{P}\lambda^4 - \mathrm{T}\lambda^3 + \mathrm{E}\lambda^2 - \rho\mathfrak{A}\omega^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(127)$$

375.] The most general case is complicated to work out, but two simple cases may be solved. If we suppose that the ends of the shaft are absolutely fixed in position, but able to change their directions *by exerting couple on the bearings* the only terminal condition *necessarily* satisfied is that $u = v = 0$ at each end. Similarly, if the ends be clamped rigidly in their initial direction, but able to *force the bearings aside* from their initial line the only *necessary* terminal condition is that $du/dz = dv/dz = 0$ at each end.

In either of these cases, we need only retain the completely periodic terms in (126), the solution being in the first case of the form

$$\left.\begin{array}{l} u = \kappa(\cos iz - \cos jz) \\ v = -\kappa(\sin iz - \sin jz) \end{array}\right\}$$

and in the second case of the form

$$\left.\begin{array}{l} u = \kappa\left(\dfrac{\cos iz}{i} - \dfrac{\cos jz}{j}\right) \\ v = -\kappa\left(\dfrac{\sin iz}{i} - \dfrac{\sin jz}{j}\right) \end{array}\right\},$$

with the condition in each case that

$$(i \sim j)\mathrm{L} = 2p\pi,$$

p being any positive integer.

The critical length in these cases is therefore given by

$$\mathrm{L}_0 = 2\pi/(i \sim j) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(128)$$

376.] In the case where the angular velocity of rotation is moderate, and the thrust and torsion couple very great, so that the effects of inertia are negligible in comparison, the quartic (127) reduces to the quadratic

$$\mathfrak{P}\lambda^2 - \mathrm{T}\lambda + \mathrm{E} = 0,$$

the roots of which, $\mathbf{E}$ being essentially negative, are real. Thus

$$i \sim j = \sqrt{\mathrm{T}^2 - 4\mathfrak{p}\mathrm{E}}/\mathfrak{p},$$

and the critical length L_0 is given by

$$\frac{\pi^2}{\mathrm{L}_0^{\,2}} = \frac{\mathrm{T}^2}{4\mathfrak{p}^2} - \frac{\mathrm{E}}{\mathfrak{p}} \qquad\qquad (129)$$

This solution is very approximately applicable to the screw shafts of large steamers, and *accurately* true in the case of *equilibrium* under thrust and twisting couple.

Naturally Curved Wires. Circular Hoops.

377.] **Equations of Motion and Terminal Conditions.** If a wire of infinitely small section have for its Central Axis a curve of any form, but everywhere of finite curvature, an elementary length of the wire can always be measured from any transverse section, such that its length is of *at least* the same order of dimensions as its greatest diameter, and yet so small that the portion is practically straight. The conditions of strain and stress in such an element may be taken to be the same as in a naturally straight beam, and by superposing one such element upon another until the curvature of their aggregate becomes sensible, it will appear that the conditions of strain in a wire of naturally finite curvature may be deduced from those of a naturally straight wire simply by substituting for "curvature due to strain" "*change of curvature due to strain*," and for "direction of the unstrained Central Axis" "*direction of the tangent to the unstrained Central Axis at any point.*"

With these changes the considerations $(i-v)$ on page 425 are applicable to naturally curved wires of equal flexibility in all directions, as are the terminal conditions on page 429, with the exception of (iv), ϖ denoting the *change of curvature* due to strain. Equations (95) and (97) also retain the same form, but equations (96) and the terminal condition (iv) become more complicated, owing to the form of the flexion couples. If

$$\lambda = \frac{\partial y}{\partial s}\frac{d^2 z}{ds^2} - \frac{\partial z}{\partial s}\frac{\partial^2 y}{\partial s^2}, \text{ etc.,} \quad \lambda' = \frac{\partial y'}{\partial s}\frac{\partial^2 z'}{\partial s^2} - \frac{\partial z'}{\partial s}\frac{\partial^2 y'}{\partial s^2}, \text{ etc.,}$$

then the natural curvature at the point (x, y, z) is approximately $\sqrt{\lambda^2+\mu^2+\nu^2}$, and the altered curvature at the corresponding point (x', y', z') is strictly $\sqrt{\lambda'^2+\mu'^2+\nu'^2}$: thus the resultant flexion couple is $\mathfrak{p}(\sqrt{\lambda'^2+\mu'^2+\nu'^2}-\sqrt{\lambda^2+\mu^2+\nu^2})$, and the direction cosines of the osculating plane of the strained Axis, in which this couple acts, are as before $\lambda'/\sqrt{\lambda'^2+\mu'^2+\nu'^2}$. The components

of the flexion couples, parallel to the arbitrary coördinate planes, are now therefore

$$\frac{\mathfrak{p}\lambda'(\sqrt{\lambda'^2+\mu'^2+\nu'^2}-\sqrt{\lambda^2+\mu^2+\nu^2})}{\sqrt{\lambda'^2+\mu'^2+\nu'^2}}, \text{ etc.,}$$

instead of simply $\mathfrak{p}\lambda'$, etc.

378.] **The Energy Method.** Owing to this complication of the ordinary equations of motion and equilibrium, problems on curved wires and hoops are generally attacked by means of the Energy Method.

Let the natural form of a uniform wire of unequal flexibilities be such that the curvature at any point of the Central Axis is $\bar{\varpi}$, the osculating plane at that point making an angle α with that principal plane of inertia in which the coefficient of flexion is $\mathfrak{p}_1$; and let the component curvatures in the principal planes be $\bar{\varpi}_1$, and $\bar{\varpi}_2$, so that $\bar{\varpi}_1 = \bar{\varpi}\cos\alpha$, $\bar{\varpi}_2 = \bar{\varpi}\sin\alpha$. Then, if the effect of the strain be to change $\bar{\varpi}$, $\bar{\varpi}_1$, $\bar{\varpi}_2$, α to ϖ, ϖ_1, ϖ_2, ϕ, and to produce at the same time extension ϵ and torsion τ, the potential energy *per unit length* of wire at the point in question will be, by § 358,

$$\mathfrak{W} = \tfrac{1}{2}\{\mathfrak{e}\epsilon^2 + \mathfrak{p}_1(\varpi_1 - \bar{\varpi}_1)^2 + \mathfrak{p}_2(\varpi_2 - \bar{\varpi}_2)^2 + \mathfrak{t}\tau^2\} \quad\ldots\ldots\ldots(130)$$

where $\qquad \varpi_1 = \varpi\cos\phi,\ \varpi_2 = \varpi\sin\phi.$

The tension, torsion couple and flexion couples will be with our previous notation

$$\left.\begin{aligned} \mathbf{E} &= \mathfrak{e}\epsilon = \frac{\partial\mathfrak{W}}{\partial\epsilon} \\ \mathbf{T} &= \mathfrak{t}\tau = \frac{\partial\mathfrak{W}}{\partial\tau} \\ \mathbf{P}_1 &= \mathfrak{p}_1(\varpi_1 - \bar{\varpi}_1) = \frac{\partial\mathfrak{W}}{\partial\varpi_1} \\ \mathbf{P}_2 &= \mathfrak{p}_2(\varpi_2 - \bar{\varpi}_2) = \frac{\partial\mathfrak{W}}{\partial\varpi_2} \end{aligned}\right\} \quad\ldots\ldots\ldots(131)$$

If the wire be of equal flexibility in all directions, the potential energy per unit length is

$$\mathfrak{W} = \tfrac{1}{2}\{\mathfrak{e}\epsilon^2 + \mathfrak{p}(\varpi - \bar{\varpi})^2 + \mathfrak{t}\tau^2\} \quad\ldots\ldots\ldots(132)$$

and the resultant flexion couple in the final osculating plane at any point is

$$\mathbf{P} = \mathfrak{p}(\varpi - \bar{\varpi}) \quad\ldots\ldots\ldots(133)$$

If ξ be any one of a system of coördinates defining the configuration of the wire in any state of strain, the resistance per unit length offered by an elementary portion of the wire to the

increase of ξ is of course $\partial\mathfrak{W}/\partial\xi$. In other words, there is a force $\partial\mathfrak{W}/\partial\xi$ if ξ is linear, or a couple $\partial\mathfrak{W}/\partial\xi$ if ξ is angular, per unit length, on each element of wire, tending to diminish ξ. From (130) and (131) we easily deduce that this action may be expressed in the form

$$\frac{\partial\mathfrak{W}}{\partial\xi}=\mathbf{E}\frac{\partial\epsilon}{\partial\xi}+\mathbf{T}\frac{\partial\tau}{\partial\xi}+\mathbf{P}_1\frac{\partial\varpi_1}{\partial\xi}+\mathbf{P}_2\frac{\partial\varpi_2}{\partial\xi}\dots\dots\dots\dots\dots(134)$$

379.] **Rotation of a wire of equal flexibility about its Central Axis.** The formula (133) shows that the energy of such a wire, whatever its natural form, depends only upon extension, torsion, and change of *resultant* curvature. If therefore the wire be set in motion in such a way that each point describes a circle about the centre of gravity of the normal transverse section in which it lies, no resistance will be offered to the motion except that due to inertia. We have thus an ideal means of transferring rotatory motion without loss of energy from one rigid axis to another in any other direction, by connecting them to the terminals of a perfectly elastic wire of uniform flexibility, so placed that in its natural form the tangents to the Central Axis at either end coincide exactly with the axes of rotation.

This result does not apply to curved wires of unequal flexibilities, because, even if the resultant curvature be maintained constant, the component curvatures in the principal planes of inertia at each point must change *periodically* during each rotation. (See § 382.)

Applications of the Energy Method.

380.] **Stretching of a uniform circular hoop of equal flexibilities in all directions.** If the Central Axis of a uniform wire be in its natural state a circle of radius r, and if the wire be stretched, without change of the circular form and without torsion, until this radius is increased to r, the length of the wire will be increased from $2\pi\mathrm{r}$ to $2\pi r$, and its curvature diminished from $1/\mathrm{r}$ to $1/r$. Thus we shall have $\epsilon=(r-\mathrm{r})/\mathrm{r}$, and

$$\mathfrak{W}=\tfrac{1}{2}\left\{\mathfrak{e}\left(\frac{r-\mathrm{r}}{\mathrm{r}}\right)^2+\mathfrak{p}\left(\frac{1}{\mathrm{r}}-\frac{1}{r}\right)^2\right\}$$
$$=\frac{(r-\mathrm{r})^2(\mathfrak{e}r^2+\mathfrak{p})}{2\mathrm{r}^2r^2}.$$

The tension will be $\mathfrak{e}(r-\mathrm{r})/\mathrm{r}$, and the flexion couple in the plane of the wire exerted across each transverse section will be $\mathfrak{p}(r-\mathrm{r})/r\mathrm{r}$.

Since the strain is expressed entirely in terms of the linear coördinate r, the resultant action on each element of the wire is a force towards the centre, of magnitude

$$\frac{d\mathfrak{W}}{dr} = \frac{(r-\mathrm{r})(\mathfrak{c}r^3+\mathfrak{p}\mathrm{r})}{\mathrm{r}^2 r^3}$$

per unit length.

381.] **Small radial vibrations in the plane of the hoop.** Let us now suppose that the wire performs small vibrations about its natural configuration, the displacement of each point being wholly radial, and the form of the Axis always circular. The velocity of each point will be $\dot{r}$, and the kinetic energy per unit length will be $\frac{1}{2}\rho\mathfrak{A}\dot{r}^2$. The principle of conservation of energy therefore gives us

$$2\pi r\left\{\tfrac{1}{2}\rho\mathfrak{A}\dot{r}^2+\frac{(r-\mathrm{r})^2(\mathfrak{c}r^2+\mathfrak{p})}{2\mathrm{r}^2r^2}\right\} = \text{constant}.$$

Differentiating as to t, and writing $r=\mathrm{r}+u$, where u is a small quantity of the first order compared with r, the equation of motion reduces to

$$\ddot{u}+u\frac{\mathfrak{c}\mathrm{r}^2+\mathfrak{p}}{\rho\mathfrak{A}\mathrm{r}^4}=0,$$

o that the periodic time of small vibrations is

$$2\pi\sqrt{\rho\mathfrak{A}\mathrm{r}^4/(\mathfrak{c}\mathrm{r}^2+\mathfrak{p})}.$$

382.] **Wire hoop of unequal flexibilities.** A hoop has aturally the form of a circle of radius r, the plane of greatest lexibility (*i.e.*, that in which the coefficient of flexion is least) at ach point making an angle α with the plane of the circle. It is tretched into a circle of radius r and held so that the plane of reatest flexibility makes everywhere an angle ϕ with the plane f the circle.

Here, if $\mathfrak{p}_1$ be the least and $\mathfrak{p}_2$ the greatest coefficient of flexion, e have with the notation of § 378,

$$\bar{\varpi}_1=\cos\alpha/\mathrm{r},\quad \bar{\varpi}_2=\sin\alpha/\mathrm{r},\quad \varpi_1=\cos\phi/r,\quad \varpi_2=\sin\phi/r,$$

nd $\epsilon=(r-\mathrm{r})/\mathrm{r}$.

Thus

$$\mathfrak{W}=\tfrac{1}{2}\left\{\mathfrak{c}\left(\frac{r-\mathrm{r}}{\mathrm{r}}\right)^2+\mathfrak{p}_1\left(\frac{\cos\phi}{r}-\frac{\cos\alpha}{\mathrm{r}}\right)^2+\mathfrak{p}_2\left(\frac{\sin\phi}{r}-\frac{\sin\alpha}{\mathrm{r}}\right)^2\right\},$$

d the action on each element of the wire consists of

(*i.*) a radial force

$$\frac{\mathfrak{c}(r-\mathrm{r})}{\mathrm{r}^2}+\frac{\mathfrak{p}_1}{r^2}\left(\frac{\cos\alpha}{\mathrm{r}}-\frac{\cos\phi}{r}\right)+\frac{\mathfrak{p}_2}{r^2}\left(\frac{\sin\alpha}{\mathrm{r}}-\frac{\sin\phi}{r}\right)\text{..........}(135)$$

er unit length, tending to restore the hoop to its natural radius, and

(*ii.*) a couple

$$\frac{\mathfrak{p}_1 \sin\phi}{r}\left(\frac{\cos\alpha}{\mathrm{r}}-\frac{\cos\phi}{r}\right)-\frac{\mathfrak{p}_2\cos\phi}{r}\left(\frac{\sin\alpha}{\mathrm{r}}-\frac{\sin\phi}{r}\right)\text{..........}(136)$$

per unit length, in the normal plane of the wire at each point, tending to turn it about its Central Axis, and restore to the angle ϕ its initial value α. This couple vanishes when

$$r=\frac{\mathrm{r}}{2}\cdot\frac{(\mathfrak{p}_2-\mathfrak{p}_1)\sin 2\phi}{\mathfrak{p}_2\cos\phi\sin\alpha-\mathfrak{p}_1\sin\phi\cos\alpha},\text{..................}(137)$$

so that, for every value of ϕ less than $\tan^{-1}(\mathfrak{p}_2\tan\alpha/\mathfrak{p}_1)$, it is possible to stretch the hoop so that its elements may experience only radial force. For *every* assigned value of r it is possible to determine ϕ so that the turning couple may vanish. If the external action be confined to keeping the hoop stretched, it will assume a form in which ϕ has one of the *four* values given by (137) when the value of r is assigned. The four positions of equilibrium thus determined are alternately stable and unstable.

If in the natural form of the hoop the plane of greatest flexibility at every point coincides with the plane of the hoop, and if the latter retain its natural radius, $\alpha=0$ and $r=\mathrm{r}$. Thus the turning couple is

$$\frac{\sin\phi}{\mathrm{r}^2}[\mathfrak{p}_1+(\mathfrak{p}_2-\mathfrak{p}_1)\cos\phi],$$

and the four positions of equilibrium are given by

$$\phi=0,\quad \phi=\pi,\quad\text{and}\quad \phi=\pi\pm\cos^{-1}[\mathfrak{p}_1/(\mathfrak{p}_2-\mathfrak{p}_1)],$$

the latter two of which are only possible when $\mathfrak{p}_2>2\mathfrak{p}_1$. The first value of ϕ represents the natural state, and the second the condition of the wire after each section has been turned through two right angles about the Central Axis—or till the plane of greatest flexibility once more coincides with the plane of the hoop.

383.] **Hoop having one very great coefficient of flexion.** If $\mathfrak{p}_2$ be enormously great in comparison with $\mathfrak{p}_1$, as in the case of an ordinary barrel hoop, the first term in (136) may be neglected in comparison with the second, and the position of equilibrium when stretched is given by $\phi=\sin^{-1}(r\sin\alpha/\mathrm{r})$; or, in other words, the change of the component curvature in the plane of small flexibility is negligible in comparison with that of the other component. The flexion couple in the plane of greatest flexibility will be

$$\mathfrak{p}_1\left(\frac{\cos\alpha}{\mathrm{r}}\sim\frac{\cos\phi}{r}\right),$$

and, since from symmetry the resultant flexion couple must be in the plane of the hoop, this latter will be

$$\frac{\mathfrak{p}_1}{\cos\phi}\left(\frac{\cos\alpha}{\mathrm{r}}\sim\frac{\cos\phi}{r}\right).$$

EXAMPLES.

[*In the following Examples all wires—unless the contrary is expressly stated—are to be supposed uniform, naturally straight, of equal flexibility in all directions, and free from the action of any impressed force or couple, except at their ends or "points of support."*]

1. Opposing couples of 10^6 centimetre-grammes are applied to the two ends of a round bar of iron (Young's modulus about 1900 million grammes weight per square centimetre) of $2\frac{1}{2}$ centimetres diameter. Find the curvature produced. Also the greatest curvature possible within the elastic limits of the material, and the couple required to produce it, assuming three million grammes weight per square centimetre to be the tenacity of the iron.

2. Show that, if a wire be subjected to tension and flexion couple of such magnitude that the product $\varpi\epsilon$ is sensible, the coefficient of flexion will be $\mathfrak{p}(1+\epsilon)$.

3. If a uniform bar, both of whose ends are fixed, be so displaced longitudinally that initially one half is uniformly extended and the other half uniformly contracted, prove that the displacement at time t will be given by

$$w = \frac{2\epsilon L}{\pi}\sum_{i=0}^{i=\infty}\frac{1}{(2i+1)^2}\cos\frac{(2i+1)\pi z}{L}\cos\frac{(2i+1)\pi\Omega_1 t}{L},$$

where ϵ is the initial extension, and Ω_1 the velocity of sound (§ 365) along the bar.

4. The extremities of a bar are attached to springs of equal strength. Show that, if

$$w_i = M_i \sin(iz/L) + N_i \cos(iz/L)$$

be an amplitude for longitudinal vibrations, then i is a root of

$$(i^2c^2 - L^2\mu^2)\tan i + 2icL\mu = 0,$$

where μ is the "strength" of either spring (*i.e.*, the ratio of the tension applied to the *increase of length* produced).

5. A steel cylinder of small elliptic section, and L centimetres in length, is clamped at one end. The gravest note when vibrating transversely in one principal plane of flexion, and the note above the gravest when vibrating in the other principal plane, both have a frequency of 256 per second. Show that the

semiaxes of the section are about $0{\cdot}00179L^2$ and $0{\cdot}000286L^2$ numerically, having given that—

(*i.*) Steel is 7·85 times as dense as water.

(*ii.*) Young's modulus $= 2{\cdot}14 \times 10^{12}$ C.G.S. units (absolute).

(*iii.*) The two smallest roots of the equation of frequency are 1·875 and 4·694.

6. A wire infinitely extended in one direction has its nearer end firmly clamped. If a series of simple harmonic transverse waves travelling along the wire be reflected at the clamped end, show that the reflected waves have the same amplitude as the incident waves, but that their phase is accelerated by one quarter of a wave length. What will be the result if the end be free instead of clamped?

7. A straight vertical wire of length L is attached at its lowest point to a cylindrical weight, the moment of inertia of which about the Axis of the wire is I. If the upper end of the wire be made to execute forced angular oscillations given by $\theta = a \sin it$, show that the oscillations of the weight will be represented by

$$\theta = \frac{a \cos\left[\tan^{-1}(iI/\sqrt{\rho\mathfrak{J}_3\mathfrak{t}})\right] \sin it}{\cos\left[\tan^{-1}(iI/\sqrt{\rho\mathfrak{J}_3\mathfrak{t}}) + iL\sqrt{\rho\mathfrak{J}_3/\mathfrak{t}}\right]},$$

the elongation of the wire being supposed negligible.

Prove that the frequency f of *natural* oscillations of the system is expressed by

$$2\pi f\sqrt{\rho\mathfrak{J}_3/\mathfrak{t}} \tan 2\pi f L\sqrt{\rho\mathfrak{J}_3/\mathfrak{t}} = \rho\mathfrak{J}_3/I.$$

[The equation of motion is by (105) $\mathfrak{t}\partial^2\theta/\partial z^2 - \rho\mathfrak{J}_3\ddot{\theta} = 0$, and the terminal conditions are (taking the origin at the lowest point) $\theta = a \sin it$ when $z = L$, and $\mathfrak{t}\partial\theta/\partial z - I\ddot{\theta} = 0$ when $z = 0$. In the case of free vibrations, i must be such that the total energy of the system remains constant, so that

$$\tfrac{1}{2}\int_0^L \left[\rho\mathfrak{J}_3\dot{\theta}^2 + \mathfrak{t}(\partial\theta/\partial z)^2\right]dz + \tfrac{1}{2}I\dot{\theta}^2_{z=0}$$

is independent of t. Evaluate this expression and equate coefficients of $\cos^2 it$ and $\sin^2 it$.]

8. A wire is held bent by suitable forces between two points A and B so that, the area between the wire and AB being given, the work expended in bending the wire may be the least possible. Show that the curvature at any point varies as $r^2 - D^2$, where $AB = 2D$, and r is the distance of the point from the middle point of AB. Show also that, if the wire be bent completely round to satisfy the same conditions, the curve assumed will be of the form $r^3 = C^3 \cos 3\theta$.

9. A square $ABCD$ is formed of four rods, each of length L, clamped together at the corners, the rod AB being elastic, and the other three rigid. If the system revolve about CD with uniform angular velocity ω, such that λL is a small quantity where λ is given by $\lambda^4 \equiv \rho\mathfrak{A}\omega^2/\mathfrak{p}$, the displacement of any point of AB at a distance z from its centre will be

$$L\frac{\sinh\frac{1}{2}\lambda L(\cos\lambda z - \cos\frac{1}{2}\lambda L) + \sin\frac{1}{2}\lambda L(\cosh\lambda z - \cosh\frac{1}{2}\lambda L)}{\sinh\frac{1}{2}\lambda L\cos\frac{1}{2}\lambda L + \sin\frac{1}{2}\lambda L\cosh\frac{1}{2}\lambda L}.$$

10. If, in the case of § 367, the rod be clamped at any one support in a position other than that which it would naturally assume when freely supported, an impossible identity will apparently be introduced. Explain this.

11. Calculate the pressure on each support of a heavy bar which rests upon four equidistant supports, two of which are at its ends.

12. A heavy beam of length L is cut in half, and one of the halves is again divided into four equal portions, which are placed upright in a row at equal distances $\frac{1}{6}L$ from one another. If the remaining half be placed upon these, and if P_1, P_2 be the thrusts on the intermediate and extreme vertical beams, show that

$$P_1 : P_2 : \tfrac{1}{8}\rho\mathfrak{A}L :: 11\mathfrak{c}L^2 + 486\mathfrak{p} : 4\mathfrak{c}L^2 + 486\mathfrak{p} : 10\mathfrak{c}L^2 + 648\mathfrak{p}.$$

3. A heavy uniform wire is supported at a series of points in the same horizontal straight line. If $\mathbf{P}_i$ be the flexion couple at the ith point of support, and D_i the distance between the $(i-\ )$th and ith supports, prove that

$$\mathbf{P}_{i-1}D_i + 2\mathbf{P}_i(D_i + D_{i+1}) + \mathbf{P}_{i+1}D_{i+1} = \tfrac{1}{4}\rho\mathfrak{A}\mathbf{g}(D_i^3 + D_{i+1}^3).$$

4. A weightless wire is supported at its two ends, and at its middle point O, and a small weight $\mathbf{W}$ is suspended from it at a point Q. Show that the downward vertical displacement u of any point P is given by

$$\begin{aligned}\mathfrak{p}L^3u = \tfrac{1}{3}\mathbf{W}\{&\tfrac{1}{4}L^3(D^3 - z^3 - z_1^3)\\ &- z^2z_1^2(z - \tfrac{3}{2}L)(z_1 - \tfrac{3}{2}L) \pm zz_1[\tfrac{1}{2}L^2(z^2 + z_1^2) + \tfrac{1}{4}L^4]\},\end{aligned}$$

where z, z_1, D are the *numerical* values of the distances OP, OQ, PQ, and the upper or lower sign is to be taken in the ambiguity according as P lies on the same side of O as Q, or on the opposite side.

1 . A rod $OABC$ is constrained to pass through three fixed points A, B, C in one straight line. Show that the deflection of the part OA, due to forces acting on that part only, is the same as if the rod had been constrained to pass through two points A and X only, where

$$AX = AB\frac{3AB + 4BC}{4AC}.$$

16. If the rod be constrained to pass through an infinite number of points, at intervals each equal to AB, the constraint—as regards the part OA—will be the same as if the rod had been constrained to pass through A and Y only, where $AY = \frac{1}{2}\sqrt{3} \,.\, AB$.

17. The maximum height for stability under gravity of a conical pole of semi-vertical angle α is

$$L_0 = \frac{3q\mathrm{i}^2 \tan^2\alpha}{16\rho\mathrm{g}},$$

where i is the least positive root of the equation $\mathbf{J}_3(\mathrm{i}) = 0$.

18. The maximum height for a paraboloid of revolution, of latus rectum D, planted with its vertex upwards, is

$$L_0 = \sqrt{\frac{qD\mathrm{i}^2}{2\rho\mathrm{g}}},$$

where i is the least positive root of the equation $\mathbf{J}_{\frac{1}{2}}(\mathrm{i}) = 0$.

19. If a rod revolve about its Central Axis under a given tension, prove that the straight form will be unstable when the number of revolutions per second exceeds the number of lateral vibrations executed in a second by the same rod under the same tension.

20. A rod of given length, securely clamped at one end and with the other end free, rotates about its Central Axis. Show that the greatest angular velocity consistent with the stability of the straight form is given by the least positive root of $\cos \mathrm{i} \,.\, \cosh \mathrm{i} = -1$, where i has the same value as in § 373.

21. If the clamp in the last example be replaced by a universal joint, and the angular velocity be such that i is a root (other than the least) of the equation $\tan \mathrm{i} = \tanh \mathrm{i}$, the Central Axis of the wire will describe the surface of revolution

$$r = M\left(\sin \mathrm{i} \sinh \frac{\mathrm{i}z}{L} + \sinh \mathrm{i} \sin \frac{\mathrm{i}z}{L}\right).$$

22. If the hoop of § 380 be cut, and the ends twisted through p complete turns and then joined again, and if the hoop be confined between two perfectly smooth parallel planes, so that the Central Axis must remain always a plane curve, the radius of the hoop in equilibrium will be a root of the quartic

$$\mathfrak{c}r^4 - \mathfrak{c}\mathrm{r}r^3 + \mathfrak{p}\mathrm{r}r - (\mathfrak{p} + p^2\mathfrak{t})\mathrm{r}^2 = 0.$$

If this radius be denoted by r_1, the time of a small vibration about the position of equilibrium will be

$$2\pi\sqrt{\frac{\rho\mathfrak{A}\mathrm{r}^2\mathrm{r}_1{}^4}{\mathfrak{c}\mathrm{r}_1{}^4 + \mathfrak{p}\mathrm{r}(3\mathrm{r} - 2\mathrm{r}_1) + 3p^2\mathfrak{t}\mathrm{r}^2}}.$$

23. A wire is twisted, and strained into the form of a helix, and its ends are then joined, so that it forms an endless spiral curve round a tubular core. Find the direction and magnitude of the resultant stress across any transverse section.

24. ξ and η are conjugate functions of x and y, the ξ curves being closed. If a wire have for its Central Axis a curve of the ξ family, and if it execute small vibrations in its own plane, so that each point moves along the principal normal to the Central Axis, and this latter remains always a curve of the same family, the time of a small oscillation will be

$$2\pi\left\{\frac{\rho\mathfrak{A}\int\frac{d\eta}{h^3}}{\mathfrak{c}\int\left(\frac{\partial h}{\partial\xi}\right)^2\frac{d\eta}{h^3}+\mathfrak{p}\int\left(\frac{\partial^2 h}{\partial\xi^2}\right)^2\frac{d\eta}{h}}\right\}^{\frac{1}{2}},$$

where the integrals are to be taken all round the curve, and ξ is to be given its initial value after differentiation.

25. Apply this result to obtain in terms of elliptic integrals the time of vibration of an elliptic wire which always retains the form of a confocal ellipse.

26. A perfectly rough bar has in its natural state the form of a circular arc of length L and curvature ϖ. A ring is formed by joining the ends of the bar, and is laid upon a smooth horizontal table, with its centre over that of a circular hole in the table. A perfectly rough cone, of very obtuse vertical angle 2β, and of weight $2\pi\mathfrak{A}g$, is placed gently on the ring, with its axis vertical and vertex downwards, and sinks under gravity. Prove that if the ring never approaches the edges of the hole, if its inertia be neglected, and if its section $\mathfrak{A}$ be supposed always to remain circular, the time of a small oscillation about a position of equilibrium is to the first approximation

$$4\pi\cot\beta\sqrt{\frac{L}{g(8\pi+L\varpi\,\mathrm{cosec}^2\beta\cos\gamma)}},$$

where

$$\gamma=\frac{L}{2\sqrt{\pi\mathfrak{A}}}\,\mathrm{cosec}^2\beta\cos\beta.$$

27. Show that if a rod be set in vibration with initial transverse displacements and velocities given by $u=\phi(z)$, $\dot{u}=\psi(z)$, the displacement at any subsequent time t may be represented by

$$u=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\phi(\xi)\cos\eta^2\omega t+\frac{\psi(\xi)}{\eta^2\omega}\sin\eta^2\omega t\right\}\cos\eta(z-\xi)\,d\xi\,d\eta,$$

where $\omega^2=\mathfrak{p}/\rho\mathfrak{A}$. (Fourier's solution.)

28. Obtain directly St. Venant's solution for the torsion of beams (§ 333) by adopting the conjugate cylindrical coördinates ξ, η, z of Example 4 (*i.*), page 258, (making p unity), and assuming $\alpha=0$, $\beta=z\tau$.

[It will be found that the general equations of equilibrium are satisfied by making $\partial w/\partial z=0$ and $\partial^2 w/\partial\xi^2+\partial^2 w/\partial\eta^2=0$, while the boundary conditions reduce to

$$\frac{\partial w}{\partial \xi}\frac{\partial \Phi}{\partial \xi}+\left(\frac{\partial w}{\partial \eta}+C^2\tau e^{2\xi}\right)\frac{\partial \Phi}{\partial \eta}=0.$$

The differential equation of the bounding surface must therefore be

$$\left(\frac{\partial w}{\partial \eta}+C^2\tau e^{2\xi}\right)d\xi-\frac{\partial w}{\partial \xi}d\eta=0\,;$$

or, if ϕ and w be any conjugate functions of ξ and η, and therefore also of x and y (Example 2, page 257),

$$\left(\frac{\partial \phi}{\partial \xi}+C^2\tau e^{2\xi}\right)d\xi+\frac{\partial \phi}{\partial \eta}d\eta=0.$$

This is satisfied by all surfaces of the form

$$\phi+\tfrac{1}{2}C^2\tau e^{2\xi}=\text{constant},$$

or

$$\phi+\tfrac{1}{2}\tau(x^2+y^2)=\text{constant}.$$

[See also Boussinesq's *Application des Potentiels à l'étude de l'équilibre et du mouvement des Solides Elastiques*, pp. 435-463.]

APPENDIX V.

Strength of Materials under Torsion and Flexion. Strain-Reversal (Nachwirkung).

Strength under Torsion. A cylindrical bar, subjected only to torsion couple applied to its ends, experiences only shearing stress (§ 330). The elastic strain produced depends therefore entirely on the *rigidity* of the material, and the only elastic limit involved is its *hardness* (page 181). To exhibit clearly the form of yielding to torsion stresses exceeding the elastic limit we will consider the simple case of a right circular cylinder (§§ 335, 341), following the account of Prof. James Thomson.*

* *Cambridge and Dublin Mathematical Journal*, Nov. 1848, sections 10-20; reproduced in Sir W. Thomson's article on Elasticity in *Encyclopædia Britannica*, section 9.

The shearing stress under torsion τ is [§ 341 (*i.*)] $\mathsf{S} = n\tau r$, where r is the distance from the Central Axis: the torsion couple (§ 335) within the elastic limits is $\mathrm{T} = \frac{1}{2}\pi n A^4\tau$, where A is the extreme radius of the cylinder.

Let us first suppose the material to be perfectly plastic (p. 170) and of solidity S. If the torsion couple be gradually increased until $\tau = \mathsf{S}/nA$ and $\mathrm{T} = \frac{1}{2}\pi A^3\mathsf{S}$, the limit of elasticity will be reached for the extreme bounding portion of the cylinder, which will then begin to flow. When still greater torsion has been produced, so that $r \equiv \mathsf{S}/n\tau$ is less than A, all that portion of the cylinder between the surfaces r and A will be in a state of flow with a uniform stress S throughout, while the portion comprised within the surface r will still be in a state of elastic strain. The torsion couple will then obviously be

$$\mathrm{T} = \int_0^r \frac{r_1\mathsf{S}}{r} \cdot r_1 \cdot 2\pi r_1 dr_1 + \int_r^A \mathsf{S} \cdot r_1 \cdot 2\pi r_1 dr_1$$
$$= \tfrac{1}{2}\pi r^3\mathsf{S} + \tfrac{2}{3}\pi(A^3 - r^3)\mathsf{S}.$$

As the torsion steadily increases, the radius r of the surface limiting the flow diminishes, until we reach a limit at which all but that portion of the cylinder in the immediate neighbourhood of the Central Axis has suffered flow. At this limit the maximum value $\mathrm{T} = \frac{2}{3}\pi A^3\mathsf{S}$ of the torsion couple is reached, which is $\frac{4}{3}$ of its value at the elastic limit. This therefore represents the *maximum resistance* to torsion of a plastic cylinder of radius A.

If now the torsion couple be gradually removed, there will be an elastic torsional recoil or untwisting of the wire, which will diminish the stress at each point by an amount proportional to its distance from the Central Axis. Suppose that, when the couple is entirely removed, the stress at distance r from the Axis is $\mathsf{S} - Cr$: then

$$\int_0^A (\mathsf{S} - Cr) \cdot r \cdot 2\pi r dr = 0,$$

or $C = \frac{4}{3}\mathsf{S}/A$. Thus the permanent stress due to *set* is a shearing stress $\mathsf{S}(1 - \frac{4}{3}r/A)$. All that portion of the bar contained *within* the surface $r = \frac{3}{4}A$ is permanently strained in the direction in which torsion took place, while all the portion *without* that surface is permanently strained in the opposite direction. The bar is, in fact, left in a very marked *state of constraint* (see page 18[illegible]).

The permanent stress being everywhere within the elastic limits, it follows from the principle of superposition that the stress produced by the application of a fresh torsion couple, *in either direction*, will be of the same form as if the bar were in a *state of ease* (see page 185), *i.e.*, proportional to r. Thus if T_1 be

the couple sufficient to produce flow, when applied *in the original direction*, we shall have

$$\int_0^A [C_1 r + \mathbf{S}(1 - \tfrac{4}{3}r/A)]r \, . \, 2\pi r dr = \mathbf{T}_1,$$

where $C_1 A - \frac{1}{3}\mathbf{S} = \mathbf{S}$; and consequently $\mathbf{T}_1 = \frac{2}{3}\pi A^3 \mathbf{S}$.

Similarly, if $\mathbf{T}_2$ be the Couple required to produce flow in the opposite direction,

$$\int_0^A [C_2 r - \mathbf{S}(1 - \tfrac{4}{3}r/A)]r \, . \, 2\pi r dr = \mathbf{T}_2,$$

where $C_2 A + \frac{1}{3}\mathbf{S} = \mathbf{S}$; and therefore $\mathbf{T}_2 = \frac{1}{3}\pi A^3 \mathbf{S}$.

Thus the *strength* of the bar under torsion is *twice as great in the direction in which it was originally twisted as in the other.*

These results are only true *numerically* of bars of perfectly plastic material, but the principle is obviously applicable also to ductile materials, the hardness of which increases during flow. Thus it is evident that the apparent strength of a bar under torsion may depend very largely upon its previous elastic history.

Strength under Flexion. In this case the strain is a longitudinal traction or pressure, proportional to the distance from the neutral plane. Taking the case of a rectangular bar of plastic material, of depth $2D$ and breadth B (see page 420), and of equal strength $\mathbf{C}$ under tension and thrust, it is easy to show that

(*i.*) The elastic limit is reached when the flexion couple amounts to $\frac{2}{3}D^2 B\mathbf{C}$.

(*ii.*) The maximum strength to resist flexion is $D^2 B\mathbf{C}$.

(*iii.*) On removal of the couple the tension of fibres distant y from the original neutral plane is $R = -\mathbf{C}(1 - 3y/2D)$, the axis of y being taken as in §§ 343-349. Thus the stress vanishes at distances $\pm\frac{2}{3}D$ from the original neutral plane.

(*iv.*) If the beam be bent again in the same direction, its strength is given by (*ii.*).

(*v.*) If it be bent in the opposite direction, its strength will be $\frac{1}{3}D^2 B\mathbf{C}$, or one third of the former.

We see therefore that in the case of flexion, as in that of torsion, the apparent strength of a bar, even of the most regular kind of material that we can imagine, depends chiefly upon the relation of the method of testing to the processes to which the bar may have been previously subjected.

Strain-Reversal (Nachwirkung). This phænomenon is described in this place because it appears most prominently, and was first observed, in connection with torsional strains. It is,

however, in all probability an invariable accompaniment of all strains which approach or surpass the elastic limits of the body.

The following account is extracted from Prof. Tait's "Properties of Matter," §§ 251, 252. The phænomenon is of purely physical interest, and no satisfactory explanation of it has yet been advanced. "All this part of our subject is still very imperfectly worked out. . . . There is no doubt that all elastic recovery in solids is gradual, so that, for instance, in . . . torsion vibrations . . . , even when there is no sensible viscous resistance, the middle point of the range does not coincide with the original untwisted position of the wire. It is always shifted towards the side to which torsion was applied, and to a greater extent the longer the wire has been kept twisted before being allowed to vibrate. With every vibration, however, it creeps slowly back towards the original undisturbed position, but usually comes to rest before reaching it. . . . These phenomena are seen in a more striking form when we dispense with oscillation. Thus, for example, suppose the wire to be kept twisted through 90° to the right for six hours, then for half an hour 90° to the left, and be then so gradually let go that there is no oscillation. When it is left to itself it turns slowly towards the right, gradually undoing part of the effect of the more recent twist, then stops, and twists still more slowly towards the left, thus undoing part of the quasi-permanent effect of the earlier twist. Thus the behaviour of such a wire, strictly speaking, is an excessively complex one, depending as it were upon its whole previous history; though of course the trace left by each stage of its treatment is less marked as the date of that stage is more remote. This subject has of late attracted great attention in Germany, and, under the name *Elastische Nachwirkung*, has been the object of numerous researches by Wiedemann, Kohlrausch, Boltzmann, etc."

A sketch of Clerk Maxwell's theory of this peculiar action is given in § 253 of the same work.

CHAPTER VIII.

PLATES AND SHELLS.

Introductory.

384.] **Definitions.** The term **Plate** will be used in this Chapter to denote a body cut from a right cylinder or right prism of any form by two (necessarily parallel) normal sections. These sections form the **Faces** of the Plate, while the intercepted portion of the original bounding surface of the prism forms the **Edge** or **Edges** of the Plate.

The **Thickness** of the Plate is the normal distance between its faces. A **Thin Plate** is one whose thickness is a small quantity of the first order compared with its least transverse dimension.

A plane drawn parallel to either face, and equidistant from both, will be called the *Median Plane*, and the section of the plate by this plane its **Median Surface.** The centre of gravity of this section is the **Centre** of the Plate. The section of the plate by any plane perpendicular to its faces is called a **Normal Surface.** The straight line drawn through the Centre, perpendicular to the faces, is the **Normal Axis.**

The *form* of the plate is determined by that of the prism from which it is cut:—thus a *Circular Plate* is derived from a right circular cylinder, a *Square Plate* from a right prism of square section, and so on.

A **Closed Shell** is a body contained by two surfaces belonging to one of a set of three orthogonal families; the surfaces being each of one sheet, and one of them entirely enclosing the other: *e.g.*—a Closed Spherical Shell is contained between two complete spherical surfaces, one of which entirely encloses the other, but which may or may not be concentric.

An **Open Shell** has for its *faces* two surfaces of one family, while its *edges* are formed by surfaces of one or both of the remaining families of an orthogonal system.

In a **Thin Shell** the thickness—here measured by the length of arc of the orthogonal curve intercepted between the faces of the shell—is a small quantity of the first order compared with its least superficial dimension.

385.] **The Class of Strains to be investigated. Exclusion of Surface Tractions on the Faces of the Plate or Shell.** Throughout the present Chapter, plates and shells will be supposed subjected to the action of Surface Tractions *on their edges only,* with or without the accompaniment of Impressed Forces. No stress will in any case be supposed to act across the faces of the plate or shell.

Closed shells will be considered as performing vibrations under normal impressed forces only.

CLEBSCH' PROBLEM: STRAINING OF A PLATE OF FINITE THICKNESS, FREE FROM IMPRESSED FORCES, BY SURFACE TRACTIONS APPLIED TO ITS EDGES ALONE, IN DIRECTIONS EVERYWHERE PARALLEL TO ITS FACES.

386.] **Statement of the Problem.** Taking the origin at the Centre, and the Normal Axis for axis of z, the boundary conditions $R=S=T=0$ must be satisfied over the entire area of either face, with the further condition $\lambda T+\mu S=0$ over the whole of the edges. Since the edges are everywhere parallel to Oz, the three stress components R, S, T are not involved in the actual surface tractions, which will therefore be none the less entirely arbitrary if we at once make the assumption * that

$$R=S=T=0 \qquad \text{.......(1)}$$

throughout the substance of the plate.

The boundary conditions will then be satisfied identically, as also will the third of the general equations of equilibrium [(103) of § 285], and it only remains to determine the most general values of u, v, and w that will satisfy the first and second of these, as well as (1) above, throughout the plate.

387.] **Solution of the Problem.** Substituting from (1) in equations (40) of § 214, we have

$$\left.\begin{aligned}\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=0 \\ (1-\sigma)\frac{\partial w}{\partial z}+\sigma\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0\end{aligned}\right\} \qquad \begin{aligned}&\text{.......(2)}\\&\text{.......(3)}\end{aligned}$$

* Compare the assumption made in (5) of Article 326, page 388.

to be satisfied concurrently with the first two of the general equations, viz.:—

$$\left.\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+(1-2\sigma)\nabla^2 u=0\\ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+(1-2\sigma)\nabla^2 v=0\end{aligned}\right\}\quad\text{.................(4)}$$

Differentiating (3) and (4) as to z, and eliminating u and v by means of (2), we obtain

$$\left.\begin{aligned}\frac{\partial^2 w}{\partial z^2}-\sigma\nabla^2 w=0\\ \frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial z^2}-(1-\sigma)\nabla^2 w\right]=0\\ \frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial z^2}-(1-\sigma)\nabla^2 w\right]=0\end{aligned}\right\}\quad\text{.........................(5)}$$

Again, differentiating the first of equations (4) as to x and the second as to y, and eliminating $\partial u/\partial x+\partial v/\partial y$ from the sum of the equations thus formed by means of (3), there results

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial w}{\partial z}=0\text{..............................(6)}$$

From (5) and (6) we deduce

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial z^2}\right)=\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial z^2}\right)=\frac{\partial}{\partial z}\left(\frac{\partial^2 w}{\partial z^2}\right)=0,\text{....................(7)}$$

whence
$$\frac{\partial^2 w}{\partial z^2}=C$$

or
$$\frac{\partial w}{\partial z}=Cz+\omega,\text{..................................(8)}$$

where C is a constant, and ω a function of x and y, which by (6) must satisfy

$$\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}=0.\text{..................................(9)}$$

Integrating (8) again, we may write

$$w=\tfrac{1}{2}Cz^2+z\omega+\chi+\tfrac{1}{2}(\varpi_1 x^2+\varpi_2 y^2+2\varpi_3 xy),$$

where χ is a second function of x and y, and ϖ_1, ϖ_2, ϖ_3 are constants. The last term is introduced on purpose to simplify the equation of condition satisfied by χ: on substituting in the first of equations (5), we see that, if we make $C=\sigma(\varpi_1+\varpi_2)/(1-\sigma)$, we shall have simply

$$\frac{\partial^2 \chi}{\partial x^2}+\frac{\partial^2 \chi}{\partial y^2}=0\text{..............................(10)}$$

Thus

$$w = \tfrac{1}{2}(\varpi_1 x^2 + \varpi_2 y^2 + 2\varpi_3 xy) + \frac{\sigma(\varpi_1 + \varpi_2)z^2}{2(1-\sigma)} + \chi + z\omega\,; \qquad (11)$$

while equations (2) give

$$\left.\begin{aligned} \frac{\partial u}{\partial z} &= -\varpi_1 x - \varpi_3 y - \frac{\partial \chi}{\partial x} - z\frac{\partial \omega}{\partial x} \\ \frac{\partial v}{\partial z} &= -\varpi_3 x - \varpi_2 y - \frac{\partial \chi}{\partial y} - z\frac{\partial \omega}{\partial y} \end{aligned}\right\},$$

or on integration

$$\left.\begin{aligned} u &= -\varpi_3 yz - \varpi_1 zx - z\frac{\partial \chi}{\partial x} - \frac{z^2}{2}\frac{\partial \omega}{\partial x} + \phi \\ v &= -\varpi_2 yz - \varpi_3 zx - z\frac{\partial \chi}{\partial y} - \frac{z^2}{2}\frac{\partial \omega}{\partial y} + \psi \end{aligned}\right\} \qquad (12)$$

where ϕ and ψ are functions of x and y only. Substitution from (11) and (12) in (3) gives

$$(1-\sigma)\omega + \sigma\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = 0, \qquad (13)$$

and the equations of condition to be satisfied by ϕ and ψ are then found from (4) to be

$$\left.\begin{aligned} (1-\sigma)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + (1+\sigma)\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) &= 0 \\ (1-\sigma)\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + (1+\sigma)\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) &= 0 \end{aligned}\right\} \qquad (14)$$

It will be found, on differentiating the first of equations (14) as to x, and the second as to y, and adding the results, that the value of ω as given by (13) satisfies (9) identically. Thus we may eliminate ω from (11) and (12) by means of (13), and the complete and most general solution of the proposed problem will finally be represented by

$$\left.\begin{aligned} u &= -\varpi_3 yz - \varpi_1 zx - z\frac{\partial \chi}{\partial x} + \phi + \frac{\sigma z^2}{2(1-\sigma)}\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) \\ v &= -\varpi_2 yz - \varpi_3 zx - z\frac{\partial \chi}{\partial y} + \psi + \frac{\sigma z^2}{2(1-\sigma)}\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) \\ w &= \tfrac{1}{2}(\varpi_1 x^2 + \varpi_2 y^2 + 2\varpi_3 xy) + \frac{\sigma(\varpi_1 + \varpi_2)z^2}{2(1-\sigma)} + \chi - \frac{\sigma z}{1-\sigma}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) \end{aligned}\right\} \ldots (15)$$

where ϕ, χ, ψ are any functions of x and y which satisfy (10) and (14) identically.

Also, since $R = 0$, it follows from equations (3) of § 253 that

$$P = q(e + \sigma f)/(1-\sigma^2), \quad Q = q(f + \sigma e)/(1-\sigma^2), \quad U = qc/2(1+\sigma).$$

Consequently the component stresses will be

$$\left.\begin{aligned}
P &= \frac{q}{1-\sigma^2}\left[-(\varpi_1+\sigma\varpi_2)z-(1-\sigma)z\frac{\partial^2\chi}{\partial x^2}+\frac{\partial\phi}{\partial x}+\sigma\frac{\partial\psi}{\partial x}+\frac{\sigma z^2}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}\right)\right]\\
Q &= \frac{q}{1-\sigma^2}\left[-(\varpi_2+\sigma\varpi_1)z-(1-\sigma)z\frac{\partial^2\chi}{\partial y^2}+\sigma\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}+\frac{\sigma z^2}{2}\frac{\partial^2}{\partial y^2}\left(\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}\right)\right]\\
U &= \frac{q}{1+\sigma}\left[-\varpi_3 z-z\frac{\partial^2\chi}{\partial x\partial y}+\tfrac{1}{2}\left(\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\right)+\frac{\sigma z^2}{2(1-\sigma)}\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}\right)\right]
\end{aligned}\right\}\text{.(16)}$$

Bearing in mind that the limiting values of z at the two faces are equal and of opposite algebraical signs, it is evident that the terms in the expressions (15) for the displacements depending upon ϖ_1, ϖ_2, ϖ_3, χ are due solely to *couples* applied to the edges in planes parallel to the Normal Axis, while the terms depending upon ϕ and ψ are due to tensions and thrusts in directions parallel to the faces, or to couples in planes perpendicular to the Normal Axis.

Flexion by Couples only.

388.] **Case of Uniform Flexion of the Median Surface.** If in equations (15) we annul all the terms but those involving the constant coefficients ϖ_1, ϖ_2, ϖ_3, they reduce to

$$\left.\begin{aligned}
u &= -\varpi_3 yz-\varpi_1 zx\\
v &= -\varpi_2 yz-\varpi_3 zx\\
w &= \tfrac{1}{2}(\varpi_1 x^2+\varpi_2 y^2+2\varpi_3 xy)+\frac{\sigma(\varpi_1+\varpi_2)z}{2(1-\sigma)}
\end{aligned}\right\}\text{..............(17)}$$

giving

$$\left.\begin{aligned}
e=-\varpi_1 z,\ f=-\varpi_2 z,\ g=\frac{\sigma(\varpi_1+\varpi_2)z}{1-\sigma}\\
a=0,\ b=0,\ c=-2\varpi_3 z
\end{aligned}\right\}\text{...............(18)}$$

and consequently

$$\left.\begin{aligned}
P &= -\frac{q(\varpi_1+\sigma\varpi_2)z}{1-\sigma^2}\\
Q &= -\frac{q(\varpi_2+\sigma\varpi_1)z}{1-\sigma^2}\\
U &= -\frac{q\varpi_3 z}{1+\sigma}
\end{aligned}\right\}\text{.........................(19)}$$

389.] **Form of the Median Surface.** The origin being at the centre of the plate, and z being zero throughout the Median Surface, the form into which this surface is strained is represented by

$$z'=\tfrac{1}{2}(\varpi_1 x'^2+\varpi_2 y'^2+2\varpi_3 x'y'),\text{.....................(20)}$$

so that it always touches the plane of xy at the origin. With certain limitations as to the magnitude of the strain, this is a surface of *uniform curvature* throughout, *i.e.*, a surface such that the curvatures of all parallel normal sections * are equal. For let (x', y', z') be any point in a normal section making an angle θ with zx, and let $(x'+\xi, y'+\eta, z'+\zeta)$ be an adjacent point in the same section. Then

$$\begin{aligned}\zeta &= \xi\frac{\partial z'}{\partial x'} + \eta\frac{\partial z'}{\partial y'} + \tfrac{1}{2}\left(\xi^2\frac{\partial^2 z'}{\partial x'^2} + \eta^2\frac{\partial^2 z'}{\partial y'^2} + 2\xi\eta\frac{\partial^2 z'}{\partial x'\partial y'}\right)\\ &= \xi(\varpi_1 x' + \varpi_3 y') + \eta(\varpi_3 x' + \varpi_2 y')\\ &\quad + \tfrac{1}{2}(\varpi_1\xi^2 + \varpi_2\eta^2 + 2\varpi_3\xi\eta).\end{aligned}$$

Let $$\xi = r\cos\theta,\quad \eta = r\sin\theta,$$

so that $$\begin{aligned}\zeta &= r[(\varpi_1 x' + \varpi_3 y')\cos\theta + (\varpi_3 x' + \varpi_2 y')\sin\theta]\\ &\quad + \tfrac{1}{2}r^2(\varpi_1\cos^2\theta + \varpi_2\sin^2\theta + 2\varpi_3\sin\theta\cos\theta)\\ &= Ar + \tfrac{1}{2}Br^2 \text{ (say).}\end{aligned}$$

Then, by the ordinary formula, the curvature of the section at (x', y', z') is $B/(1+A^2)^{\frac{3}{2}}$, and consequently, if the strain be so limited that $\varpi_1 x + \varpi_3 y$ and $\varpi_3 x + \varpi_2 y$ are infinitely small throughout the limits of the Median Surface, this curvature will be $\varpi_1\cos^2\theta + \varpi_2\sin^2\theta + 2\varpi_3\sin\theta\cos\theta$; and this varies only with θ.

Assuming these limitations to hold, the curvature of any normal section of the surface into which the Median Surface of the plate is strained may be put into the form

$$\tfrac{1}{2}(\varpi_1 + \varpi_2) + \tfrac{1}{2}(\varpi_1 - \varpi_2)\cos 2\theta + \varpi_3\sin 2\theta.$$

The terms involving ϖ_1 and ϖ_2 may evidently be analysed into

(*i.*) a cylindrical curvature ϖ_1 in all normal sections parallel to zx, together with a cylindrical curvature ϖ_2 in all normal sections parallel to yz, or

(*ii.*) a *spherical* curvature $\frac{1}{2}(\varpi_1+\varpi_2)$ of all normal sections in any direction, together with a cylindrical curvature $\frac{1}{2}(\varpi_1-\varpi_2)$ of normal sections parallel to zx and an equal and opposite † cylindrical curvature $\frac{1}{2}(\varpi_2-\varpi_1)$ of all normal sections parallel to yz. This last pair of cylindrical curvatures are said to form an *anticlastic system* of curvature of *amount* $\frac{1}{2}(\varpi_1-\varpi_2)$.

The term involving ϖ_3 consists of a cylindrical curvature ϖ_3 of all normal sections parallel to the bisector of the positive

* Strictly speaking, this should be—of all normal sections whose traces on the unstrained Median Surface are parallel. But since, in an infinitely small strain, all normal sections of the strained Median Surface are infinitely nearly parallel to Oz, the two statements are practically identical.

† The sign of the curvature will be taken *positive* when the radius of curvature is on the same side of the surface as the positive direction of the Normal Axis Oz.

angle between zx and yz, and an equal and opposite cylindrical curvature $-\varpi_3$ of all normal sections parallel to the other bisector. This term therefore also represents an anticlastic system of amount ϖ_3.

The most general mode of uniform curvature may therefore be analysed into a spherical or *synclastic* system and two anticlastic systems.

390.] **Transformation to the Principal Axes of the Strain.** If the tangents to the lines of curvature of the strained Median Surface at its centre be taken for axes of x and y, the *Indicatrix* * of the curvature will be referred to its principal axes, and the surface will take the form

$$z' = \tfrac{1}{2}(\Pi_1 x'^2 + \Pi_2 y'^2) \qquad (21)$$

With these axes of reference, equations (17), (18), (19) reduce to

$$\left.\begin{aligned} u &= -\Pi_1 zx, \quad v = -\Pi_2 yz \\ w &= \tfrac{1}{2}(\Pi_1 x^2 + \Pi_2 y^2) + \frac{\sigma(\Pi_1 + \Pi_2)z^2}{2(1-\sigma)} \end{aligned}\right\} \qquad (22)$$

$$\left.\begin{aligned} e = -\Pi_1 z,\ f = -\Pi_2 z,\ g &= \frac{\sigma(\Pi_1 + \Pi_2)z}{1-\sigma} \\ a = b = c &= 0 \end{aligned}\right\} \qquad (23)$$

$$P = -\frac{q(\Pi_1 + \sigma\Pi_2)z}{1-\sigma^2},\quad Q = -\frac{q(\Pi_2 + \sigma\Pi_1)z}{1-\sigma^2},\quad U = 0; \qquad (24)$$

so that the shear disappears, and the new axes of x and y are the principal axes of the strain.

It is obvious that the Median Surface is a *Neutral Plane* (§ 347), *i.e.*, it simply suffers warping without strain of any kind.

The analysis of § 347 will sufficiently explain the nature of the strain.

391.] **The Flexion Couples.**† Returning to the arbitrarily directed axes Ox, Oy of § 388, the components of the stress, at any point (x, y, z), across a Normal Surface of the plate the perpendicular on which from the Centre makes an angle θ with Ox, are

$$\left.\begin{aligned} F &= -\frac{qz[(\varpi_1 + \sigma\varpi_2)\cos\theta + (1-\sigma)\varpi_3 \sin\theta]}{1-\sigma^2} \\ G &= -\frac{qz[(\varpi_2 + \sigma\varpi_1)\sin\theta + (1-\sigma)\varpi_3 \cos\theta]}{1-\sigma^2} \end{aligned}\right\} \qquad (25)$$

* Frost's *Solid Geometry*, Article 382.

† The formulæ of this Article are proved synthetically in a paper by Mr. R. R. Webb, *Messenger of Mathematics*, vol. XI.

Integrating F, G, zF, zG from $z=+\frac{1}{2}\tau$ to $z=-\frac{1}{2}\tau$ (where τ is the thickness of the plate), we see that the total stress action across any length of a Normal Surface of the plate reduces to a couple, the components of which—per unit length of the Surface, measured parallel to the Faces—about axes parallel to Ox and Oy, in the standard directions are

$$\left.\begin{aligned} &+\frac{q\tau^3[(\varpi_2+\sigma\varpi_1)\sin\theta+(1-\sigma)\varpi_3\cos\theta]}{12(1-\sigma^2)} \\ &-\frac{q\tau^3[(\varpi_1+\sigma\varpi_2)\cos\theta+(1-\sigma)\varpi_3\sin\theta]}{12(1-\sigma^2)} \end{aligned}\right\}\quad\text{.............(26)}$$

These give a Flexion Couple proper, in a plane perpendicular to that of the Normal Surface and parallel to Oz, of amount

$$\frac{q\tau^3}{24(1-\sigma^2)}\{(1+\sigma)(\varpi_1+\varpi_2)+(1-\sigma)[(\varpi_1-\varpi_2)\cos 2\theta+2\varpi_3\sin 2\theta]\}\text{....(27)}$$

per unit length, and a couple in the plane of the Normal Surface, of amount

$$\frac{q\tau^3}{24(1+\sigma)}[(\varpi_2-\varpi_1)\sin 2\theta+2\varpi_3\cos 2\theta]\text{.................(28)}$$

per unit length (compare § 352). The former couple consists of two parts, due respectively to synclastic and anticlastic flexion: the latter is due to anticlastic flexion alone. The directions of these Normal Surfaces across which the flexion couple is a maximum or a minimum are given by

$$\tan 2\theta=\frac{2\varpi_3}{\varpi_1-\varpi_2},\text{ ....(29)}$$

and for Surfaces in these directions the second couple vanishes. These are of course the directions of the principal axes of the strain, as may be deduced directly from §§ 389, 390 by the properties of the Indicatrix-conic.

If now we write

$$\mathfrak{s}=\frac{q\tau^3}{12(1-\sigma)},\quad \mathfrak{a}=\frac{q\tau^3}{12(1+\sigma)},\text{ .... (30)}$$

the expression (27) for the flexion couple proper may be written

$$\mathfrak{s}\cdot\tfrac{1}{2}(\varpi_1+\varpi_2)+\mathfrak{a}\cdot[\tfrac{1}{2}(\varpi_1-\varpi_2)\cos 2\theta+\varpi_3\sin 2\theta]\text{...........(31)}$$

By § 389 the coefficient of $\mathfrak{s}$ is the curvature of "fibres" of the Median Surface due to synclastic flexion of the plate, and the coefficient of $\mathfrak{a}$ is the curvature of fibres of the Median Surface perpendicular to the Normal Surface across which the flexion couple acts (or parallel to the plane of the couple) due to anti-

clastic flexion of the plate. Thus, with a notation analogous to that of § 349, $\mathfrak{s}$ and $\mathfrak{a}$ may be termed *Coefficients of Synclastic* and *Anticlastic Flexion* respectively.* If we write

$$\mathbf{P}_1 = \frac{\mathfrak{s}(\varpi_1 + \sigma\varpi_2)}{1+\sigma}, \quad \mathbf{P}_2 = \frac{\mathfrak{s}(\varpi_2 + \sigma\varpi_1)}{1+\sigma}, \quad \mathbf{P}_3 = \mathfrak{a}\varpi_3, \quad\ldots\ldots\ldots\ldots(32)$$

the components (26) of the couple per unit length across any Normal Surface perpendicular to Ox will be $\mathbf{P}_3$ and $-\mathbf{P}_1$, while those of the couple per unit length across Normal Surfaces perpendicular to Oy will be $\mathbf{P}_2$ and $-\mathbf{P}_3$: all being reckoned in the standard directions. The flexion couple proper (31), across any Normal Surface, is

$$\tfrac{1}{2}(\mathbf{P}_1 + \mathbf{P}_2) + \tfrac{1}{2}(\mathbf{P}_1 - \mathbf{P}_2)\cos 2\theta + \mathbf{P}_3 \sin 2\theta \quad\ldots\ldots\ldots\ldots(33)$$

per unit length. Equation (29) may be written in the equivalent form

$$\tan 2\theta = \frac{2\mathbf{P}_3}{\mathbf{P}_1 - \mathbf{P}_2} \quad\ldots\ldots\ldots\ldots(34)$$

392.] **The Potential Energy.** By equation (20) of § 199 we have

$$\left.\begin{aligned} W &= \tfrac{1}{2}\iiint(Pe + Qf + Uc)\,dx\,dy\,dz \\ &= \frac{q\tau^3\mathfrak{A}}{24(1-\sigma^2)}[\varpi_1{}^2 + \varpi_2{}^2 + 2\sigma\varpi_1\varpi_2 + 2(1-\sigma)\varpi_3{}^2] \\ &= \frac{q\tau^3\mathfrak{A}}{24(1-\sigma^2)}[(\varpi_1 + \varpi_2)^2 - 2(1-\sigma)(\varpi_1\varpi_2 - \varpi_3{}^2)] \end{aligned}\right\} \quad\ldots\ldots\ldots(35)$$

where $\mathfrak{A}$ is the area of the unstrained Median Surface. The latter form exhibits W in terms of the Invariants of the Indicatrix, and we may deduce from this, or directly from § 390, that

$$\left.\begin{aligned} W &= \frac{q\tau^3\mathfrak{A}}{24(1-\sigma^2)}[(\Pi_1 + \Pi_2)^2 - 2(1-\sigma)\Pi_1\Pi_2] \\ &= \frac{q\tau^3\mathfrak{A}}{24(1-\sigma^2)}(\Pi_1{}^2 + \Pi_2{}^2 + 2\sigma\Pi_1\Pi_2) \end{aligned}\right\} \quad\ldots\ldots\ldots(36)$$

where Π_1 and Π_2 are the principal curvatures.

If δW be the increase of energy due to a small increment of each of the curvatures,

$$\begin{aligned} \delta W &= \frac{q\tau^3\mathfrak{A}}{12(1-\sigma^2)}[(\varpi_1 + \sigma\varpi_2)\delta\varpi_1 + (\varpi_2 + \sigma\varpi_1)\delta\varpi_1 + 2(1-\sigma)\varpi_3\delta\varpi_3] \\ &= \mathbf{P}_1\delta\varpi_1 + \mathbf{P}_2\delta\varpi_2 + 2\mathbf{P}_3\delta\varpi_3 \quad\ldots\ldots\ldots\ldots(37) \end{aligned}$$

* These coefficients, since they occur in expressions for couples *per unit length of surface*, are one linear dimension below those of Article 349.

If therefore the configuration of the strained Median Section be expressed in terms of any number of independent coördinates, of which ξ is one, the resistance offered to an increase of ξ will be

$$\frac{\partial W}{\partial \xi} = \mathbf{P}_1 \frac{\partial \varpi_1}{\partial \xi} + \mathbf{P}_2 \frac{\partial \varpi_2}{\partial \xi} + 2\mathbf{P}_3 \frac{\partial \varpi_3}{\partial \xi} \quad \ldots\ldots\ \ldots\ldots\ \ldots\ldots (38)$$

(Compare § 378.)

393.] **Case of Non-uniform Anticlastic Flexion.** The terms depending upon χ are

$$u = -z\frac{\partial \chi}{\partial x}, \quad v = -z\frac{\partial \chi}{\partial y}, \quad w = \chi, \quad \ldots\ldots\ldots\ldots\ldots\ldots (39)$$

where χ satisfies (10). If we analyse this strain by the method of § 389, we find that if $x\frac{\partial^2 \chi}{\partial x^2} + y\frac{\partial^2 \chi}{\partial x \partial y}$ and $x\frac{\partial^2 \chi}{\partial x \partial y} + y\frac{\partial^2 \chi}{\partial y^2}$ be both infinitely small within the limits of the plate, the curvature of any normal section of the strained Median Surface of the plate is given by

$$\tfrac{1}{2}\left(\frac{\partial^2 \chi}{\partial x^2} - \frac{\partial^2 \chi}{\partial y^2}\right)\cos 2\theta + \frac{\partial^2 \chi}{\partial x \partial y}\sin 2\theta,$$

which consists of two systems of anticlastic curvature. Neither of these can vanish for any position of the axes of x and y, unless χ is a quadratic function of these coördinates, in which case the flexion is uniform, and the strain is included in the type just considered.

Straining without Flexion Couple.

394.] **The Remaining Terms.** The terms depending upon ϕ and ψ in the general equations (15) represent a strain which is due solely to tensions and thrusts applied round the Edges of the plate in directions parallel to the Faces, and couples in planes perpendicular to the Normal Axis.

It leaves the Median Surface absolutely unchanged, and produces anticlastic curvature in all parallel surfaces, proportional to their distance from that surface.

We shall not further concern ourselves with this strain.

EQUILIBRIUM OF A THIN PLATE UNDER IMPRESSED FORCES THROUGHOUT ITS MASS, AND SURFACE TRACTIONS APPLIED TO ITS EDGES, SUCH THAT THE COMPONENT FORCES PARALLEL TO THE FACES ON ANY PORTION OF THE PLATE BOUNDED BY NORMAL SURFACES ARE EITHER EVANESCENT OR REDUCIBLE TO COUPLES.

395.] **Preliminary.** The results obtained in §§ 388-392 for the case of uniform flexion of a plate of any thickness by surface tractions applied to its edges in directions parallel to its faces, and everywhere reducible to couples or evanescent, are extended to the case of a thin plate subject to impressed forces and surface tractions *the components of which parallel to the faces* satisfy this condition, by a procedure very much like that of § 360. We assume in fact that the stress due to the applied couples will be everywhere of the *form* of that just discussed, only that the curvatures will *vary from point to point* of the Median Surface, and that the applied forces (necessarily perpendicular to the faces) on any portion of the plate bounded by normal surfaces will introduce shearing stresses in the same direction across those surfaces. The plate may be considered geometrically as coincident with its Median Surface.

396.] **Equations of Equilibrium.*** If X, Y, Z be the components of the impressed force per unit mass at (x, y, z), the restriction imposed upon the form of the resultant force acting on any part of the plate bounded by Normal Surfaces requires that

$$\int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} X dz = \int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} Y dz = 0, \qquad (40)$$

Let

$$\int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} Z dz = \tau \mathfrak{Z}, \qquad (41)$$

and let the components of the impressed couple on a rectangular element $\tau dxdy$ of the plate, about axes through its centre $(x, y, 0)$ parallel to Ox and Oy, be $\rho\tau\mathfrak{L}dxdy$, $\rho\tau\mathfrak{M}dxdy$. The sole impressed force on the element is of course $\rho\tau\mathfrak{Z}dxdy$, acting through its centre parallel to Oz.

Let A, B be the shearing forces per unit length at $(x, y, 0)$ across Normal Surfaces drawn through that point perpendicular to Ox, Oy respectively. The components of the flexion

* This and the two following articles are taken, with merely a change of notation, from Thomson and Tait's *Natural Philosophy*, Articles 644-648.

couple at $(x, y, 0)$ across these surfaces, per unit length, are by § 391 $\mathbf{P}_3$ and $-\mathbf{P}_1$, $\mathbf{P}_2$ and $-\mathbf{P}_3$ respectively. Hence the equations of equilibrium are easily seen to be

$$\left.\begin{aligned}
&\rho\tau\mathfrak{Z}dxdy + \left(A + \tfrac{1}{2}dx\frac{\partial A}{\partial x}\right)dy - \left(A - \tfrac{1}{2}dx\frac{\partial A}{\partial x}\right)dy \\
&\qquad + \left(B + \tfrac{1}{2}dy\frac{\partial B}{\partial y}\right)dx - \left(B - \tfrac{1}{2}dy\frac{\partial B}{\partial y}\right)dx = 0 \\
&\rho\tau\mathfrak{L}dxdy + Bdxdy + \left(\mathbf{P}_3 + \tfrac{1}{2}dx\frac{\partial \mathbf{P}_3}{\partial x}\right)dy - \left(\mathbf{P}_3 - \tfrac{1}{2}dx\frac{\partial \mathbf{P}_3}{\partial x}\right)dy \\
&\qquad + \left(\mathbf{P}_2 + \tfrac{1}{2}dy\frac{\partial \mathbf{P}_2}{\partial y}\right)dx - \left(\mathbf{P}_2 - \tfrac{1}{2}dy\frac{\partial \mathbf{P}_2}{\partial y}\right)dx = 0 \\
&\rho\tau\mathfrak{M}dxdy - Adxdy - \left(\mathbf{P}_1 + \tfrac{1}{2}dx\frac{\partial \mathbf{P}_1}{\partial x}\right)dy + \left(\mathbf{P}_1 - \tfrac{1}{2}dx\frac{\partial \mathbf{P}_1}{\partial x}\right)dy \\
&\qquad - \left(\mathbf{P}_3 + \tfrac{1}{2}dy\frac{\partial \mathbf{P}_3}{\partial y}\right)dx + \left(\mathbf{P}_3 - \tfrac{1}{2}dy\frac{\partial \mathbf{P}_3}{dy}\right)dx = 0
\end{aligned}\right\}$$

or, on simplification,

$$\left.\begin{aligned}
\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \rho\tau\mathfrak{Z} &= 0 \\
\frac{\partial \mathbf{P}_3}{\partial x} + \frac{\partial \mathbf{P}_2}{\partial y} + B + \rho\tau\mathfrak{L} &= 0 \\
\frac{\partial \mathbf{P}_1}{\partial x} + \frac{\partial \mathbf{P}_3}{\partial y} + A - \rho\tau\mathfrak{M} &= 0
\end{aligned}\right\} \quad \text{.....................(42)}$$

It may be shown, as in § 389, that if $x\partial^2w/\partial x^2 + y\partial^2w/\partial x\partial y$ and $x\partial^2w/\partial x\partial y + y\partial^2w/\partial y^2$ are infinitely small within the limits of the plate,

$$\varpi_1 = \frac{\partial^2 w}{\partial x^2},\quad \varpi_2 = \frac{\partial^2 w}{\partial y^2},\quad \varpi_3 = \frac{\partial^2 w}{\partial x\partial y} \quad \text{.....................(43)}$$

Substituting from (43) in (32),

$$\mathbf{P}_1 = \frac{\mathfrak{s}}{1+\sigma}\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right),\quad \mathbf{P}_2 = \frac{\mathfrak{s}}{1+\sigma}\left(\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right),\quad \mathbf{P}_3 = \mathfrak{a}\frac{\partial^2 w}{\partial x\partial y}, \quad \text{...(44)}$$

whence, since $(1+\sigma)\mathfrak{a} = (1-\sigma)\mathfrak{s}$, equations (42) may be written

$$\left.\begin{aligned}
\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \rho\tau\mathfrak{Z} &= 0 \\
\frac{\mathfrak{s}}{1+\sigma}\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + B + \rho\tau\mathfrak{L} &= 0 \\
\frac{\mathfrak{s}}{1+\sigma}\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + A - \rho\tau\mathfrak{M} &= 0
\end{aligned}\right\} \quad \text{.................(45)}$$

On elimination of A and B between these three equations, we obtain

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 w=\frac{(1+\sigma)\rho\tau}{\mathfrak{s}}\left(\mathfrak{Z}+\frac{\partial\mathfrak{M}}{\partial x}-\frac{\partial\mathfrak{L}}{\partial y}\right), \quad\ldots\ldots\ldots\ldots(46)$$

a linear partial differential equation of the fourth order, to be satisfied by w in all cases of equilibrium under strain of the kind supposed.

397.] **The Boundary Conditions.** Poisson's three boundary conditions are easily obtained by considering the equilibrium of a triangular element of the plate, bounded by planes of length dx, dy parallel to zx, yz and an element of the edge of length ds. Let the outward normal to ds make an angle θ with Ox; let H be the surface traction on the edge parallel to Oz, and let

$$\int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau} H dz=\tau\mathfrak{h}$$

so that $\tau\mathfrak{h}ds$ is the shearing force on the element τds of edge. Let $\mathbf{P}ds$ and $\mathfrak{q}ds$ be the couples on the element in the plane perpendicular to it and in its own plane. Then, *on the assumption that the force and couples acting across the edge must be of the same form as on any Normal Surface within the substance of the plate,* we have first

$$\tau\mathfrak{h}ds=Ady+Bdx$$

or

$$\tau\mathfrak{h}=A\cos\theta+B\sin\theta\,; \quad\ldots\ldots\ldots\ldots(47)$$

and further by (33), (28) and (32)

$$\left.\begin{aligned}\mathbf{P}&=\tfrac{1}{2}(\mathbf{P}_1+\mathbf{P}_2)+\tfrac{1}{2}(\mathbf{P}_1-\mathbf{P}_2)\cos 2\theta+\mathbf{P}_3\sin 2\theta \quad\ldots\ldots(48)\\ \mathfrak{q}&=\tfrac{1}{2}(\mathbf{P}_2-\mathbf{P}_1)\sin 2\theta+\mathbf{P}_3\cos 2\theta \quad\ldots\ldots(49)\end{aligned}\right\}$$

These are Poisson's three conditions. Kirchhoff, however, has shown that the assumption involved in them (expressed in italics above) is not necessarily fulfilled, so that they *express too much.* The proof of this statement depends upon the fact (to be proved in the next Article) that if we apply, all round the edge of the plate, a shearing force parallel to Oz of amount $\tau(\mathfrak{H}-\mathfrak{h})$, per unit length and couple round axes everywhere parallel to the Median Surface and perpendicular to the edge, of amount $(\mathbf{Q}-\mathfrak{q})$ per unit length, such that

$$\tau(\mathfrak{H}-\mathfrak{h})=\frac{d}{ds}(\mathbf{Q}-\mathfrak{q}) \quad\ldots\ldots\ldots\ldots(50)$$

no modification of the strain whatever will be produced, except at points infinitely near the edge.

Thus we may suppose $\tau\mathfrak{H}$ to be the shearing force per unit length and $\mathbf{Q}$ the couple per unit length in the tangent plane to the edge *actually applied* at each point, where $\mathfrak{H}$ and $\mathbf{Q}$ may be *any quantities connected by the relation* (50), $\mathfrak{h}$ and $\mathfrak{q}$ being given by (47) and (49).

Eliminating $\mathfrak{h}$ and $\mathfrak{q}$ by means of (47) and (49), and A and B by means of (42) from (50), we obtain

$$\tau[\mathfrak{H} + \rho(\mathfrak{L}\sin\theta - \mathfrak{M}\cos\theta)] - \frac{d\mathbf{Q}}{ds} + \frac{\partial}{\partial s}[\tfrac{1}{2}(\mathbf{P}_2 - \mathbf{P}_1)\sin 2\theta + \mathbf{P}_3\cos 2\theta]$$
$$+\left(\frac{\partial \mathbf{P}_1}{\partial x} + \frac{\partial \mathbf{P}_3}{\partial y}\right)\cos\theta + \left(\frac{\partial \mathbf{P}_3}{\partial x} + \frac{\partial \mathbf{P}_2}{\partial y}\right)\sin\theta = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(51)$$

Equations (48) and (51) are Kirchhoff's *two* boundary conditions. If they be regarded as determining the values of w at the edge, $\mathbf{P}$, $\mathbf{Q}$ and $\mathfrak{H}$ may be treated as entirely arbitrary couples and force applied to the edge.

398.] **Proof of Kirchhoff's Boundary Theorem.** "The proposition stated" in the last Article "is equivalent to this:—that a certain distribution of normal* shearing force on the bounding edge of a finite plate may be determined which shall produce the same effect as any given distribution of couples round axes everywhere perpendicular to the Normal Surface supposed to constitute the edge. To prove this let equal forces act in opposite directions in lines EF, $E'F'$ on each side of the middle line† and parallel to it, constituting the supposed distribution of couple. It must be understood that the forces are actually distributed along their lines of action, and not, as in the abstract dynamics of ideal rigid bodies, applied indifferently at any points of these lines; but the amount of the force per unit length, though equal in the neighbouring parts of the two lines, must differ from point to point along the edge, to constitute any other than a uniform distribution of couple. Lastly, we may suppose the forces in the opposite directions to be not confined to two lines, as shown in the diagram, but to be diffused over the two halves of the edge on the two sides of its middle line; and further, the amount of them in equal infinitely small breadths at different distances from the middle line must be proportional to these distances" [see formulæ (25) of § 391] "if the given distribution of couple is to be thoroughly such as" $\mathfrak{q}$ of § 397.

"Let now the whole edge be divided into infinitely small rectangles, such as $ABCD$ in Figure 61, by lines drawn per-

* *I.e.*, parallel to the Normal Axis of the plate.

† The line in which the edge is cut by the Median Surface of the plate.

pendicularly across it.* In one of these rectangles apply a balancing system of couples consisting of a diffused couple equal and opposite to the part of the given distribution of couple belonging to the area of the rectangle, and a couple of single forces in the lines AD, CB, of equal and opposite moment. This balancing system obviously cannot cause any sensible disturbance (stress or strain) in the plate, except within a distance comparable with the sides of the rectangle; and, therefore, when the same thing is done in all the rectangles into which the edge is divided, the plate is only disturbed to an infinitely small distance from the edge inwards all round. But the given distribution of couple is thus removed (being directly balanced by a system of diffused force equal and opposite everywhere to that constituting it), and there remains only the set of forces applied in the cross lines. Of these there are two in each cross line, derived from the operations performed in the two rectangles of which it is a common side, and their difference alone remains

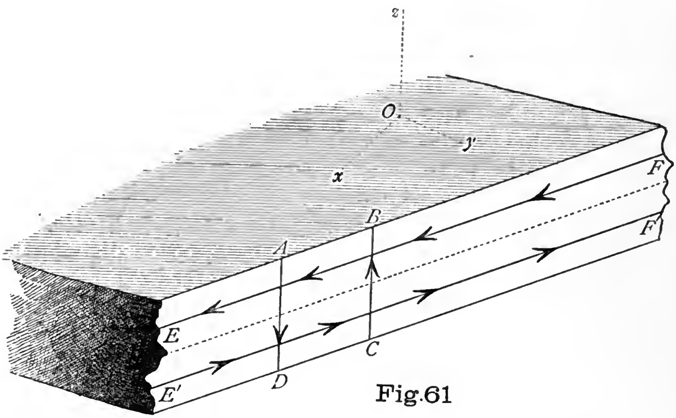


Fig.61

effective. Thus we see that *if the given distribution of couple be uniform along the edge, it may be removed without disturbing the condition of the plate except infinitely near the edge.*"

Otherwise, "*a distribution of couple on the edge of a plate, round axes everywhere in the plane of the plate* (*i.e.*, in the plane of the unstrained Median Surface), *of any given amount per unit of length of the edge, may be removed, and, instead, a distribution of force perpendicular to the plate, equal in amount per unit length of the edge, to the rate of variation per unit length of the amount of the couple, without altering the flexion of the plate as a whole, or producing any disturbance in its*

* To the unstrained Median Surface.

stress or strain except infinitely near the edge." For, in Figure 61, let $AB=ds$, the arc s being measured from A towards B. Then, $\mathsf{q}-\mathsf{Q}$ being the amount of the given couple per unit length, the amount of it on the rectangle $ABCD$ is $(\mathsf{q}-\mathsf{Q})ds$. Thus the forces introduced along AD, CB to form the balancing system must be of amount $\mathsf{q}-\mathsf{Q}$. Similarly, the amount of the forces introduced along BC and the next transverse line is $\mathsf{q}-\mathsf{Q}+ds\frac{d}{ds}(\mathsf{q}-\mathsf{Q})$, and finally we are left with a force of amount $ds\frac{d(\mathsf{q}-\mathsf{Q})}{ds}$ along BC, and a similar force in the negative direction of the Normal Axis along every other such transverse line. And obviously we may substitute for forces of amount $\frac{d(\mathsf{q}-\mathsf{Q})}{ds}ds$ at infinitesimal intervals ds a continuous distribution of force of amount $\frac{d(\mathsf{q}-\mathsf{Q})}{ds}$ per unit length round the whole edge, without causing disturbance in the plate except at infinitely small distances from the edge. Hence, finally, we have the result stated in equation (50) of § 397.

Equilibrium and Normal Vibrations of Thin Plates Subject to any Distribution of Normal Impressed Force, but free from Impressed Couple except at the Edges; Treated by the Energy Method.

399.] **The Total Energy.** The second of the expressions (3) for the potential energy of a plate of area $\mathfrak{A}$ subjected to uniform flexion, may be written by means of (30) and (43) in the form

$$W=\mathfrak{A}\left\{\frac{\mathfrak{s}+\mathfrak{a}}{4}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-\mathfrak{a}\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}\quad\ldots.(52)$$

The flexion of an element $\tau dxdy$ may be considered uniform under any circumstances, so that we deduce for the entire potential energy of a plate subject to non-uniform flexion

$$W=\iint\left\{\frac{\mathfrak{s}+\mathfrak{a}}{4}(\nabla^2 w)^2-\mathfrak{a}\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}dxdy,\ldots(53)$$

where $\qquad \nabla^2\equiv\partial^2/\partial x^2+\partial^2/\partial y^2.$

If the plate be executing normal vibrations, the entire kinetic energy will be

$$\mathfrak{T} = \tfrac{1}{2}\rho\tau\iint \dot{w}^2 dxdy, \text{..........................}(54)$$

and the total energy of the plate will of course be $W + \mathfrak{T}$.

400.] **The Variational Equation of Motion, and the Boundary Conditions.** Let us suppose the plate to be either in equilibrium, or executing normal vibrations freely or under normal forces only, and that its edges are either free or "supported" or "clamped" (§ 366) all round. In the most general case, the small amount of work done in producing the increment δw of normal displacement will be, with the notation of §§ 396-397,

$$\iint \rho\tau\mathfrak{Z}\delta w dxdy + \int \left[\left(\tau\mathfrak{H} - \frac{d\mathbf{Q}}{ds}\right)\delta w + \mathbf{P}\frac{\delta\partial w}{\partial\nu}\right]ds,$$

where τds is the element of edge, and $d\nu$ the element of outward drawn normal to it. [The work done by the couple $\mathbf{Q}$, as couple, that is in producing flexion about axes perpendicular to the edge, is $-\int \mathbf{Q}\frac{\delta\partial w}{\partial s}ds$, which, since s is necessarily a closed curve, vanishes identically. This is the analytical justification of Kirchhoff's principle.] Thus the variational equation of motion is

$$\iint \left\{ \frac{\mathfrak{S}+\mathfrak{N}}{4}\delta(\nabla^2 w)^2 - \mathfrak{N}\delta\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right] + \rho\tau(\ddot{w} - \mathfrak{Z})\delta w \right\} dxdy$$
$$- \int \left[\left(\tau\mathfrak{H} - \frac{d\mathbf{Q}}{ds}\right)\delta w + \mathbf{P}\frac{\delta\partial w}{\partial\nu}\right]ds = 0. \text{...............}(55)$$

Taking the first term separately, and making use of the general theorem

$$\iint (\phi\nabla^2\psi - \psi\nabla^2\phi)dxdy = \int \left(\phi\frac{\partial\psi}{\partial\nu} - \psi\frac{\partial\phi}{\partial\nu}\right)ds, \text{............}(56)$$

where the double integral is taken over the entire area of the plate, and the single integral round the whole of its boundary edge, we have

$$\iint \delta(\nabla^2 w)^2 dxdy = 2\iint \nabla^2 w \delta\nabla^2 w dxdy$$
$$= 2\iint \nabla^2\nabla^2 w \,.\, \delta w \,.\, dxdy + 2\int \left(\nabla^2 w \,.\, \frac{\delta\partial w}{\partial\nu} - \frac{\partial\nabla^2 w}{\partial\nu} \,.\, \delta w\right)ds. \text{...}(57)$$

Again, the second term of (55), on being integrated twice by parts, gives

$$\iint \delta\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dxdy$$

$$= \iint \left[\frac{\partial^2 w}{\partial y^2}\frac{\delta\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\frac{\delta\partial^2 w}{\partial y^2} - 2\frac{\partial^2 w}{\partial x\partial y}\frac{\delta\partial^2 w}{\partial x\partial y}\right] dxdy$$

$$= \int \left\{ \left(\cos\theta\frac{\partial^2 w}{\partial y^2} - \sin\theta\frac{\partial^2 w}{\partial x\partial y}\right)\frac{\delta\partial w}{\partial x} + \left(\sin\theta\frac{\partial^2 w}{\partial x^2} - \cos\theta\frac{\partial^2 w}{\partial x\partial y}\right)\frac{\delta\partial w}{\partial y} \right\} ds$$

$$= \int \left\{ \left(\cos\theta\frac{\partial^2 w}{\partial y^2} - \sin\theta\frac{\partial^2 w}{\partial x\partial y}\right)\left(\cos\theta\frac{\delta\partial w}{\partial \nu} - \sin\theta\frac{\delta\partial w}{\partial s}\right) + \left(\sin\theta\frac{\partial^2 w}{\partial x^2} - \cos\theta\frac{\partial^2 w}{\partial x\partial y}\right)\left(\sin\theta\frac{\delta\partial w}{\partial \nu} + \cos\theta\frac{\delta\partial w}{\partial s}\right) \right\} ds$$

$$= \int \left\{ \left[\sin\theta\cos\theta\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}\right) + (\sin^2\theta - \cos^2\theta)\frac{\partial^2 w}{\partial x\partial y}\right]\frac{\delta\partial w}{ds} + \left[\sin^2\theta\frac{\partial^2 w}{\partial x^2} + \cos^2\theta\frac{\partial^2 w}{\partial y^2} - 2\sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y}\right]\frac{\delta\partial w}{\partial \nu} \right\} ds,$$

where θ has the same meaning as in § 397. Integrating the first term again by parts, and neglecting the integrated portion (s being necessarily a closed curve, and the function to be integrated necessarily single-valued), we have

$$\int \left\{ \frac{\partial}{\partial s}\left[\sin\theta\cos\theta\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right) + (\cos^2\theta - \sin^2\theta)\frac{\partial^2 w}{\partial x\partial y}\right]\delta w + \left[\sin^2\theta\frac{\partial^2 w}{\partial x^2} + \cos^2\theta\frac{\partial^2 w}{\partial y^2} - 2\sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y}\right]\frac{\delta\partial w}{\partial \nu} \right\} ds \quad\text{.......(58)}$$

Finally, collecting the results of (57) and (58), multiplying them by their proper coefficients in (55), and adding the remaining terms of that expression, the complete variational equation is

$$\iint \left\{ (\mathfrak{s}+\mathfrak{a})\nabla^2\nabla^2 w + 2\rho\tau(\ddot{w} - \mathfrak{Z}) \right\} . \delta w . dxdy$$
$$- \int \left\{ (\mathfrak{s}+\mathfrak{a})\frac{\partial\nabla^2 w}{\partial \nu} + 2\mathfrak{a}\frac{\partial}{\partial s}\left[\sin\theta\cos\theta\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right) + (\cos^2\theta - \sin^2\theta)\frac{\partial^2 w}{\partial x\partial y}\right] + 2\left(\tau\mathfrak{Z} - \frac{dQ}{ds}\right) \right\} . \delta w . ds$$
$$+ \int \left\{ (\mathfrak{s}+\mathfrak{a})\nabla^2 w - 2\mathfrak{a}\left[\sin^2\theta\frac{\partial^2 w}{\partial x^2} + \cos^2\theta\frac{\partial^2 w}{\partial y^2} - 2\sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y}\right] - 2P \right\} . \frac{\delta\partial w}{\partial \nu} . ds = 0 \quad\text{..............(59)}$$

Thus the general equation of vibration, to be satisfied at every point of the plate, is

$$(\mathfrak{s}+\mathfrak{a})\nabla^2\nabla^2 w + 2\rho\tau(\ddot{w} - \mathfrak{Z}) = 0, \quad\text{.........................(60)}$$

while the *two* boundary conditions are

$$\left\{ (\mathfrak{s}+\mathfrak{a})\frac{\partial\nabla^2 w}{\partial\nu} + 2\mathfrak{a}\frac{\partial}{\partial s}\left[\sin\theta\cos\theta\left(\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w}{\partial x^2}\right)\right.\right.$$

$$\left.\left. + (\cos^2\theta-\sin^2\theta)\frac{\partial^2 w}{\partial x\partial y}\right] + 2\left(\tau\mathfrak{H} - \frac{d\mathbf{Q}}{ds}\right)\right\}\delta w = 0 \ldots\ldots\ldots(61)$$

and

$$\left\{ (\mathfrak{s}+\mathfrak{a})\nabla^2 w - 2\mathfrak{a}\left[\sin^2\theta\frac{\partial^2 w}{\partial x^2} + \cos^2\theta\frac{\partial^2 w}{\partial y^2}\right.\right.$$

$$\left.\left. - 2\sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y}\right] - 2\mathbf{P}\right\}\frac{\delta\partial w}{\partial\nu} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(62)$$

If the edge is *clamped* all round we have $\delta w = 0$, $\delta\partial w/\partial r = 0$ everywhere, and (61) and (62) are necessarily satisfied.

If the edge is only *supported* $\delta w = 0$, and we must have

$$(\mathfrak{s}+\mathfrak{a})\nabla^2 w - 2\mathfrak{a}\left[\sin^2\theta\frac{\partial^2 w}{\partial x^2} + \cos^2\theta\frac{\partial^2 w}{\partial y^2} - 2\sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y}\right] = 2\mathbf{P},$$

or

$$(\mathfrak{s}-\mathfrak{a})\nabla^2 w + 2\mathfrak{a}\left[\cos^2\theta\frac{\partial^2 w}{\partial x^2} + \sin^2\theta\frac{\partial^2 w}{\partial^2} + 2\sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y}\right] = 2\mathbf{P} \ldots(63)$$

If the edge is *free*, we must have, in addition to (63),

$$(\mathfrak{s}+\mathfrak{a})\frac{\partial\nabla^2 w}{\partial\nu} + 2\mathfrak{a}\frac{\partial}{\partial s}\left[\sin\theta\cos\theta\left(\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w}{\partial x^2}\right) + (\cos^2\theta-\sin^2\theta)\frac{\partial^2 w}{\partial x\partial y}\right]$$

$$= 2\left(\frac{\partial\mathbf{Q}}{ds} - \tau\mathfrak{H}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(64)$$

It is easy to show that, on making $\mathfrak{L}$, $\mathfrak{M}$, zero, and writing $\mathfrak{Z} - \dot{w}$ for $\mathfrak{Z}$, equations (46), (48), and (51) reduce to (60), (63) and (64) respectively.

401.] **Transformation to Conjugate Cylindrical Coördinates.** Let ξ and η be conjugate functions of x and y, and let the form of the edge be such that it can be represented by the equation $\xi = constant$. It is obvious that the equations of the last Article will be much more readily applicable if they can be transformed from x and y to ξ and η. It is an excellent example of the methods of Chapter V. to effect this transformation *ab initio*.

The principal curvatures Π_1, Π_2 of any surface $\Phi(x, y, z) = 0$ are the roots of the quadratic *

$$\mathfrak{h}^4\Pi^2 \pm \mathfrak{h}\Pi[A^2a + B^2b + C^2c + 2BCa' + 2CAb' + 2ABc' - \mathfrak{h}^2\nabla^2\Phi]$$

$$+ A^2(bc - a'^2) + B^2(ca - b'^2) + C^2(ab - c'^2)$$

$$+ 2BC(b'c' - aa') + 2CA(c'a' - bb') + 2AB(a'b' - cc') = 0 \ldots\ldots\ldots(65)$$

* Frost's *Solid Geometry*, Article 608.

where $A = \partial\Phi/\partial x$......, $a = \partial^2\Phi/\partial x^2$......, $a' = \partial^2\Phi/\partial y\partial z$......, and $h^2 = A^2 + B^2 + C^2$. Putting $\Phi = z + w$, and transforming (65) from (x, y, z) to (ξ, η, z) by the formulæ of §§ 230, 231, 245, we obtain

$$\left.\begin{aligned}
\varpi_1 + \varpi_2 &= \Pi_1 + \Pi_2 = h^2\left(\frac{\partial^2 w}{\partial\xi^2} + \frac{\partial^2 w}{\partial\eta^2}\right)\\
\varpi_1\varpi_2 - \varpi_3{}^2 &= \Pi_1\Pi_2 = h^4\left[\frac{\partial^2 w}{\partial\xi^2}\frac{\partial^2 w}{\partial\eta^2} - \left(\frac{\partial^2 w}{\partial\xi\partial\eta}\right)^2\right]\\
&\quad - h^3\left(\frac{\partial h}{\partial\xi}\frac{\partial w}{\partial\xi} - \frac{\partial h}{\partial\eta}\frac{\partial w}{\partial\eta}\right)\left(\frac{\partial^2 w}{\partial\xi^2} - \frac{\partial^2 w}{\partial\eta^2}\right)\\
&\quad - 2h^3\left(\frac{\partial h}{\partial\eta}\frac{\partial w}{\partial\xi} + \frac{\partial h}{\partial\xi}\frac{\partial w}{\partial\eta}\right)\frac{\partial^2 w}{\partial\xi\partial\eta}\\
&\quad - h^2\left[\left(\frac{\partial h}{\partial\xi}\right)^2 + \left(\frac{\partial h}{\partial\eta}\right)^2\right]\left[\left(\frac{\partial w}{\partial\xi}\right)^2 + \left(\frac{\partial w}{\partial\eta}\right)^2\right]
\end{aligned}\right\}\text{............(66)}$$

Substituting in (35) we obtain an expression for W analogous to (53), the element of surface being $d\xi d\eta/h^2$, and the arbitrary variation of this must be equated to

$$\rho\tau\iint(\mathfrak{Z} - \ddot{w}) \,.\, \delta w \,.\, \frac{d\xi d\eta}{h^2} + \int\left[\left(\tau\mathfrak{J} - h\frac{d\mathbf{Q}}{d\eta}\right)\delta w + \mathbf{P}h\frac{\delta\partial w}{\partial\xi}\right]\frac{d\eta}{h}.$$

On integrating by parts and rearranging terms, we have finally the general equation of vibration *

$$(\mathfrak{s} + \mathfrak{a})\nabla^2\nabla^2 w + 2\rho\tau(\ddot{w} - \mathfrak{Z}) = 0 \text{........ ....(67)}$$

where

$$\nabla^2 = h^2(\partial^2/\partial\xi^2 + \partial^2/\partial\eta^2);$$

together with the boundary conditions

$$(\mathfrak{s} - \mathfrak{a})\nabla^2 w + 2\mathfrak{a}\left[h^2\frac{\partial^2 w}{\partial\xi^2} + h\left(\frac{\partial h}{\partial\xi}\frac{\partial w}{\partial\xi} - \frac{\partial h}{\partial\eta}\frac{\partial w}{\partial\eta}\right)\right] = 2\mathbf{P} \text{......(68)}$$

and

$$(\mathfrak{s} + \mathfrak{a})\frac{\partial\nabla^2 w}{\partial\xi} + 2\mathfrak{a}\frac{\partial}{\partial\eta}\left[h^2\frac{\partial^2 w}{\partial\xi\partial\eta} + h\left(\frac{\partial h}{\partial\eta}\frac{\partial w}{\partial\xi} + \frac{\partial h}{\partial\xi} + \frac{\partial w}{\partial\eta}\right)\right] = 2\left(\frac{d\mathbf{Q}}{d\eta} - \frac{\tau\mathfrak{J}}{h}\right) \text{.(69)}$$

Of these (68) must be satisfied round a *supported* edge, and both round a *free* edge.

For examples on Normal Vibrations of Plates the student is referred to Lord Rayleigh's "Theory of Sound," Chapter X., and for examples on equilibrium to Thomson and Tait's "Natural Philosophy," §§ 649-657, 719-729. We shall here confine ourselves to a single example of the latter class, to exhibit the convenience of curvilinear coördinates in cases of symmetrical strain.

* There is apparently a residual double integral from the second term of W but this vanishes with $\nabla^2 \log h$ which is *identically* zero for all conjugate functions. See Note at end of volume.

Circular Plate Symmetrically loaded and supported.

402.] **The General Expression for the Displacement.** A circular plate of radius C is placed so that its unstrained plane is horizontal, and loaded and supported in a perfectly symmetrical manner about its centre: required the general expression for the vertical downward displacement of any point.

If Oz be directed vertically downwards through the centre it is evident that the load, and the boundary forces and couples (if any) must be functions of r only (§ 244). Thus taking the conjugate coördinates of Example 4 (*i*), page 258,

$$\xi = \log(r/C), \quad \eta = \theta, \qquad (70)$$

all the quantities involved will be independent of η. The general equation (67) of equilibrium thus reduces to

$$h^2\frac{d^2}{d\xi^2} \cdot h^2\frac{d^2w}{d\xi^2} = \frac{2\rho\tau\mathfrak{Z}}{\mathfrak{s}+\mathfrak{a}},$$

or, since $h=1/r$ and $d/d\xi = r \,.\, d/dr$,

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr} = \frac{2\rho\tau\mathfrak{Z}}{\mathfrak{s}+\mathfrak{a}} \qquad (71)$$

Integrating four times we have

$$w = \frac{2\rho\tau}{\mathfrak{s}+\mathfrak{a}}\int_0^r \frac{dr}{r}\int_0^r r\,dr\int_0^r \frac{dr}{r}\int_0^r r\mathfrak{Z}\,dr$$
$$+ \tfrac{1}{4}C'r^2(\log r - 1) + \tfrac{1}{4}C''r^2 + C'''\log r + C'''', \qquad (72)$$

where C', C'', C''', C'''' are arbitrary constants. Since however it is clear from symmetry that the tangent plane to the strained plate at the centre will be horizontal, we must put

$$C''' = 0 \qquad (73)$$

403.] **The Boundary Conditions.** Equations (70) and (71) reduce in this case to

$$\mathbf{P} = \rho\tau\left[\int_0^C \frac{dr}{r}\int_0^r r\mathfrak{Z}\,dr - \frac{2\mathfrak{a}}{(\mathfrak{s}+\mathfrak{a})C^2}\int_0^C r\,dr\int_0^r \frac{dr}{r}\int_0^r r\mathfrak{Z}\,dr\right]$$
$$+ \tfrac{1}{2}C'(\mathfrak{s}\log C + \tfrac{1}{2}\mathfrak{a}) + \tfrac{1}{2}\mathfrak{s}C'' \qquad (74)$$

$$\mathfrak{H} = -\frac{\rho}{C}\int_0^C r\mathfrak{Z}\,dr - \frac{(\mathfrak{s}+\mathfrak{a})C'}{2\tau C} \qquad (75)$$

404.] **Plate under Gravity, supported by its Centre only.** In this case $\mathfrak{Z} = \mathfrak{g}$, and $w=0$ when $r=0$; also, since the edge is free, $\mathbf{P}=0$, $\mathfrak{H}=0$. Thus $C''''=0$, $C' = -\mathfrak{g}\rho\tau C^2/(\mathfrak{s}+\mathfrak{a})$, $C'' = \mathfrak{g}\rho\tau C^2[\log C - \frac{1}{2} + 4\mathfrak{a}/\mathfrak{s}]/(\mathfrak{s}+\mathfrak{a})$, and finally

$$w = \frac{\mathfrak{g}\rho\tau r^2}{4(\mathfrak{s}+\mathfrak{a})}\left[r^2 - C^2\left(\log\frac{r}{C} - \frac{\mathfrak{s}+8\mathfrak{a}}{2\mathfrak{s}}\right)\right] \qquad (76)$$

Normal Vibrations of Thin Shells under Normal Forces.

405.] **Formation of the Variational Equation of Motion.** We now advance from the consideration of thin plates to that of thin shells, subject only to normal impressed forces and (if open shells) to surface tractions applied to the edges only, and such that the component tensions in the tangent plane to the shell at each point of its edge reduce to couples.

A portion of a shell, as defined in § 384, may be taken of such small superficial dimensions that in its natural state it is practically plane, while the *change of curvature* produced in it by the strain is practically uniform. Thus, if the principal curvatures at any point of the shell be increased from $\bar{\Pi}_1$, $\bar{\Pi}_2$, to Π_1, Π_2, we are led, as in § 377, to the assumptions (*i.*) that the couples per unit length across the principal normal surfaces at any point of a thin shell are

$$\left.\begin{aligned} \mathbf{P}_1 &= \frac{q\tau^3}{12(1-\sigma^2)}[\Pi_1 - \bar{\Pi}_1 + \sigma(\Pi_2 - \bar{\Pi}_2)] \\ \mathbf{P}_2 &= \frac{q\tau^3}{12(1-\sigma^2)}[\Pi_2 - \bar{\Pi}_2 + \sigma(\Pi_1 - \bar{\Pi}_1)] \end{aligned}\right\} \quad \text{.............(77)}$$

where τ is the thickness of the shell *at the point*, and (*ii.*) that the potential energy, per unit of unstrained superficial area of the plate, is

$$\mathfrak{W} = \frac{q\tau^3}{24(1-\sigma^2)}\{(\Pi_1 - \bar{\Pi}_1 + \Pi_2 - \bar{\Pi}_2)^2 - 2(1-\sigma)(\Pi_1 - \bar{\Pi}_1)(\Pi_2 - \bar{\Pi}_2)\} \quad \text{..........(78)}$$

Since the thickness τ is, in general, a function of the position of the point on the plate, it is convenient to change our notation for the coefficients of flexion. With the notation of Chapter V., let the surfaces of the plate be represented by $\xi = C$, $\xi = C + \kappa$, where κ is a small quantity of the first order in comparison with the range of value of η and ζ over the surfaces of the plate (§ 384). Then the thickness τ at the point (C, η, ζ) will be (§ 230)

$$\tau = \kappa / h_1, \quad \text{..................................(79)}$$

h_1 having of course the value it assumes when $\xi = C$, and being in consequence a function of η and ζ. Thus if we write

$$\mathrm{s} = \frac{q\kappa^3}{12(1-\sigma)}, \quad \mathrm{a} = \frac{q\kappa^3}{12(1+\sigma)}, \quad \text{.........................(80)}$$

s and a will be absolute constants. We also have, by equations (10) of § 232,

$$\bar{\Pi}_1 = {}_\xi\varpi_\eta = \frac{h_1}{h_2}\frac{\partial h_2}{\partial \xi}, \quad \bar{\Pi}_2 = {}_\xi\varpi_\zeta = \frac{h_1}{h_3}\frac{\partial h_3}{\partial \xi}, \quad \text{.................(81)}$$

where we again are to make $\xi = C$ after differentiation, and (78) may now be written

$$\mathfrak{W} = \frac{\mathrm{s}+\mathrm{a}}{4h_1^3}(\Pi_1 + \Pi_2 - {}_\xi\varpi_\eta - {}_\xi\varpi_\zeta)^2$$
$$- \frac{\mathrm{a}}{h_1^3}(\Pi_1 - {}_\xi\varpi_\eta)(\Pi_2 - {}_\xi\varpi_\zeta). \qquad (82)$$

The vibrations being supposed normal, η and ζ will remain constant for each point, and the only effect of the strain will be to change the value of ξ from C to $C+a$, where a is a small quantity of the first order, and in general a function of η and ζ. The normal velocity at time t will be (§ 237) $\dot{a}/h_1$, and the kinetic energy per unit of unstrained superficial area will be $\rho\kappa\dot{a}^2/2h_1^3$. Thus, the element of surface (§ 230) being $d\eta d\zeta/h_2h_3$, if we write for the normal impressed force per unit area

$$\frac{1}{h_1}\int_C^{C+\kappa} \Xi d\xi = \kappa\mathfrak{X}/h_1, \qquad (83)$$

the variational equation of motion will be

$$\iint \{\tfrac{1}{4}(\mathrm{s}+\mathrm{a})\delta(\Pi_1 + \Pi_2 - {}_\xi\varpi_\eta - {}_\xi\varpi_\zeta)^2 - \mathrm{a}\delta(\Pi_1 - {}_\xi\varpi_\eta)(\Pi_2 - {}_\xi\varpi_\zeta)$$
$$+ \rho\kappa(\ddot{a} - h_1\mathfrak{X})\delta a\}\frac{d\eta d\zeta}{h_1^3h_2h_3} - \int\left[\left(\frac{\kappa\mathfrak{H}}{h_1} - \frac{d\mathbf{Q}}{ds}\right)\frac{\delta a}{h_1} + \mathbf{P}h_1\frac{\partial}{\partial\nu}\left(\frac{\delta a}{h_1^2}\right)\right]ds = 0, \quad (84)$$

where $\kappa\mathfrak{H}/h_1$ is the shearing force per unit length applied to the edge in a direction perpendicular to the faces, $\mathbf{P}$ is the couple per unit length in the plane parallel to this force and perpendicular to the edge (flexion couple), and $\mathbf{Q}$ is the couple per unit length in the tangent plane to the edge. In an edge formed by a portion of an η surface $ds = d\zeta/h_3$, $d\nu = d\eta/h_2$, and in an edge formed by a portion of a ζ surface $ds = d\eta/h_2$, $d\nu = d\zeta/h_3$.

In default of general formulæ, analogous to (66), giving the sum and product of the increments of the principal curvatures in terms of a and its derivatives as to η and ζ, the equation of motion and boundary conditions cannot be obtained in general terms, but each case must be solved separately from this point.

406.] **Case in which the surfaces of the shell remain always members of the family to which they initially belong.*** If we suppose the vibration to be of this kind, a will

* Examples:—(*i.*) a shell bounded by concentric spheres performing normal vibrations symmetrical about the centre, (*ii.*) an ellipsoidal shell with confocal surfaces, vibrating normally so that the surfaces remain confocal with their initial forms, etc.

of course be independent of η and ζ (§ 242), and we shall have simply

$$\left.\begin{aligned}\Pi_1 &= {}_\xi\varpi_\eta + a\frac{\partial\,{}_\xi\varpi_\eta}{\partial\xi}\\ \Pi_2 &= {}_\xi\varpi_\zeta + a\frac{\partial\,{}_\xi\varpi_\zeta}{\partial\xi}\end{aligned}\right\}\text{..............................}(85)$$

so that equation (84) reduces to

$$\rho\kappa\ddot{a}\iint\frac{d\eta d\zeta}{h_1{}^3h_2h_3}$$
$$+\,a\iint\left\{\frac{\mathrm{s}+\mathrm{a}}{2}\left[\frac{\partial({}_\xi\varpi_\eta+{}_\xi\varpi_\zeta)}{\partial\xi}\right]^2-2\mathrm{a}\frac{\partial\,{}_\xi\varpi_\eta}{\partial\xi}\frac{\partial\,{}_\xi\varpi_\zeta}{\partial\xi}\right\}\frac{d\eta d\zeta}{h_1{}^3h_2h_3}$$
$$-\,\rho\kappa\iint\frac{\mathfrak{X}d\eta d\zeta}{h_1{}^2h_2h_3}=0\text{..............................}(86)$$

The boundary condition to be satisfied, in all cases in which the assumed mode of vibration does not require the edge to remain fixed, is

$$\kappa\mathfrak{Z}-h_1\frac{d\mathbf{Q}}{ds}-2\mathbf{P}\frac{\partial h_1}{\partial\nu}=0\text{..............................}(86a)$$

The *periods* of possible vibrations of this kind are independent of the impressed forces (unless these are periodic), and can be ascertained when the form and dimensions of the shell are given.

Example.

407.] **A spherical shell of radius C and small uniform thickness τ, performs free radial vibrations symmetrical about a diameter, the amplitude of the displacement being proportional to a zonal harmonic. Required the periodic time of the vibration.*** Let Oz be the axis of symmetry, and u the radial displacement. Then, with the notation of § 243, u is independent of ω, and equation (84) of § 405 reduces to

$$\int_0^\pi\{\tfrac{1}{2}(\mathrm{s}+\mathrm{a})\delta(\Pi_1+\Pi_2-2/C)^2-\mathrm{a}\delta(\Pi_1-1/C)(\Pi_2-1/C)$$
$$+\,\rho\tau\ddot{u}\delta u\}\sin\theta\delta\theta=0\text{..............................}(87)$$

But if P be the point $(C+u,\,\theta)$ on the strained shell, and PG the normal at P, meeting Oz in G, and making an angle ψ with Oz we have $r=OP=C+u$,

$$\Pi_1=\frac{1}{r}+\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right),\quad \Pi_2=\frac{1}{PG}.$$

* Professor C. Niven, Mathematical Tripos, 1878.

$$\therefore \qquad \Pi_1 = \frac{1}{C}\left(1 - \frac{u}{C}\right) - \frac{1}{C^2}\frac{d^2u}{d\theta^2},$$

and
$$\Pi_2 = \frac{1}{r}\frac{\sin(\theta+\psi)}{\sin\theta} = \frac{1}{r}(1+\psi\cot\theta)$$
$$= \frac{1}{r}\left(1 - \cot\theta\frac{dr}{rd\theta}\right) = \frac{1}{C}\left(1 - \frac{u}{C}\right) - \frac{\cot\theta}{C^2}\frac{du}{d\theta}.$$

Thus, if we write
$$\cos\theta = p, \quad p\frac{du}{dp} - u = \phi,$$

we shall have
$$\left.\begin{aligned} \Pi_1 - \frac{1}{C} &= -\frac{1}{C^2}\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{1}{C^2}\left[(1-p^2)\frac{d^2u}{dp^2} - p\frac{du}{dp} + u\right] \\ &= -\frac{1}{C^2}\left(\frac{d^2u}{dp^2} - p\frac{d\phi}{dp} - \phi\right) \\ \Pi_2 - \frac{1}{C} &= -\frac{1}{C^2}\left(\cot\theta\frac{du}{d\theta} + u\right) = \frac{1}{C^2}\left(p\frac{du}{dp} - u\right) = \frac{\phi}{C^2}. \end{aligned}\right\},$$

and consequently
$$\left.\begin{aligned} \left(\Pi_1 - \frac{1}{C}\right)\left(\Pi_2 - \frac{1}{C}\right) &= -\frac{1}{C^4}\left[\left(p\frac{du}{dp} - u\right)\frac{d^2u}{dp^2} - p\phi\frac{d\phi}{dp} - \phi^2\right] \\ \left(\Pi_1 + \Pi_2 - \frac{2}{C}\right) &= -\frac{1}{C^2}\left[(1-p^2)\frac{d^2u}{dp^2} - 2p\frac{du}{dp} + 2u\right] \\ &= -\frac{1}{C^2}\left\{\frac{d}{dp}\left[(1-p^2)\frac{du}{dp}\right] + 2u\right\} \end{aligned}\right\};$$

so that equation (87) may be written
$$\frac{\mathrm{s}+\mathrm{a}}{4C^4}\int_{-1}^{1}\delta\left\{\frac{d}{dp}\left[(1-p^2)\frac{du}{dp}\right] + 2u\right\}^2 dp$$
$$+\frac{\mathrm{a}}{C^4}\int_{-1}^{1}\delta\left\{\left(p\frac{du}{dp} - u\right)\frac{d^2u}{dp^2} - p\phi\frac{d\phi}{dp} - \phi^2\right\}dp$$
$$+\rho\tau\int_{-1}^{1}\ddot{u}\delta u\,dp = 0.$$

Since $1-p^2=0$ at both limits, the first line reduces, after integration by parts, to
$$\frac{\mathrm{s}+\mathrm{a}}{2C^4}\int_{-1}^{1}\left\{\frac{d}{dp}\left[(1-p^2)\frac{d}{dp}\right] + 2\right\}^2 u\,.\,\delta u\,dp,$$

while the second line is equal to
$$\frac{\mathrm{a}}{2C^4}\delta\left\{u^2_{p=-1} - u^2_{p=1} + \int_{-1}^{1}\left[\left(\frac{du}{dp}\right)^2 - \phi^2\right]dp\right\},$$

or $$\frac{\mathrm{a}}{2C^4}\delta\left\{u^2_{p=-1} - u^2_{p=1} + \int_{-1}^{1}\left[(1-p^2)\left(\frac{du}{dp}\right)^2 + 2pu\frac{du}{dp} - u^2\right]dp\right\},$$

or $$\frac{\mathrm{a}}{2C^4}\int_{-1}^{1}\delta\left\{(1-p^2)\left(\frac{du}{dp}\right)^2 - 2u^2\right\}dp,$$

or $$-\frac{\mathrm{a}}{C^4}\int_{-1}^{1}\left\{\frac{d}{dp}\left[(1-p^2)\frac{d}{dp}\right] + 2\right\}u\,.\,\delta u\,dp.$$

Thus the equation of motion is

$$(\mathrm{s}+\mathrm{a})\left\{\frac{d}{dp}\left[(1-p^2)\frac{d}{dp}\right] + 2\right\}^2 u$$
$$- 2\mathrm{a}\left\{\frac{d}{dp}\left[(1-p^2)\frac{d}{dp}\right] + 2\right\}u + 2C^4\rho\tau\ddot{u} = 0.$$

If we now assume that the displacement is of the form

$$u = A\sin it\,.\,\mathbf{P}_j(\cos\theta)$$

where $\mathbf{P}$ denotes a Legendre's coefficient, we have $\ddot{u} = -i^2u$ and

$$\left\{\frac{d}{dp}\left[(1-p^2)\frac{d}{dp}\right] + 2\right\}u = -(j-1)(j+2)u,$$

so that the equation of motion will be satisfied if

$$i^2 = \frac{(j-1)(j+2)}{2C^4\rho\tau}[(\mathrm{s}+\mathrm{a})(j-1)(j+2) + 2\mathrm{a}]$$
$$= \frac{q\tau^2(j-1)(j+2)}{12(1-\sigma^2)\rho C^4}[(j-1)(j+2) + 1 - \sigma],$$

whence we deduce the required periodic time $2\pi/i$.

EXAMPLES.

1. The most general solution of Clebsch' Problem, consistent with the absence of shear, is of the form

$$u = A_1x - \varpi_1zx + A_2xy - \tfrac{1}{2}B_2(\sigma x^2 + y^2 - \sigma z^2) - \tfrac{1}{2}C_3x(\tfrac{1}{3}\sigma x^2 + y^2 - \sigma z^2)$$
$$v = B_1y - \varpi_2yz + B_2xy - \tfrac{1}{2}A_2(x^2 + \sigma y^2 - \sigma z^2) + \tfrac{1}{2}C_3y(x^2 + \tfrac{1}{3}\sigma y^2 - \sigma z^2)$$
$$w = -\frac{\sigma(A_1+B_1)z}{1-\sigma} + \tfrac{1}{2}(\varpi_1x^2 + \varpi_2y^2) + \frac{\sigma(\varpi_1+\varpi_2)z^2}{2(1-\sigma)}$$
$$- \sigma A_2yz - \sigma B_2zx - \tfrac{1}{2}\sigma C_3z(x^2 - y^2).$$

2. Find the vertical depression under gravity of the centre of a uniform circular plate, whose edge is supported all round by a rigid horizontal circular frame.

3. If a thin circular plate be supported by its centre so that the tangent plane there is horizontal, the ratio of the vertical depression of the edge when the weight is uniformly distributed over the plate to that when the weight is concentrated uniformly round the edge is $7+3\sigma : 12+4\sigma$.

4. A uniform rectangular board, of length $2L$, breadth $2C$ and weight $\mathbf{W}$, is hinged all along its four edges to a fixed rigid horizontal frame. Show that a possible form of equilibrium is given by

$$32\mathfrak{s}LCw = (1+\sigma)\rho\tau\mathbf{W}(L^2-x^2)(C^2-y^2)$$

the origin being at the centre of the frame, and Ox, Oy being parallel to the edges.

Find the distribution of couple which must be applied to the edges to maintain this configuration.

5. A uniform plate of infinite length, of breadth C and thickness τ, is fixed along the middle line of one of its edges, and is acted on along the opposite edge by a tangential force perpendicular to the plane of the plate, the resultant magnitude of which per unit length is $\tau\mathfrak{H}$. Verify that, if the fixed line be taken for the axis of y, the conditions of equilibrium are satisfied by the displacements

$$\left.\begin{aligned} u &= -\frac{\tau\mathfrak{H}}{\mathfrak{n}}\left[\frac{(1-\sigma)(2Cx-x^2)z}{2}+\frac{(2-\sigma)z^3}{6}+\frac{\tau^2 z}{4}\right] \\ v &= 0 \\ w &= \frac{\tau\mathfrak{H}}{\mathfrak{n}}\left[\frac{\sigma(C-x)z^2}{2}+(1-\sigma)\left(\frac{Cx^2}{2}-\frac{x^3}{6}\right)\right] \end{aligned}\right\}.$$

6. A spherical shell of radius A, and uniform thickness τ, performs small radial vibrations symmetrical about its centre. Show that the periodic time is $\pi A^3\sqrt{2\rho\tau/\mathfrak{s}}$.

7. A thin shell is contained by two confocal spheroids of revolution, whose major and minor semi-axes are A, B; $\sqrt{A^2-\kappa}$, $\sqrt{B^2-\kappa}$ respectively. If the shell performs small normal vibrations in such a manner that its surfaces remain always confocal with their initial forms show that the periodic time is

$$8\pi A^3\sqrt{\rho\kappa\lambda^3(3+4\lambda^2+8\lambda^4)/30(\mathfrak{s}+\mathfrak{a})C_1}$$

or

$$8\pi A^3\sqrt{\rho\kappa\lambda^6(8+4\lambda^2+3\lambda^4)/30(\mathfrak{s}+\mathfrak{a})C_2},$$

according as the shell is *oblate* or *prolate*, where $\lambda = B/A$ and

$$\left.\begin{aligned} C_1 &= (\tfrac{17}{24} + 2\sigma)\lambda - (\tfrac{11}{4} - 3\sigma)\lambda^3 + (1 - 2\sigma)\lambda^5 + \tfrac{2}{3}\lambda^7 \\ &\quad + \frac{\cos^{-1}\lambda}{\sqrt{1-\lambda^2}}[\tfrac{11}{8} - (1 + 5\sigma)\lambda^2 + 2(1 + 2\sigma)\lambda^4], \\ C_2 &= \tfrac{2}{3} + (1 - 2\sigma)\lambda^2 - (\tfrac{11}{4} - 3\sigma)\lambda^4 + (\tfrac{17}{24} + 2\sigma)\lambda^6 \\ &\quad + \frac{\log\dfrac{1+\sqrt{1-\lambda^2}}{\lambda}}{\sqrt{1-\lambda^2}}[2(1 + 2\sigma)\lambda^4 - (1 + 5\sigma)\lambda^6 + \tfrac{11}{8}\lambda^8] \end{aligned}\right\}.$$

[Employ the equation of § 406, with the notation of § 251. The unstrained surfaces of the shell will be given by $\xi = 0$, $\xi = \kappa$. See *Proceedings* of the Cambridge Philosophical Society, Vol. V., pp. 68 and 312.]

CHAPTER IX.

IMPACT.

408.] **Definitions and Fundamental Principles.** Under the general term *Impact* we include all cases of sudden change of strain, of sudden application of stress to a body hitherto in its natural state (such as may be caused by the shock of contact with another body), and of sudden release from strain of a body hitherto in equilibrium under stress.

It is of course impossible for any but infinitely great forces to produce *finite* strain instantaneously, or in an infinitely short time, and hence it follows that a finite stress requires a finite time for its application or removal. We may however suppose that strains and stresses of such magnitude as we have dealt with in this work may be applied or removed, by continuous increase from or decrease to zero, in periods of time quite insignificant in comparison with the finite times of their subsequent application, or during which their effects last: and all such cases may be treated *analytically* as if stress were applied instantaneously, or as if it were within the order of magnitudes which we are discussing from the very moment that its effects begin.

These effects take the form of small straining vibrations, into the kinetic energy of which is transformed a portion of the initial energy possessed by the body before the impact—whether potential energy of strain (as in the case of a body suddenly released), or kinetic energy of bodily translation (as in the case of a body suffering collision).

Such rapid applications or removals of stress as we here suppose will in reality (§§ 21-26) generate local variations of temperature, and consequently cause dissipation of energy by conduction of heat, etc. For reasons, however, which have already been fully discussed in Chapter I., we leave out of account all disturbances due to changes of temperature, and we are therefore reduced to the artificial assumption that *The total energy of any system of perfectly elastic* (or rigid) *bodies between which impacts take place—reckoned by summing up for all the bodies of the system* (i.) *the kinetic energy of each*

due to any motion of translation or rotation of the body as a whole, as well as to small straining vibrations propagated through its mass, and (ii.) *the potential energy of each due to the strain at any instant—is an absolutely constant quantity, unaffected by any number of impacts between bodies exclusively belonging to the system.* This is the fundamental Principle of the Conservation of Energy.

Of course, in all cases in which impressed forces act upon all or any of the bodies in the system, the work done by or against them must be taken into account in applying this principle.

Again, the measure of the impact, in cases of collision, is the amount of *Momentum* * impulsively communicated to one of the colliding bodies and taken from the other during the period of contact. We shall confine ourselves exclusively to cases of *direct* collision; in which the colliding bodies move as a whole in parallel directions, the surfaces of contact being normal to the direction of motion. In all such cases the impact is entirely in the direction of initial motion, and consequently *the resultant momentum of either body in any perpendicular direction will be zero after, as before, the impact.* Further, since the momentum in the direction of impact lost by one body is gained by the other (impact being a purely mutual reaction, like all other results of stress) *the resultant momentum of the system in the direction of initial motion is unaffected by impact.* These two statements express the fundamental Principle of the Conservation of Momentum.

In all cases of collision with ideally rigid and fixed obstacles, or of release from rigid fixed supports, the elastic body retains the equivalents of its entire initial energy. In such cases its subsequent state of motion may be called the state of Equivalent Motion, or—if there be no translation of the body as a whole—of equivalent vibration. In these cases also, for obvious reasons, we can only utilise the principle of conservation of momentum in directions perpendicular to that of impact.

Example of Sudden Release.

409.] **A uniform rod of length L is stretched to a uniform extension ϵ and held thus in equilibrium; required the effect of suddenly letting one end go, the other remaining fixed.** Obviously there will be no tendency to torsion or flexion, and the displacement of any point in the central axis of the rod will therefore be purely longitudinal. Taking Oz

* The momentum of an elastic body must of course be taken as the algebraic sum of the momenta of all its elements.

coincident with this axis, O being at the permanently fixed end, and expressing by w the excess of the distance from O of any point in the axis, at time t from release, over its distance in the natural state of the bar, w will evidently satisfy the equations of § 365, so that

$$\Omega_1^2\frac{\partial^2 w}{\partial z^2}-\frac{\partial^2 w}{\partial t^2}=0 \quad\ldots\ldots(1)$$

Since t is reckoned from the instant at which motion begins, we must obtain a solution of (1) which will make $w=\epsilon z$, $\dot{w}=0$, when $t=0$, for all values of z from O to L; and we are also to have $w=0$ when $z=0$, and $\partial w/\partial z=0$ when $z=L$, for all values of t.

The form of the solution is clearly

$$w=\sum_{i=0}^{i=\infty}A_i\sin\frac{(2i+1)\pi z}{2L}\cos\frac{(2i+1)\pi\Omega_1 t}{2L},\quad\ldots\ldots(2)$$

for this satisfies (1) and three of the limiting conditions identically, and we have now only to determine A_i so that

$$\sum_{i=0}^{i=\infty}A_i\sin\frac{(2i+1)\pi z}{2L}=\epsilon z,$$

for all values of z from O to L. Hence we find, by Fourier's theory, that

$$A_i=\frac{8L\epsilon}{\pi^2}\frac{(-1)^i}{(2i+1)^2},$$

and consequently

$$w=\frac{8L\epsilon}{\pi^2}\sum_{i=0}^{i=\infty}\frac{(-1)^i}{(2i+1)^2}\sin\frac{(2i+1)\pi z}{2L}\cos\frac{(2i+1)\pi\Omega_1 t}{2L}\quad\ldots\ldots(3)$$

[It may be observed that, when $t=0$ and $z=L$, this expression reduces to

$$\begin{aligned}w&=\frac{8L\epsilon}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)\\&=\epsilon L,\end{aligned}$$

as of course it should.]

The equation expressing the principle of conservation of energy is

$$\int_0^L\left\{\frac{\rho}{2}\left(\frac{\partial w}{\partial t}\right)^2+\frac{q}{2}\left(\frac{\partial w}{\partial z}\right)^2\right\}\mathfrak{A}\,dz=\tfrac{1}{2}q\epsilon^2 L\mathfrak{A},$$

where $\mathfrak{A}$ is the transverse section; or

$$\int_0^L\left\{\left(\frac{\partial w}{\partial t}\right)^2+\Omega_1^2\left(\frac{\partial w}{\partial z}\right)^2\right\}dz=L\epsilon^2\Omega_1^2\quad\ldots\ldots(4)$$

This is easily reduced, on substitution from (3), to

$$\frac{16\epsilon^2\Omega_1^{\,2}}{\pi^2}\sum_{i=0}^{i=\infty}\frac{1}{(2i+1)^2}\int_0^L\left\{\sin^2\frac{(2i+1)\pi z}{2L}\cos^2\frac{(2i+1)\pi\Omega_1 t}{2L}\right.$$
$$\left.+\cos^2\frac{(2i+1)\pi z}{2L}\sin^2\frac{(2i+1)\pi\Omega_1 t}{2L}\right\}dz = L\epsilon^2\Omega_1^{\,2},$$

or

$$\frac{8\epsilon^2 L\Omega_1^{\,2}}{\pi^2}\sum_{i=0}^{i=\infty}\frac{1}{(2i+1)^2} = L\epsilon^2\Omega_1^{\,2},$$

which is an identity.

Whenever the time from release is an odd multiple of L/Ω_1, or the time required for a sound vibration to travel the length of the rod, $w=0$ for all values of z and the rod passes through its *natural* state. Whenever the time is an even multiple of $2L/\Omega_1$, $w=\epsilon z$, and the rod passes through its initial state of strain. Whenever the time is an odd multiple of $2L/\Omega_1$, $w=-\epsilon z$ and the initial state of strain is reversed.

The traction on the fixed end of the bar is

$$q\frac{\partial w}{\partial z}_{(z=0)} = \frac{4q\epsilon}{\pi}\sum_{i=0}^{i=\infty}\frac{(-1)^i}{2i+1}\cos\frac{(2i+1)\pi\Omega_1 t}{2L}.$$

This is equal to $q\epsilon$ from $t=0$ to $t=L/\Omega_1$, when it suddenly changes sign (§ 408) and is equal to $-q\epsilon$ from $t=L/\Omega_1$ to $t=2L\Omega'_1$, this cycle being repeated indefinitely in equal periods of time.

Example of Direct Collision with a Fixed Rigid Obstacle.

410.] **A Rod of length L moving with velocity U in the direction of its length, comes into direct collision with a fixed rigid wall. Required the subsequent motion.**
During the whole time that the rod is in contact with the wall the end in contact will be absolutely fixed. Thus if we take that end for origin, and the Axis of the rod for Oz, we have $\dot{w}=0$ when $t=0$ from $z=0$ to $z=L$, and $w=0$, $\dot{w}=0$ when $z=0$ during whole time of contact. Also, since the further end is free, $\partial w/\partial z=0$ when $z=L$ for all time.

The form of the displacement during contact with the wall is evidently

$$w = -\Sigma A_i\sin\frac{(2i+1)\pi z}{2L}\sin\frac{(2i+1)\pi\Omega_1 t}{2L},$$

for this satisfies all the foregoing conditions. The constant coefficients may be determined by the consideration that at the instant of impact every point in the body is moving with a velocity U towards the wall, and consequently, for an unappreciable interval following that instant every point in the body has

a velocity $\mathbf{U}$ *relative to the end O*. Thus we must have $\dot{w} = -\mathbf{U}$ when $t=0$ for all values of z between O and L, including the limit $z=L$ but excluding the limit $z=0$ (where $\dot{w}=0$).

Now the series

$$-\frac{4\mathbf{U}}{\pi}\sum \frac{1}{2i+1} \sin \frac{(2i+1)\pi z}{2L}$$

$= -\mathbf{U}$ from $z=0$ to $z=2L$, exclusive of both limits, and vanishes when $z=0$. Thus we shall satisfy this condition by making

$$-\frac{\pi\Omega_1}{2L}\Sigma(2i+1)A_i \sin \frac{(2i+1)\pi z}{2L} = -\frac{4\mathbf{U}}{\pi}\sum\frac{1}{2i+1}\sin\frac{(2i+1)\pi z}{2L},$$

or

$$A_i = \frac{8L\mathbf{U}}{\pi^2\Omega_1}\cdot\frac{1}{(2i+1)^2},$$

and consequently

$$w = -\frac{8L\mathbf{U}}{\pi^2\Omega_1}\sum_{i=0}^{i=\infty}\frac{1}{(2i+1)^2}\sin\frac{(2i+1)\pi z}{2L}\sin\frac{(2i+1)\pi\Omega_1 t}{2L} \quad \text{........(5)}$$

The equation of conservation of energy is

$$\tfrac{1}{2}\rho\mathfrak{A}\int_0^L \left\{ \Omega_1^2\left(\frac{\partial w}{\partial z}\right)^2 + \dot{w}^2 \right\} dz = \tfrac{1}{2}\rho\mathfrak{A}\mathbf{U}^2 L$$

which reduces to the identity

$$\frac{8\mathbf{U}^2 L}{\pi^2}\sum\frac{1}{(2i+1)^2} = \mathbf{U}^2 L.$$

At the instant when $t=2L/\Omega_1$ $w=0$ throughout, and

$$\dot{w} = \frac{4\mathbf{U}}{\pi}\sum\frac{1}{2i+1}\sin\frac{(2i+1)\pi z}{2L}$$
$$= \mathbf{U} \text{ throughout.}$$

At that moment therefore the rod is instantaneously in its natural state, and is moving bodily from the wall with velocity $\mathbf{U}$. Contact consequently ceases after a period $2L/\Omega_1$ from the first impact.

Confining ourselves for the present to the period of contact, we deduce from (5)

$$\begin{aligned}\frac{\partial w}{\partial z} &= -\frac{4\mathbf{U}}{\pi\Omega_1}\sum\frac{1}{2i+1}\sin\frac{(2i+1)\pi z}{2L}\sin\frac{(2i+1)\pi\Omega_1 t}{2L}\\ &= -\frac{4\mathbf{U}}{\pi\Omega_1}\left\{\sum\frac{1}{2i+1}\sin\frac{(2i+1)\pi(z+\Omega_1 t)}{2L}\right.\\ &\qquad \left.-\sum\frac{1}{2i+1}\sin\frac{(2i+1)\pi(z-\Omega_1 t)}{2L}\right\} \quad \text{.....................(6)}\end{aligned}$$

Theoretically, the first of these series $=\frac{1}{4}\pi$ from $z+\Omega_1 t=0$ to $z+\Omega_1 t=2L$, and $=-\frac{1}{4}\pi$ from $z+\Omega_1 t=-2L$ to $z+\Omega_1 t=0$; also the second series $=\frac{1}{4}\pi$ from $z-\Omega_1 t=0$ to $z-\Omega_1 t=2L$, and $=-\frac{1}{4}\pi$ from $z-\Omega_1 t=-2L$ to $z-\Omega_1 t=0$. Thus, within the limits 0 and L for z, and 0 and $2L/\Omega_1$ for t, we see that:

(*i.*) If $t<L/\Omega_1$ the first series $=\frac{1}{4}\pi$ from $z=0$ to $z=L$, while the second series $=-\frac{1}{4}\pi$ from $z=0$ to $z=\Omega_1 t$ and $=\frac{1}{4}\pi$ from $z=\Omega_1 t$ to $z=L$.

(*ii.*) If $t>L/\Omega_1$, we get by putting $t=L/\Omega_1+t'$ and $z=L-z'$ in (6)

$$\frac{\partial w}{\partial z}=-\frac{2U}{\pi\Omega_1}\left\{\sum\frac{1}{2i+1}\sin\frac{(2i+1)\pi(z'-\Omega_1 t')}{2L}\right.$$
$$\left.+\sum\frac{1}{2i+1}\sin\frac{(2i+1)\pi(z'+\Omega_1 t')}{2L}\right\}.$$

The first of these series $=-\frac{1}{4}\pi$ from $z'=0$ to $z'=\Omega_1 t-L$ and $=\frac{1}{4}\pi$ from $z'=\Omega_1 t-L$ to $z'=L$, while the second series $=\frac{1}{4}\pi$ from $z'=0$ to $z'=L$. Thus the first series $=\frac{1}{4}\pi$ from $z=0$ to $z=2L-\Omega_1 t$ and $=-\frac{1}{4}\pi$ from $z=2L-\Omega_1 t$ to $z=L$, while the second series $=\frac{1}{4}\pi$ from $z=0$ to $z=L$.

Summing up results, it follows that

(*i.*) from $t=0$ to $t=L/\Omega_1$ $\frac{\partial w}{\partial z}=-\frac{\mathbf{U}}{\Omega_1}$ from $z=0$ to $z=\Omega_1 t$, and $\frac{\partial w}{\partial z}=0$ from $z=\Omega_1 t$ to $z=L$.

(*ii.*) from $t=L/\Omega_1$ to $t=2L/\Omega_1$, $\frac{\partial w}{\partial z}=-\frac{\mathbf{U}}{\Omega_1}$ from $z=0$ to $z=2L-\Omega_1 t$, and $\frac{\partial w}{\partial z}=0$ from $z=2L-\Omega_1 t$ to $z=L$.

Thus a portion of the rod next the wall suffers uniform compression of amount $\mathbf{U}/\Omega_1$, the remainder being free from strain; and the geometrical surface separating the two portions advances from the fixed end with uniform velocity Ω_1 along the rod, is reflected at the free end, and returns with the same velocity, reaching the wall again at the instant $(t=2L/\Omega_1)$ when contact ceases.

The thrust on the end in contact with the wall is $\epsilon\mathbf{U}/\Omega_1$ or $\mathfrak{A}\mathbf{U}\sqrt{qp}$ throughout the duration of contact.

At the instant when the rod leaves the wall it is unstrained, and every point is moving with the initial velocity $\mathbf{U}$ reversed. Since no forces act on the rod, its centre of gravity will continue to move with the same velocity $\mathbf{U}$, and the kinetic energy due to the motion of its centre of gravity alone will be equal to its kinetic energy before the impact. Hence the kinetic energy of

motion of the parts of the rod relatively to its centre of gravity is zero, and consequently no such motion can take place. The rod therefore retreats in its initial unstrained condition and with its initial speed **U**.

EXAMPLES.

1. Two uniform heavy beams AB, CD, equal in every respect, are connected by a weightless inelastic string BC; the beam AB lies unstrained on a smooth rigid horizontal table, while CD is suspended at rest under the action of gravity by the string which, being held at B, passes over a small smooth pulley at the edge of the table, and in one line with AB produced. Investigate the motion of the string when set free; prove that its tension, after being instantaneously diminished by one half, remains constant, and that its velocity receives equal increments at equal intervals.

2. Example 3 on Chapter VII. (page 449) may be treated as a case of sudden release by the method of § 409.

3. Prove that if we make $\gamma_i = 0$ in equation (80) of § 271 it will represent the vibrations excited in an infinite plate of thickness l, moving with velocity **U**, on its median plane being instantaneously brought to rest.

4. A uniform elastic bar is suspended vertically by one end, and to the other is attached a weight **W**, which is supported so that the bar is unstrained (the effect of gravity upon it being neglected). If the weight be suddenly set free, investigate the motion of the system.

5. Prove that if an elastic bar, of length L, impinges directly with velocity **U** on a longer bar, of length pL and the same cross section, the first bar will be reduced to rest by the impact, while the second bar will appear to move forward by successive advances of the ends with velocity **U** for intervals of time $2L/\Omega_1$, alternating with intervals of rest of duration $2(p-1)L/\Omega_1$.

6. If the revolution of the square described in Example 9 on Chapter VII. (page 451) be suddenly stopped by its sides striking simultaneously a smooth fixed rigid plane, prove that the displacement at any subsequent time t from the impact will be given by

$$\frac{4\omega^{\frac{5}{2}}}{\lambda}\sum\frac{\tan(i\lambda L/2\omega^{\frac{1}{2}})}{i(\omega^2 - i^4)}\left\{\frac{\cosh(i\lambda z/\omega^{\frac{1}{2}})}{\cosh(i\lambda L/2\omega^{\frac{1}{2}})} - \frac{\cos(i\lambda z/\omega^{\frac{1}{2}})}{\cos(i\lambda L/2\omega\ )}\right\}\cos i^2 t,$$

the summation extending to all values of i given by the equation

$$\tan \frac{i\lambda L}{2\omega^{\frac{1}{2}}} + \tanh \frac{i\lambda L}{2\omega^{\frac{1}{2}}} = 0.$$

7. A uniform circular disc is rotating about an axis through its centre, perpendicular to its plane. It is suddenly stopped by all the part within a concentric circle being rigidly clamped. Show that the strain at any point is a pure shear, and that the disc will have a tendency to split from the inner circle outwards, commencing at an angle of 45° with the radius.

Several practical examples on Impact will be found at the end of Chapter XVI. of Prof. Cotterill's "Applied Mechanics," which the student is strongly recommended to consult.

CHAPTER X

VISCOSITY.

411.] **Analytical Expression of the effects of Viscosity.** We have seen in Appendix IV. (pages 175-177) that the effect of viscosity, in an elastic body undergoing changing shear, is to introduce a shearing stress depending only upon the rate at which the shear increases, and, when this rate is small, directly proportional to it. We have also seen that a mere cubical dilatation or compression does not call any viscous resistance into play.

Since the *elastic* shearing stress is simply proportional to the absolute amount of shear, it is evident that the effect of viscosity will be taken into account if we replace the elastic shearing stress Σ by $\Sigma+\frac{\nu}{n}\frac{\partial\Sigma}{\partial t}$, where ν is the modulus of viscosity (page 177). We have seen in §§ 210-213 that the most general small strain can be resolved into dilatation and shears, and that, in the expressions for the stresses in an isotropic solid, the modulus of compression k appears only as a coefficient of dilatations, and the modulus of rigidity n only as a coefficient of shears. If then we express every coefficient in our linear equations of motion in terms of k and n, and then replace the coefficient n by the operator $\left(n+\nu\frac{\partial}{\partial t}\right)$, we shall have taken viscosity fully into account. It follows that the coefficient m must be replaced by the operator $\left(m+\frac{\nu}{3}\frac{\partial}{\partial t}\right)$.

412.] **Equations of Motion and Boundary Conditions for a Viscous Solid in Motion.** Taking equations (48) of § 239 as the most general form, and modifying them as just described, we have

$$\left.\begin{aligned}
&\left(m+n+\frac{4}{3}\nu\frac{\partial}{\partial t}\right)h_1\frac{\partial\Delta}{\partial\xi}-2\left(n+\nu\frac{\partial}{\partial t}\right)h_2h_3\left[\frac{\partial}{\partial\eta}\left(\frac{\Theta_3}{h_3}\right)-\frac{\partial}{\partial\zeta}\left(\frac{\Theta_2}{h_2}\right)\right]\\
&\qquad\qquad+\rho(\Xi-\ddot{u})=0\\
&\left(m+n+\frac{4}{3}\nu\frac{\partial}{\partial t}\right)h_2\frac{\partial\Delta}{\partial\eta}-2\left(n+\nu\frac{\partial}{\partial t}\right)h_3h_1\left[\frac{\partial}{\partial\zeta}\left(\frac{\Theta_1}{h_1}\right)-\frac{\partial}{\partial\xi}\left(\frac{\Theta_3}{h_3}\right)\right]\\
&\qquad\qquad+\rho(H-\ddot{v})=0\\
&\left(m+n+\frac{4}{3}\nu\frac{\partial}{\partial t}\right)h_3\frac{\partial\Delta}{\partial\zeta}-2\left(n+\nu\frac{\partial}{\partial t}\right)h_1h_2\left[\frac{\partial}{\partial\xi}\left(\frac{\Theta_2}{h_2}\right)-\frac{\partial}{\partial\eta}\left(\frac{\Theta_1}{h_1}\right)\right]\\
&\qquad\qquad+\rho(Z-\ddot{w})=0
\end{aligned}\right\}\text{........(1)}$$

and, similarly, the boundary conditions (45) of § 238 become

$$\left.\begin{aligned}
&\left[(m-n)\Delta+2ne-2\nu\left(\frac{\dot\Delta}{3}-\dot e\right)\right]h_1\frac{\partial\Phi}{\partial\xi}+(nc+\nu\dot c)h_2\frac{\partial\Phi}{\partial\eta}\\
&\qquad\qquad+(nb+\nu\dot b)h_3\frac{\partial\Phi}{\partial\zeta}=\mathfrak{h}\Xi'\\
&(nc+\nu\dot c)h_1\frac{\partial\Phi}{\partial\xi}+\left[(m-n)\Delta+2nf-2\nu\left(\frac{\dot\Delta}{3}-\dot f\right)\right]h_2\frac{\partial\Phi}{\partial\eta}\\
&\qquad\qquad+(na+\nu\dot a)h_3\frac{\partial\Phi}{\partial\zeta}=\mathfrak{h}H'\\
&(nb+\nu\dot b)h_1\frac{\partial\Phi}{\partial\xi}+(na+\nu\dot a)h_2\frac{\partial\Phi}{\partial\eta}\\
&\qquad\qquad+\left[(m-n)\Delta+2ng-2\nu\left(\frac{\dot\Delta}{3}-\dot g\right)\right]h_3\frac{\partial\Phi}{\partial\zeta}=\mathfrak{h}Z'
\end{aligned}\right\}\text{...(2)}$$

where $e, f, g, \Delta, a, b, c, \Theta_1, \Theta_2, \Theta_3$ are given in terms of u, v, w by equations (26), (27), (29), (31) of § 235, and $\mathfrak{h}$ by equation (19) of § 233.

Example.

413.] **Torsional Vibrations of a viscous cylindrical rod of circular section.** With the notation of § 244, let the surface of the rod be given by $r=C$, the origin being at one end of the central axis, and the length of the rod being L.

From the analogy of § 335 we are led to assume that the torsional motion will consist in a bodily twisting of each transverse section about the axis in its own plane; or, analytically, that in pure torsion $u=w=0$ and $v=r\phi$, where ϕ is a function only of z and t.

On this assumption we have

$$\left.\begin{aligned}
&e=0,\ f=0,\ g=0,\ \Delta=0,\\
&a=r\frac{\partial\Phi}{\partial z},\ b=0,\ c=0,\\
&\Theta_1=-\frac{r}{2}\frac{\partial\phi}{\partial z},\ \Theta_2=0,\ \Theta_3=\phi
\end{aligned}\right\},$$

and the equations of motion (1) reduce to the single equation

$$\left(n + \nu\frac{\partial}{\partial t}\right)\frac{\partial^2\phi}{\partial z^2} = \rho\frac{\partial^2\phi}{\partial t^2} \quad\dots\dots\dots\dots\dots\dots (3)$$

the conditions (2) for freedom of the lateral surface being satisfied identically.

414.] **Free Oscillations.** We can now solve completely the case in which the end $z=0$ is fixed and the end $z=L$, after having been held twisted through an angle τL till the rod assumes the configuration of equilibrium $\phi=\tau z$ (§ 335), is set free. We have

(*i.*) when $t=0$, $\phi=\tau z$, $\dot\phi=0$.
(*ii.*) when $z=0$, $\phi=0$.
(*iii.*) when $z=L$, $\partial\phi/\partial z=0$.

Thus the appropriate solution of (3) will evidently be of the form

$$\phi = \Sigma A_i \sin pz\, e^{-jt}\cos it,$$

substitution giving us the relations

$$j = p^2\nu/2\rho,\quad i^2 = j^2 + (n - j\nu)p^2/\rho\,;$$

or, if $\Omega' = \sqrt{n/\rho}$ as in § 266

$$\phi = \Sigma A_p \sin pz\,.\,e^{-\nu p\Omega'^2 t/2n}\cos p\Omega'\sqrt{1 - p^2\nu^2\Omega'^2/4n^2}\,.\,t \dots\dots\dots (4)$$

To satisfy the remaining boundary conditions we must have $p=(2i+1)\pi/2L$, where i is any integer, and

$$\tau z = \Sigma A_p \sin\frac{(2i+1)\pi z}{2L},$$

or
$$A_p = \frac{8L\tau}{\pi^2}\cdot\frac{(-1)^i}{(2i+1)^2}.$$

Thus finally

$$\phi = \frac{8L\tau}{\pi^2}\sum_{i=0}^{i=\infty}\frac{(-1)^i}{(2i+1)^2}\sin\frac{(2i+1)\pi z}{2L}\,.\exp\left[-\frac{(2i+1)^2\pi^2\nu\Omega'^2 t}{8nL^2}\right]$$
$$\times\cos\left[\frac{(2i+1)\pi\Omega'}{2L}\sqrt{1-\frac{(2i+1)^2\pi^2\nu^2\Omega'^2}{16n^2L^2}}\,.\,t\right].$$

The effect of viscosity, in increasing the periodic times and steadily diminishing the amplitudes of the vibrations, is obvious.

Extension to Viscous Liquids.

415.] **Equations of Motion.** If we regard a liquid as a limiting form of the solid state in which the elastic rigidity is absolutely zero, we can deduce the equations of motion of a viscous liquid from (41) of § 237 by simply making $n=0$, and $P=Q=R=-\Pi$, where Π is the hydrostatic pressure, *i.e.*, the only *elastic* stress that can exist in such a body (see Appendix IV., pages 169-180).

We have then in general

$$\nu\frac{\partial}{\partial t}\left\{2h_1h_2h_3\frac{\partial}{\partial\xi}\left(\frac{e}{h_2h_3}\right)+2f\frac{h_1}{h_2}\frac{\partial h_2}{\partial\xi}+2g\frac{h_1}{h_3}\frac{\partial h_3}{\partial\xi}\right.$$
$$\left.-\tfrac{2}{3}h_1\frac{\partial\Delta}{\partial\xi}+h_1{}^2h_2h_3\left[\frac{\partial}{\partial\eta}\left(\frac{c}{h_3h_1{}^2}\right)+\frac{\partial}{\partial\zeta}\left(\frac{b}{h_1{}^2h_2}\right)\right]\right\}$$
$$+\rho(\Xi-\ddot{u})-h_1\frac{\partial\Pi}{\partial\xi}=0,\text{ etc.}$$

The relative displacements in a liquid may however be indefinitely great consistently with infinitely small strain (except when they are such as to produce cubical dilatation or compression), and it is in general impossible to identify their magnitudes or directions. All that we are concerned with practically is the relative *velocity of displacement* of different parts of the liquid, and this must of course be small in order that the viscous resistances may be small. If we change our notation, making u, v, w represent the *velocities* of displacement parallel to fixed axes, and e, f, g, a, b, c, Δ the *rates of increase* of the longitudinal extensions, shears and cubical dilatation, the equations of motion will be, when u, v, w are very small,

$$\left.\begin{aligned}
&\nu\left\{2h_2h_3\frac{\partial}{\partial\xi}\left(\frac{e}{h_2h_3}\right)+\frac{2f}{h_2}\frac{\partial h_2}{\partial\xi}+\frac{2g}{h_3}\frac{\partial h_3}{\partial\xi}-\tfrac{2}{3}\frac{\partial\Delta}{\partial\xi}\right.\\
&\qquad\left.+h_1h_2h_3\left[\frac{\partial}{\partial\eta}\left(\frac{c}{h_3h_1{}^2}\right)+\frac{\partial}{\partial\zeta}\left(\frac{b}{h_1{}^2h_2}\right)\right]\right\}+\frac{\rho}{h_1}\left(\Xi-\frac{\partial u}{\partial t}\right)=\frac{\partial\Pi}{\partial\xi}\\
&\nu\left\{2h_3h_1\frac{\partial}{\partial\eta}\left(\frac{f}{h_3h_1}\right)+\frac{2g}{h_3}\frac{\partial h_3}{\partial\eta}+\frac{2e}{h_1}\frac{\partial h_1}{\partial\eta}-\tfrac{2}{3}\frac{\partial\Delta}{\partial\eta}\right.\\
&\qquad\left.+h_1h_2h_3\left[\frac{\partial}{\partial\zeta}\left(\frac{a}{h_1h_2{}^2}\right)+\frac{\partial}{\partial\xi}\left(\frac{c}{h_2{}^2h_3}\right)\right]\right\}+\frac{\rho}{h_2}\left(H-\frac{\partial v}{\partial t}\right)=\frac{\partial\Pi}{\partial\eta}\\
&\nu\left\{2h_1h_2\frac{\partial}{\partial\zeta}\left(\frac{g}{h_1h_2}\right)+\frac{2e}{h_1}\frac{\partial h_1}{\partial\zeta}+\frac{2f}{h_2}\frac{\partial h_2}{\partial\zeta}-\tfrac{2}{3}\frac{\partial\Delta}{\partial\zeta}\right.\\
&\qquad\left.+h_1h_2h_3\left[\frac{\partial}{\partial\xi}\left(\frac{b}{h_2h_3{}^2}\right)+\frac{\partial}{\partial\eta}\left(\frac{a}{h_3{}^2h_1}\right)\right]\right\}+\frac{\rho}{h_3}\left(Z-\frac{\partial w}{\partial t}\right)=\frac{\partial\Pi}{\partial\zeta}
\end{aligned}\right\}\quad\ldots(5)$$

The quantities e, f, g, Δ, a, b, c will still be given in terms of u, v, w by the linear equations (26), (27) and (29) of § 235.

416.] Liquids treated as "Incompressible." In all mobile liquids the numerical value of k is so very great in comparison with that of ν, that it is usual to neglect $\nu\Delta$ in comparison with Π. In fact, if the hydrostatic pressure be supposed of the same order of small quantities as the shearing stresses due to viscosity, the rate of cubical compression will be very small in comparison with the rate of shearing. This treatment of liquids as "incompressible" is of course only an approximation, intended solely to reduce the great analytical difficulties introduced into hydrodynamics by taking viscosity into account.

On this assumption the equations of motion may be written in the form

$$\left.\begin{aligned}
&\nu h_1h_2h_3\left\{2h_2\frac{\partial}{\partial\xi}\left(\frac{f}{h_2{}^2h_3}\right)+2h_3\frac{\partial}{\partial\xi}\left(\frac{g}{h_2h_3{}^2}\right)-h_1\frac{\partial}{\partial\eta}\left(\frac{c}{h_3h_1{}^2}\right)\right.\\
&\qquad\left.-h_1\frac{\partial}{\partial\zeta}\left(\frac{b}{h_1{}^2h_2}\right)\right\}=\rho\left(\Xi-\frac{\partial u}{\partial t}\right)-h_1\frac{\partial\Pi}{\partial\xi}\\
&\nu h_1h_2h_3\left\{2h_3\frac{\partial}{\partial\eta}\left(\frac{g}{h_3{}^2h_1}\right)+2h_1\frac{\partial}{\partial\eta}\left(\frac{e}{h_3h_1{}^2}\right)-h_2\frac{\partial}{\partial\zeta}\left(\frac{a}{h_1h_2{}^2}\right)\right.\\
&\qquad\left.-h_2\frac{\partial}{\partial\xi}\left(\frac{c}{h_2{}^2h_3}\right)\right\}=\rho\left(\mathrm{H}-\frac{\partial v}{\partial t}\right)-h_2\frac{\partial\Pi}{\partial\eta}\\
&\nu h_1h_2h_3\left\{2h_1\frac{\partial}{\partial\zeta}\left(\frac{e}{h_1{}^2h_2}\right)+2h_2\frac{\partial}{\partial\zeta}\left(\frac{f}{h_1h_2{}^2}\right)-h_3\frac{\partial}{\partial\xi}\left(\frac{b}{h_2h_3{}^2}\right)\right.\\
&\qquad\left.-h_3\frac{\partial}{\partial\eta}\left(\frac{a}{h_3{}^2h_1}\right)\right\}=\rho\left(\mathrm{Z}-\frac{\partial w}{\partial t}\right)-h_3\frac{\partial\Pi}{\partial\zeta}
\end{aligned}\right\}\quad\text{......(6)}$$

with the further condition

$$e+f+g=0.\quad\text{..............................(7)}$$

When referred to Cartesians these equations take the simple forms

$$\left.\begin{aligned}
\nu\nabla^2u-\frac{\partial\Pi}{\partial x}+\rho\left(X-\frac{\partial u}{\partial t}\right)&=0\\
\nu\nabla^2v-\frac{\partial\Pi}{\partial y}+\rho\left(Y-\frac{\partial v}{\partial t}\right)&=0\\
\nu\nabla^2w-\frac{\partial\Pi}{\partial z}+\rho\left(Z-\frac{\partial w}{\partial t}\right)&=0
\end{aligned}\right\}\quad\text{.....................(8)}$$

with

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0;\quad\text{...........................(9)}$$

so that, in the case of conservative impressed forces, derived from a potential Ψ, we obtain by elimination

$$\nabla^2(\Pi-\rho\Psi)=0.\quad\text{..............................(10)}$$

[Compare this with equation (164) of § 303.]

417.] **Boundary Conditions.** In practice, the bounding surfaces of a liquid must be either (*i.*) free, (*ii.*) subject to the uniform normal pressure of a gas, or (*iii.*) in contact with a solid or another liquid; and in all but the first of these cases the contact must be maintained throughout the motion.

Thus in cases (*ii.*) and (*iii.*) we have the purely kinematic condition that the *normal* velocity of every point in the surface of a liquid is equal to that of the point in the surface of the other body (of whatever nature) in contact with it.

The dynamical condition is when relative motion takes place between the two bodies, tangentially to the dividing surface, the shearing stress exerted on either is proportional to the relative velocity, and in a direction tending directly to retard it. Thus, let u be the velocity of any point in the surface of the liquid resolved in the tangent plane to the surface, and u' the surface velocity of the body in contact with it: then u satisfies the equation

$$\frac{\partial u}{\partial n} + \mu(u - u') = 0 \quad\ldots\ldots\ldots\ldots(11)$$

where n is the element of normal to the surface, measured outwards from the liquid, and μ is a new constant—the "modulus of contact viscosity."

In the more mobile liquids (ether, alcohol, etc.) the value of μ is so great that practically no relative slipping takes place at the surfaces of contact, so that the surface velocities of the liquid, *in all directions*, are the same as those of the body limiting it.

In case (i) the boundary conditions are to be found by writing $n = 0$, $\Delta = 0$, $\Xi' = \mathrm{H}' = Z' = 0$, $P = Q = R = -\Pi$ in equations (2). Thus they become with our new notation

$$\left.\begin{aligned} (2\nu e - \Pi)h_1\frac{\partial\Phi}{\partial\xi} + \nu c h_2\frac{\partial\Phi}{\partial\eta} + \nu b h_3\frac{\partial\Phi}{\partial\zeta} &= 0 \\ \nu c h_1\frac{\partial\Phi}{\partial\xi} + (2\nu f - \Pi)h_2\frac{\partial\Phi}{\partial\eta} + \nu a h_3\frac{\partial\Phi}{\partial\zeta} &= 0 \\ \nu b h_1\frac{\partial\Phi}{\partial\xi} + \nu a h_2\frac{\partial\Phi}{\partial\eta} + (2\nu g - \Pi)h_3\frac{\partial\Phi}{\partial\zeta} &= 0 \end{aligned}\right\} \quad\ldots\ldots\ldots(12)$$

Examples of the motion of Viscous Liquids will be found in Professor Lamb's "Motion of Fluids," Chapter IX.

APPENDIX VI.

Economy of Material in Nature.

A few simple examples of economy of material—*i.e.*, the principle of producing the greatest possible elastic strength under specified types of strain, with the least expenditure of a given material—have already been discussed in Chapter VII. Numerous beautiful applications of this principle are to be found among organic structures, and in fact they may be looked for with confidence wherever great strength in proportion to the material available, or great lightness in proportion to strength is an advantage.

Good examples in the vegetable kingdom are to be found in the stems of the grasses and the order Umbelliferæ. These plants grow thickly together, or force their way among other thickly growing plants, and often on very poor soils. They are all enormously reproductive, and bear their seeds in heavy masses. It is therefore of the utmost importance to them to use the least possible material in building up their stems, and at the same time to make them strong enough to resist considerable vertical thrust and flexion couple. They all have largely hollowed cylindrical stems.

Very young trees, which have to struggle for food with the surrounding grasses, etc., have most of their mass concentrated in an external cylindrical layer of the stem, the axial portion being occupied by a soft and light pith. As growth proceeds, however, and their leaves in the one direction, and their roots in the other, emerge from the sphere of close competition, they accumulate material beyond the strict needs of economy, and it is largely devoted to hardening of the axial portion of the stem. Consequently in old trees the "heart-wood" is the more durable and valuable portion of the trunk.

The stem of the common rush, on the other hand, composed of a thin but very tough outer rind, requiring some exertion of strength to break it, and a pith of large relative volume but very small mass, is a good instance of the attainment of extreme lightness without too great a sacrifice of strength.

It is, however, in the complex structure of the bones of the higher animals that we find the most consistent and remarkable application of the principle of economy. It is of course advisable, in order that the muscular power may be fully utilised, that the bones, which from a mechanical point of view are simply an inert system of levers, should be as light as possible, and at the same time the exertion of that very power exposes them habitually to

considerable stresses. In order to explain how these varied requirements are met, we will describe the structure of the human thigh-bone as a typical example. This is a long bone, the principal office of which is to transmit half the weight of the trunk and head to the knee-joint, and thence to the ground. The principal stress to which it is subject is therefore one of longitudinal thrust. It is however also subjected, especially in walking or running, to considerable flexion couple and slight torsion couple. The bone consists of two terminal articular masses, which receive the complicated stresses from the joints and muscles, and a connecting shaft, almost the only function of which is to transmit stress from one articular mass to the other. The shaft, which for our purposes may be regarded as approximately cylindrical, thus receives almost its entire stress across its end surfaces, and, in accordance with the principles of §§ 329, 336, and 355, it is extensively hollowed out throughout its length, the hard, rigid and heavy bone-substance being compactly arranged, mainly in the form of longitudinal fibres, as a cylindrical casing, and the interior space being filled with light and semi-fluid marrow, which for practical purposes may be said to offer resistance only to cubical compression. The structure of the articular masses, which are subject to very varied stresses over the greater portion of their surfaces, is naturally much more complicated. Broadly speaking, it may be said that the rigid bone-substance of the shaft-cylinder divides on entering the terminal mass into a thin outer casing, and a series of thin laminæ which in the main take the form of the principal surfaces of the most severe form of strain to which the mass is subject, the small orthogonal spaces enclosed by these laminæ being filled with marrow. Under the specified strain the laminæ are in the proper position to transmit directly the principal normal stresses, and are only subject to cubical compression, and the interspaces to change of volume, to which their contents offer a resistance comparable with that of a solid. On the other hand, the composite structure admits readily of small deformations under accidental shocks in unaccustomed directions. The advantages of this arrangement over a solid bony structure, either of the same strength or of the same weight, are obvious.

Figure 62 exhibits a slightly diagrammatic view of the lines of stress* in a section of the upper portion of the thigh-bone, cut vertically from right to left and looked at from the front. It will be seen that the "head" AB has a considerable inclination inwards, like the head of a crane. The direct thrust, due to the weight of the body, falls exclusively upon the surface A, the tensions on the surfaces B, C, D, E, H and J being due to liga-

* Of course they are not really plane curves.

ments and muscles exerting the couple necessary to maintain the upright posture. The muscles arising from F and G assist in keeping the knee-joint rigid. It is evident that the main thrust will be transmitted by strut lines down the inner side of the shaft, while the orthogonal tensions required to support the "head" will act along tie lines arising from the outer side. The details of the arrangement are shown in the figure.

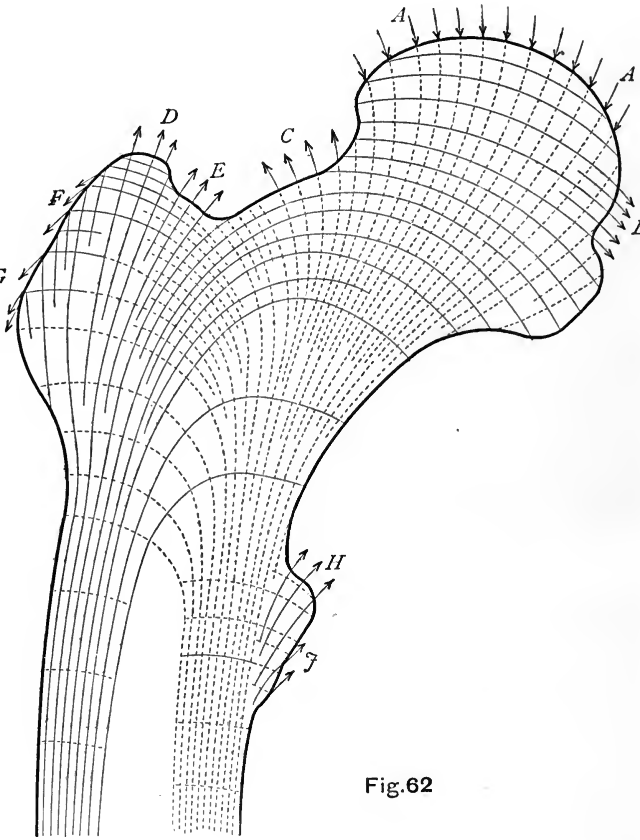


Fig.62

On comparing this with Figure 63, which is *from a photograph of an actual section of the same bone,* the reader cannot fail to be struck by the extraordinary closeness with which the sections of the bony laminæ correspond to the theoretical lines of stress.

The small bones of the body, such as those of the spine, the wrist, and the ankle and heel, are practically in the position of

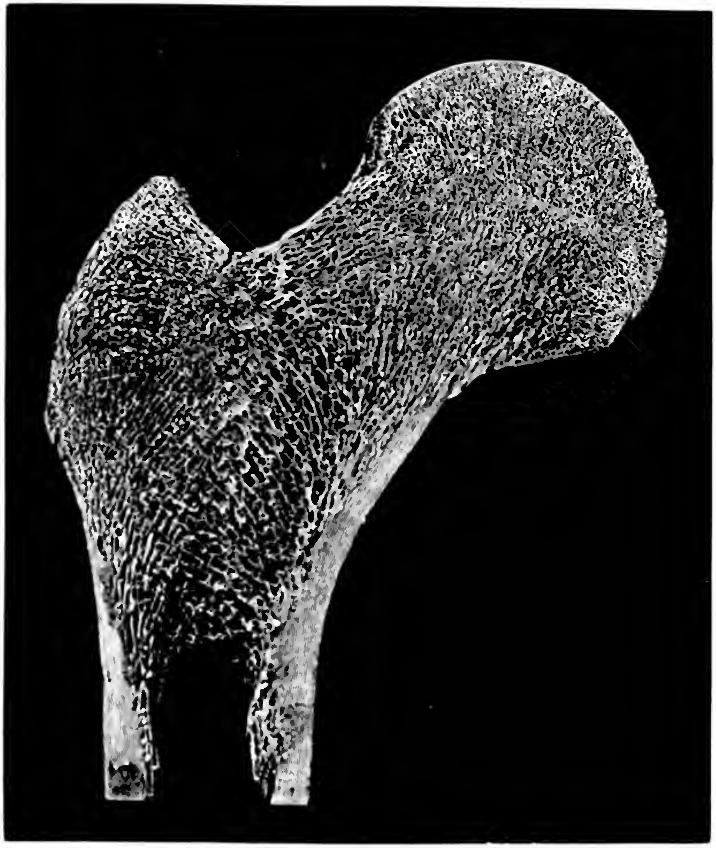


Fig. 63.

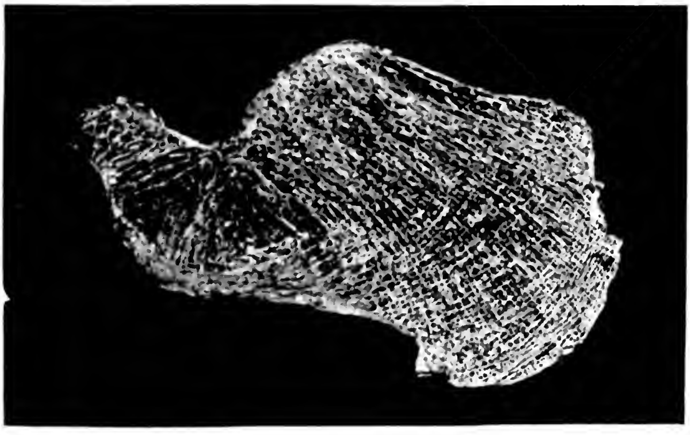


Fig. 65.

articular masses without shafts, and are constructed on the same principle. Figure 64 represents the theoretical lines of stress in

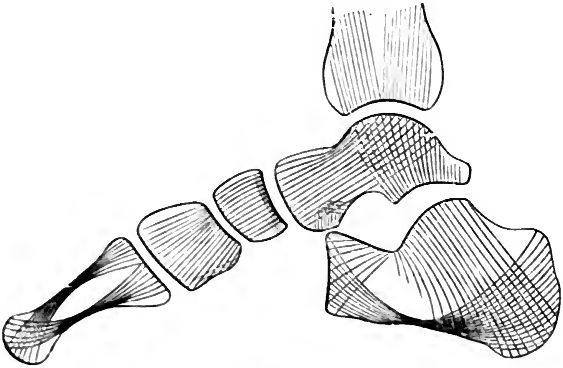


Fig. 64.

the bones articulating at the ankle-joint, and Figure 65 is from a photograph of a section of the heel bone.*

* The student should read a charming paper, entitled "How a Bone is Built," by Dr. Donald Macalister, in the "English Illustrated Magazine" for July, 1884 (Volume I., pages 640-649). Figures 63, 64, 65 are taken from that paper, by the kind permission of the Publishers. The photographs are by Zaaijer, engraved by J. D. Cooper, and the diagram 64 is from a drawing by Professor Hermann Meyer.

ADDITIONAL NOTES.

I.—§ 3, page 2.

Sphere of Action of the Intermolecular Forces.

Some very interesting cases are known of the appreciable action between the molecules of bodies whose surfaces can be brought into really intimate contact. Professor Tait supplies the following instances ("Properties of Matter"):—

(*i.*) Finely powdered graphite is re-solidified in the manufacture of lead pencils by the application of powerful pressure. (*ii.*) Two freshly cut lead surfaces, pressed firmly together with a screwing motion, will adhere very strongly to one another. (*iii.*) Sir Joseph Whitworth's steel planes are so true that, when pressed together, they offer a resistance to separation markedly greater than can be accounted for by the pressure of the atmosphere. (*iv.*) The surfaces of marble blocks may be so truly worked that, on being pressed together, either can be lifted suspended from the other, even *in vacuo* (if its weight be not too great in comparison with the area of contact). (*v.*) All the processes of gilding, silver-plating, etc., as well as the properties of gum and glue, depend upon the cohesive forces between molecules brought within insensible distances of one another.

II.—§ 123, page 56.

Expressions for the Component Strains and Rotations, to the second power of small quantities.

Let the coördinates of the points P, Q, R in the natural state be (x, y, z), $(x+dx, y, z)$, $(x, y+dy, z)$, and let P', Q', R' be the strained positions of these points. Then, if the component displacements of P be u, v, w, the coördinates of Q' relative to P' will be

$$\left(1+\frac{\partial u}{\partial x}\right)dx,\quad \frac{\partial v}{\partial x}dx,\quad \frac{\partial w}{\partial x}dx.$$

But $P'Q' = (1+e)PQ = (1+e)dx$, and therefore

$$(1+e)^2 = \left(1+\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 :$$

thus to the second order of approximation

$$\left.\begin{aligned} e &= \frac{\partial u}{\partial x} + \tfrac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\ f &= \frac{\partial v}{\partial y} + \tfrac{1}{2}\left[\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \\ g &= \frac{\partial w}{\partial z} + \tfrac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] \end{aligned}\right\}.$$

Again the projections of $P'R'$ upon the axes are

$$\frac{\partial u}{\partial y}dy, \quad \left(1+\frac{\partial v}{\partial y}\right)dy, \quad \frac{\partial w}{\partial y}dy,$$

and $\qquad P'R' = (1+f)dy$, so that

$\sin c = \cos(\tfrac{1}{2}\pi - c) = \cos Q'P'R'$

$$= \frac{\left(1+\dfrac{\partial u}{\partial x}\right)\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\left(1+\dfrac{\partial v}{\partial y}\right) + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}}{\left\{1+\dfrac{\partial u}{\partial x} + \tfrac{1}{2}\left[\left(\dfrac{\partial v}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial x}\right)^2\right]\right\}\left\{1+\dfrac{\partial v}{\partial y} + \tfrac{1}{2}\left[\left(\dfrac{\partial w}{\partial y}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2\right]\right\}}$$

and ultimately

$$\left.\begin{aligned} a &= \frac{\partial w}{\partial y}\left(1-\frac{\partial v}{\partial y}\right) + \frac{\partial v}{\partial z}\left(1-\frac{\partial w}{\partial z}\right) + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} \\ b &= \frac{\partial u}{\partial z}\left(1-\frac{\partial w}{\partial z}\right) + \frac{\partial w}{\partial x}\left(1-\frac{\partial u}{\partial x}\right) + \frac{\partial v}{\partial z}\frac{\partial v}{\partial x} \\ c &= \frac{\partial v}{\partial x}\left(1-\frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial y}\left(1-\frac{\partial v}{\partial y}\right) + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{aligned}\right\}.$$

Finally, if $P'Q'$, $P'R'$ make angles ϕ, ψ with the plane of zx,

$$\theta_3 = \tfrac{1}{2}(\phi+\psi) - \frac{\pi}{4};$$

$$\therefore \qquad \sin\theta_3 = \tfrac{1}{2}\left[\sqrt{1-\cos(\phi+\psi)} - \sqrt{1+\cos(\phi+\psi)}\right].$$

Now $\qquad \tan\phi = \dfrac{\partial v}{\partial x}\Big/\left(1+\dfrac{\partial u}{\partial x}\right), \quad \tan\psi = \left(1+\dfrac{\partial v}{\partial y}\right)\Big/\dfrac{\partial u}{\partial y},$

and therefore

$$\cos(\phi+\psi) = -\frac{\dfrac{\partial v}{\partial x}\left(1+\dfrac{\partial v}{\partial y}\right) - \dfrac{\partial u}{\partial y}\left(1+\dfrac{\partial u}{\partial x}\right)}{1+\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\tfrac{1}{2}\left[\left(\dfrac{\partial v}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2+2\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y}\right]}.$$

Thus ultimately

$$\left.\begin{aligned}
2\theta_1 &= \frac{\partial w}{\partial y}\left(1+\frac{\partial w}{\partial z}\right)-\frac{\partial v}{\partial z}\left(1+\frac{\partial v}{\partial y}\right)\\
2\theta_2 &= \frac{\partial u}{\partial z}\left(1+\frac{\partial u}{\partial x}\right)-\frac{\partial w}{\partial x}\left(1+\frac{\partial w}{\partial z}\right)\\
2\theta_3 &= \frac{\partial v}{\partial x}\left(1+\frac{\partial v}{\partial y}\right)-\frac{\partial u}{\partial y}\left(1+\frac{\partial u}{\partial x}\right)
\end{aligned}\right\}.$$

III.—§ 235, page 222.

Transformation of the Component Rotations.

With the notation of Chapter V.,

$$\begin{aligned}
2\theta_1 &= \frac{\partial}{\partial y}\Big[\nu_1 u+\nu_2 v+\nu_3 w\Big]-\frac{\partial}{\partial z}\Big[\mu_1 u+\mu_2 v+\mu_3 w\Big]\\
&= \frac{\partial}{\partial y}\left[\frac{u}{h_1}\frac{\partial\xi}{\partial z}+\frac{v}{h_2}\frac{\partial\eta}{\partial z}+\frac{w}{h_3}\frac{\partial\zeta}{\partial z}\right]-\frac{\partial}{\partial z}\left[\frac{u}{h_1}\frac{\partial\xi}{\partial y}+\frac{v}{h_2}\frac{\partial\eta}{\partial y}+\frac{w}{h_3}\frac{\partial\zeta}{\partial y}\right]\\
&= \frac{\partial\xi}{\partial z}\frac{\partial}{\partial y}\left(\frac{u}{h_1}\right)-\frac{\partial\xi}{\partial y}\frac{\partial}{\partial z}\left(\frac{u}{h_1}\right)+\frac{\partial\eta}{\partial z}\frac{\partial}{\partial y}\left(\frac{v}{h_2}\right)-\frac{\partial\eta}{\partial y}\frac{\partial}{\partial z}\left(\frac{v}{h_2}\right)\\
&\qquad+\frac{\partial\zeta}{\partial z}\frac{\partial}{\partial y}\left(\frac{w}{h_3}\right)-\frac{\partial\zeta}{\partial y}\frac{\partial}{\partial z}\left(\frac{w}{h_3}\right)\\
&= \left[\nu_1 h_1\left(\mu_1\frac{\partial}{\partial s_1}+\mu_2\frac{\partial}{\partial s_2}+\mu_3\frac{\partial}{\partial s_3}\right)-\mu_1 h_1\left(\nu_1\frac{\partial}{\partial s_1}+\nu_2\frac{\partial}{\partial s_2}+\nu_3\frac{\partial}{\partial s_3}\right)\right]\left(\frac{u}{h_1}\right)\\
&\quad+\left[\nu_2 h_2\left(\mu_1\frac{\partial}{\partial s_1}+\mu_2\frac{\partial}{\partial s_2}+\mu_3\frac{\partial}{\partial s_3}\right)-\mu_2 h_2\left(\nu_1\frac{\partial}{\partial s_1}+\nu_2\frac{\partial}{\partial s_2}+\nu_3\frac{\partial}{\partial s_3}\right)\right]\left(\frac{v}{h_2}\right)\\
&\quad+\left[\nu_3 h_3\left(\mu_1\frac{\partial}{\partial s_1}+\mu_2\frac{\partial}{\partial s_2}+\mu_3\frac{\partial}{\partial s_3}\right)-\mu_3 h_3\left(\nu_1\frac{\partial}{\partial s_1}+\nu_2\frac{\partial}{\partial s_2}+\nu_3\frac{\partial}{\partial s_3}\right)\right]\left(\frac{w}{h_3}\right)\\
&= h_1\left(\lambda_2\frac{\partial}{\partial s_3}-\lambda_3\frac{\partial}{\partial s_2}\right)\left(\frac{u}{h_1}\right)+h_2\left(\lambda_3\frac{\partial}{\partial s_1}-\lambda_1\frac{\partial}{\partial s_3}\right)\left(\frac{v}{h_2}\right)\\
&\qquad+h_3\left(\lambda_1\frac{\partial}{\partial s_2}-\lambda_2\frac{\partial}{\partial s_1}\right)\left(\frac{w}{h_3}\right);
\end{aligned}$$

and so for θ_2 and θ_3.

But $\Theta_1=\lambda_1\theta_1+\mu_1\theta_2+\nu_1\theta_3$; and therefore

$$\begin{aligned}
2\Theta_1 &= h_3\frac{\partial}{\partial s_2}\left(\frac{w}{h_3}\right)-h_2\frac{\partial}{\partial s_3}\left(\frac{v}{h_2}\right)\\
&= h_2 h_3\left[\frac{\partial}{\partial\eta}\left(\frac{w}{h_3}\right)-\frac{\partial}{\partial\zeta}\left(\frac{v}{h_2}\right)\right],
\end{aligned}$$

and so on.

IV.—§ 239, page 229.

Lamé's transformation of the General Equations.

Multiplying equations (52a) of § 218 by λ_1, μ_1, ν_1 respectively and adding, we obtain

$$(m+n)\left(\lambda_1\frac{\partial\Delta}{\partial x}+\mu_1\frac{\partial\Delta}{\partial y}+\nu_1\frac{\partial\Delta}{\partial z}\right)$$
$$-2n\left[\lambda_1\left(\frac{\partial\theta_3}{\partial y}-\frac{\partial\theta_2}{\partial z}\right)+\mu_1\left(\frac{\partial\theta_1}{\partial z}-\frac{\partial\theta_3}{\partial x}\right)+\nu_1\left(\frac{\partial\theta_2}{\partial x}-\frac{\partial\theta_1}{\partial y}\right)\right]$$
$$+\rho[\lambda_1(X-\ddot{u})+\mu_1(Y-\ddot{v})+\nu_1(Z-\ddot{w})]=0,$$

or, with the notation of Chapter V.,

$$(m+n)\frac{\partial\Delta}{\partial s_1}-2n\left[\lambda_1\left(\frac{\partial\theta_3}{\partial y}-\frac{\partial\theta_2}{\partial z}\right)+\mu_1\left(\frac{\partial\theta_1}{\partial z}-\frac{\partial\theta_3}{\partial x}\right)\right.$$
$$\left.+\nu_1\left(\frac{\partial\theta_2}{\partial x}-\frac{\partial\theta_1}{\partial y}\right)\right]+\rho(\Xi-\ddot{u})=0.$$

Now
$$\theta_2=\frac{\Theta_1}{h_1}\frac{\partial\xi}{\partial y}+\frac{\Theta_2}{h_2}\frac{\partial\eta}{\partial y}+\frac{\Theta_3}{h_3}\frac{\partial\zeta}{\partial y},$$
$$\theta_3=\frac{\Theta_1}{h_1}\frac{\partial\xi}{\partial z}+\frac{\Theta_2}{h_2}\frac{\partial\eta}{\partial z}+\frac{\Theta_3}{h_3}\frac{\partial\zeta}{\partial z},$$

and therefore, by the results of the last Note,

$$\frac{\partial\theta_3}{\partial y}-\frac{\partial\theta_2}{\partial z}=h_1\left(\lambda_2\frac{\partial}{\partial s_3}-\lambda_3\frac{\partial}{\partial s_2}\right)\left(\frac{\Theta_1}{h_1}\right)$$
$$+h_2\left(\lambda_3\frac{\partial}{\partial s_1}-\lambda_1\frac{\partial}{\partial s_3}\right)\left(\frac{\Theta_2}{h_2}\right)+h_3\left(\lambda_1\frac{\partial}{\partial s_2}-\lambda_2\frac{\partial}{\partial s_1}\right)\left(\frac{\Theta_3}{h_3}\right),$$

and so on. Consequently

$$\lambda_1\left(\frac{\partial\theta_3}{\partial y}-\frac{\partial\theta_2}{\partial z}\right)+\mu_1\left(\frac{\partial\theta_1}{\partial z}-\frac{\partial\theta_3}{\partial x}\right)+\nu_1\left(\frac{\partial\theta_2}{\partial x}-\frac{\partial\theta_1}{\partial y}\right)$$
$$=h_3\frac{\partial}{\partial s_2}\left(\frac{\Theta_3}{h_3}\right)-h_2\frac{\partial}{\partial s_3}\left(\frac{\Theta_2}{h_2}\right),$$

and the transformed equations are

$$(m+n)\frac{\partial\Delta}{\partial s_1}-2n\left[h_3\frac{\partial}{\partial s_2}\left(\frac{\Theta_3}{h_3}\right)-h_2\frac{\partial}{\partial s_3}\left(\frac{\Theta_2}{h_2}\right)\right]+\rho(\Xi-\ddot{u})=0,$$

etc., etc.

V.—§ 241, page 231.

Differential Equations of the Lines of Stress, referred to any Curvilinear system.

It follows at once from § 163 that the principal stresses N_1, N_2, N_3, at any point are the roots of the cubic in ϕ

$$\begin{vmatrix} P-\phi & U & T \\ U & Q-\phi & S \\ T & S & R-\phi \end{vmatrix} = 0$$

and that the directions of the principal axes are given by

$$\frac{P\lambda + U\mu + T\nu}{\lambda} = \frac{U\lambda + Q\mu + S\nu}{\mu} = \frac{T\lambda + S\mu + R\nu}{\nu} = N,$$

where λ, μ, ν are the cosines of the angles made by the principal axis corresponding to N with the elements ds_1, ds_2, ds_3, and the notation is throughout that of Chapter V.

Now $ds_1/\lambda = ds_2/\mu = ds_3/\nu = ds$, where ds is the element of the principal axis, so that these latter equations may be written

$$\frac{Pds_1 + Uds_2 + Tds_3}{ds_1} = \frac{Uds_1 + Qds_2 + Sds_3}{ds_2} = \frac{Tds_1 + Sds_2 + Rds_3}{ds_3} = N.$$

These then are the differential equations of the Lines of Stress. See § 293 for an example.

VI.—§ 401, page 477.

A Theorem in Conjugate Functions.

If ξ, η be conjugate functions of x and y, and if

$$h = \sqrt{\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2} = \sqrt{\left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2}$$

it is required to show that $\nabla^2 \log h = 0$.

Whatever function h may be of x and y,

$$2h^4\nabla^2 \log h = h^2\nabla^2 h^2 - \left(\frac{\partial h^2}{\partial x}\right)^2 - \left(\frac{\partial h^2}{\partial y}\right)^2$$

identically. But

$$h^2 = \left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2,$$

and therefore

$$h^4\nabla^2 \log h = \left[\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2\right]\left\{\frac{\partial\xi}{\partial x}\frac{\partial^3\xi}{\partial x^3} + \left(\frac{\partial^2\xi}{\partial x^2}\right)^2\right.$$

$$\left. + \frac{\partial\xi}{\partial y}\frac{\partial^3\xi}{\partial x^2\partial y} + \frac{\partial\xi}{\partial x}\frac{\partial^3\xi}{\partial x\partial y^2} + 2\left(\frac{\partial^2\xi}{\partial x\partial y}\right)^2 + \frac{\partial\xi}{\partial y}\frac{\partial^3\xi}{\partial y^3} + \left(\frac{\partial^2\xi}{\partial y^2}\right)^2\right\}$$

$$- 2\left(\frac{\partial\xi}{\partial x}\frac{\partial^2\xi}{\partial x^2} + \frac{\partial\xi}{\partial y}\frac{\partial^2\xi}{\partial x\partial y}\right)^2 - 2\left(\frac{\partial\xi}{\partial x}\frac{\partial^2\xi}{\partial x\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial^2\xi}{\partial y^2}\right)^2$$

$$= \left[\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2\right]\left[\frac{\partial\xi}{\partial x}\frac{\partial\nabla^2\xi}{\partial \xi} + \frac{\partial\xi}{\partial y}\frac{\partial\nabla^2\xi}{\partial y}\right]$$

$$- \left\{\left[\left(\frac{\partial\xi}{\partial x}\right)^2 - \left(\frac{\partial\xi}{\partial y}\right)^2\right]\left(\frac{\partial^2\xi}{\partial x^2} - \frac{\partial^2\xi}{\partial y^2}\right) + 4\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}\frac{\partial^2\xi}{\partial x\partial y}\right\}\nabla^2\xi$$

$= 0$ identically, since $\nabla^2\xi = 0$.

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[*The Arabic numbers refer to the Articles.*]

ROBERT MACLEHOSE, UNIVERSITY PRESS, GLASGOW.